Algebraic Structures Derived from Essential Surfaces and Foams

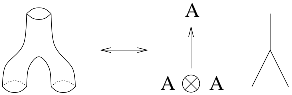
Masahico Saito (U. of South Florida)
Coauthor: J. Scott Carter

Waseda University, December 25, 2009

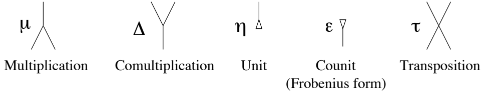
Outline

Background

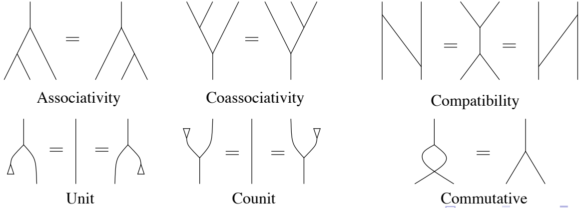
$\{\ (1+1)\text{-TQFTs}\ \} \overset{1-1}{\longleftrightarrow} \{\ \text{Commutative Frobenius algebras}\ \}$



Diagrammatics of Frobenius algebras, generating operators:

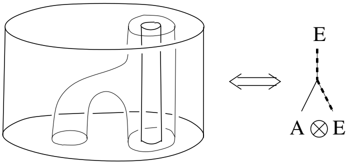


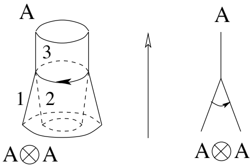
Relations:



Problem

Problem: Characterize TQFTs of essential surface cobordisms and foams.





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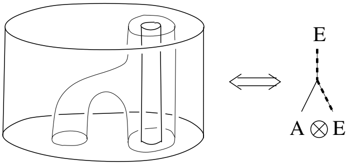

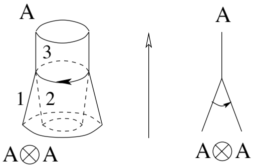

We focus on thickened surfaces $F \times [0,1]$ and $sl(3)$-foams, and investigate their algebraic structures.

Motivations

- A natural question as generalizations of $(1+1)$-TQFT.
- Essential cobordisms are used in global Khovanov homology by Asaeda–Przytycki-Sikora [APS] (for thickened surfaces $F \times [0,1]$), Turaev-Turner [TT] (unoriented TQFT) and Ishii-Tanaka [IT] (for virtual knots, cf. [Manturov]). They should have algebraic structures.
- Foams appear in $sl(3)$ Khovanov homology [Mackaay-Vaz].
- By considering such algebraic structures of generalized TQFTs, new generalizations of KhoHo may be found.
- Possible refinements of Bar-Natan modules by Asaeda-Frohman, Kaiser.

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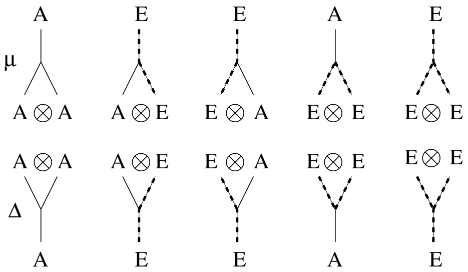
Outline

Generating maps

Generating maps for some saddle points:



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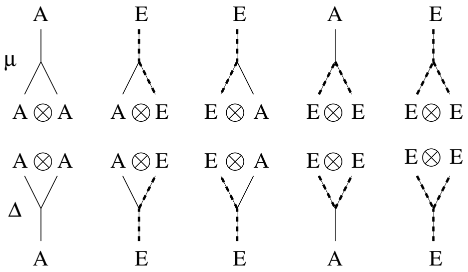

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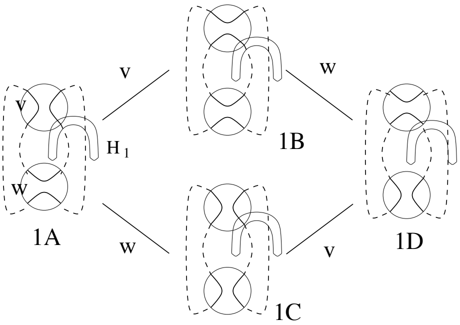
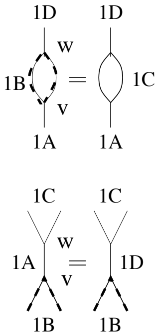
Relations

Relations are checked case by case.
For example, if a handle is present as depicted, we obtain a relation:





Outline

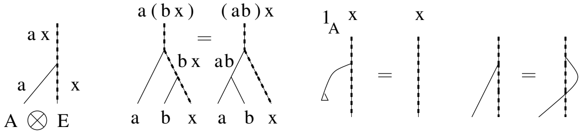
Frobenius pairs

By checking all conditions in [APS],

Definition

We propose a *commutative Frobenius pair* (A, E):

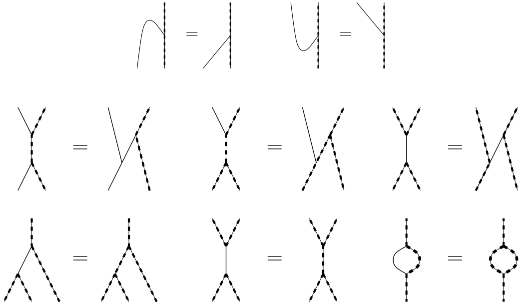
(i) $A = (m_A, \Delta_A, \iota_A, \epsilon_A)$ is a commutative Frobenius algebra over k.
(ii) E is an A-bimodule and A-bicomodule, with the same right and left actions and coactions.



(1) (2) (3) (4)

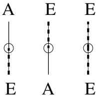
Frobenius pairs (cont.)

These should satisfy variety of conditions:

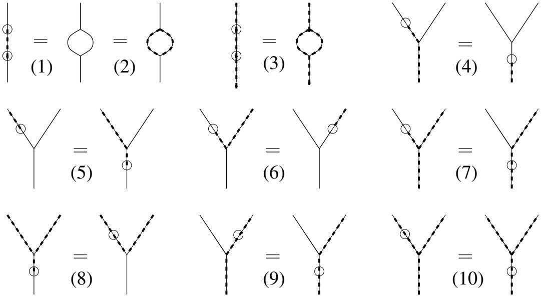


Roughly, the dotted line does not end at a trivalent vertex, while a solid line can. All relations that satisfy this condition hold.

Möbius maps

Three maps 

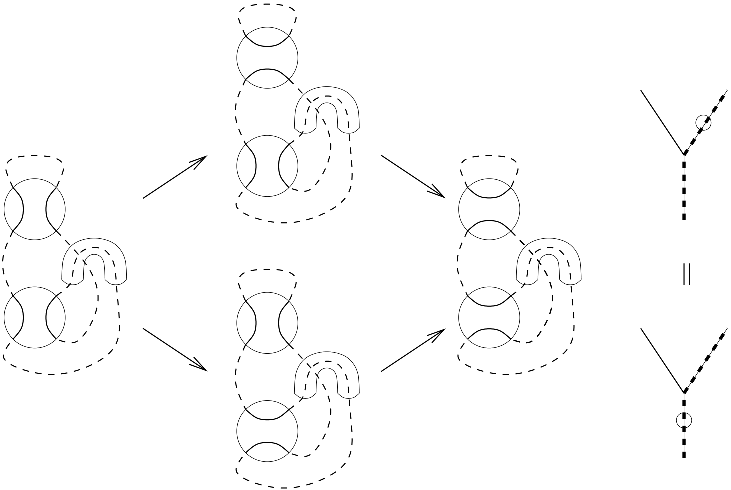
are called *Möbius maps* if they satisfy:



corresponding to non-orientable cobordisms.

Möbius maps

For example, Case (4) of [APS] gives rise to one of the conditions:



Examples

Example

For [APS]: $A = \mathbb{Z}[X]/(X^2)$, $E = \langle Y, Z \rangle_{\mathbb{Z}}$ with correspondence: $1 \leftrightarrow -$, $X \leftrightarrow +$, $Y \leftrightarrow -0$, $Z \leftrightarrow +0$.
Multiplications: $XY = XZ = Y^2 = Z^2 = 0$ and $YZ = X$.
Comultiplications:

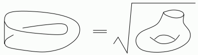
$$\Delta(1) = 1 \otimes X + X \otimes 1 + Y \otimes Z + Z \otimes Y,$$

$$\Delta(X) = X \otimes X, \ \Delta(Y) = X \otimes Y, \ \Delta(Z) = X \otimes Z.$$

Möbius maps: $\nu(1) = Y + Z$, $\nu(X) = 0$, $\nu(Y) = \nu(Z) = X$.

Example

For [TT]: $A = E = \mathbb{Z}_2[X, \lambda^{\pm 1}]/(X^2 - hX)$ with $h = \lambda^2$. All mutiplications and comultiplications are those of A. All Möbius maps are defined by multiplication by λ.

Note: $(\mu\Delta)(1) = h = \lambda^2$, .

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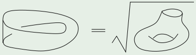
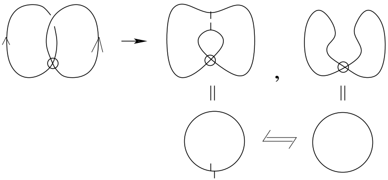
Note: $(\mu\Delta)(1) = h = \lambda^2$, .

Virtual circles with poles

Smoothings for the Miyazawa polynomial



Smoothings of a virtual Hopf link

A circle with pairs of opposite poles are considered essential. The two smoothings correspond to a cobordism between essential and inessential circles.

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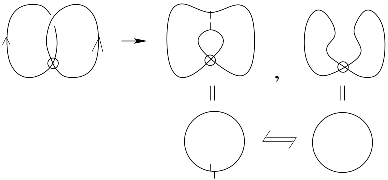

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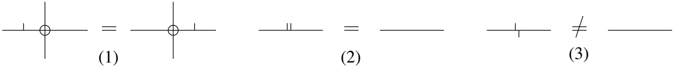
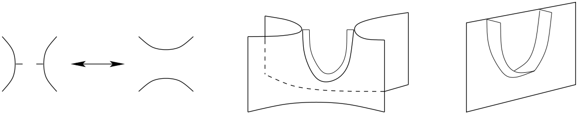


Relations among poles in the Miyazawa polynomial



(1) (2) (3)

Hemmed cobordisms formed by poles

Cobordisms of virtual circles with poles have a structure of commutative Frobenius pairs with Möbius maps.

Outline

Constructions

A generalization of [APS]:

Theorem

Let $A = \mathbb{Z}[X, h, t]/(X^2 - hX - t)$ and $E = \langle Y, Z \rangle$. If (A, E) is a commutative Frobenius pair with Möbius maps such that $\mu^E_{E,E} = \Delta^{E,E}_E = 0$, then A must be $A = \mathbb{Z}[X, a]/(X - a)^2$. For this A, operations are defined by

$$XY = aY,\ XZ = aZ, \qquad Y^2 = c_{YY}(X - a),$$
$$YZ = c_{YZ}(X - a), \qquad Z^2 = c_{ZZ}(X - a),$$
$$\Delta^{A,E}_E(Y) = (X - a) \otimes Y,\ \Delta^{A,E}_E(Z) = (X - a) \otimes Z,$$
$$\Delta^{E,E}_A(1) = d_{YY} Y \otimes Y + d_{YZ} Y \otimes Z + d_{YZ} Z \otimes Y$$
$$+ d_{ZZ} Z \otimes Z,\ \Delta^{E,E}_A(X) = a\, \Delta^{E,E}_A(1),$$

for some constants that satisfy certain conditions.

Constructions

	[APS]	New
A	$\mathbb{Z}[X]/(X^2)$	$\mathbb{Z}[X,a]/(X-a)^2$
XY	0	aY
XZ	0	aZ
Y^2	0	$c_{YY}(X-a)$
Z^2	0	$c_{ZZ}(X-a)$
YZ	0	$c_{YZ}(X-a)$
$\Delta_E^{A,E}(Y)$	$X \otimes Y$	$(X-a) \otimes Y$
$\Delta_E^{A,E}(Z)$	$X \otimes Z$	$(X-a) \otimes Z$
$\Delta_A^{E,E}(1)$	$Y \otimes Z + Z \otimes Y$	$d_{YY} Y \otimes Y + d_{YZ} Y \otimes Z$ $+ d_{YZ} Z \otimes Y + d_{ZZ} Z \otimes Z$
$\Delta_A^{E,E}(X)$	0	$a\ \Delta_A^{E,E}(1)$
ν_E^E	0	0
$\nu_E^A(Y)$	X	$e_Y(X-a)$
$\nu_E^A(Z)$	X	$e_Z(X-a)$
$\nu_A^E(1)$	$Y + Z$	$f_Y Y + f_Z Z$

Constructions

A generalization of [TT]:

Theorem

Let A be a commutative Frobenius algebra with handle element ϕ, such that there exists an element $\xi \in A$ with $\xi^2 = \phi$. Then there exists a commutative Frobenius pair with Möbius maps.

Corollary

Let $A = E = k[X]/(X^2 - hX - t)$, where $k = \mathbb{Z}[a^{\pm 1}, b^{\pm 1}]$, and $h = -2b^{-1}(a - b^{-1})$, $t = -b^{-2}(a^2 + h)$. Then (A, E) gives rise to a commutative Frobenius pair with Möbius maps.

Proof. Let $\xi = a + bX$, and one computes that $\xi^2 = \phi = 2X - h$.

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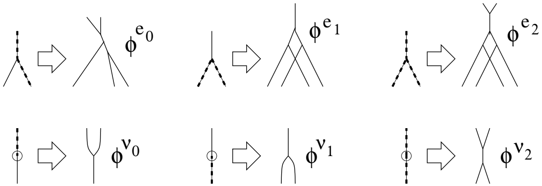
Constructions

A generalization of [IT]:

Theorem

Let A be a commutative Frobenius algebra with handle element ϕ such that its handle element $\phi \in A$ is invertible, then there exists a commutative Frobenius pair (A, E) with Möbius maps.

Idea of Proof: Take $E = A \otimes A$ and define maps as follows:



There is a solution for constants: $e_0 = -1$, $e_1 = e_2 = -2$, $\nu_0 = 1$, $\nu_1 = -1$ and $\nu_2 = 0$.

Outline

Example

In [Mackaay-Vaz], the $sl(3)$-invariant was described by foams, with Frobenius algebra structure defined by
$A = \mathbb{Z}[a, b, c][X]/(X^3 - aX^2 - bX - c)$
with the multiplication and the unit are defined by those for polynomials,
the Frobenius form (counit) ϵ is defined by
$\epsilon(1) = \epsilon(X) = 0$, $\epsilon(X^2) = -1$,
the comultiplication is computed as

$$\begin{aligned}
\Delta(1) &= -(1 \otimes X^2 + X \otimes X + X^2 \otimes 1) \\
&\quad +a(1 \otimes X + X \otimes 1) + b(1 \otimes 1), \\
\Delta(X) &= -(X \otimes X^2 + X^2 \otimes X) + a(X \otimes X) - c(1 \otimes 1), \\
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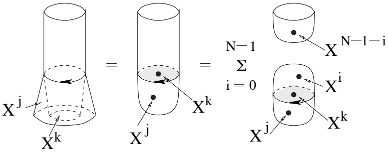
Example (cont.)

Following our approach, evaluate the branch line operation by skein relation. For $A = \mathbb{Z}[X]/(X^N)$, for example, it looks like:



$$[X^j, X^k] = \sum_{i=0}^{N-1} \theta(X^i, X^j, X^k) X^{N-1-i}$$

Theorem

The branch curve operation $[\ ,\ ]$ is skew-symmetric and satisfies the Jacobi identity:

$$[U,[V,W]] + [V,[W,U]] + [W,[U,V]].$$

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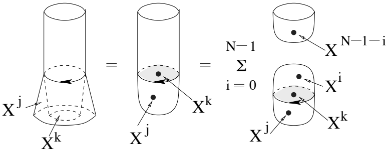

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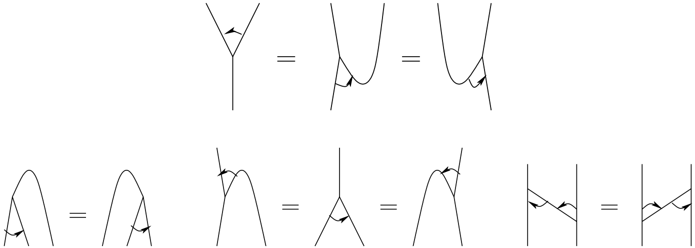
Example (cont.)

Using the Frobenius form, define $\mathbf{\Delta} : A \to A \otimes A$ by $\mathbf{\Delta}(V) = \sum [V, 1_{(1)}] \otimes 1_{(2)}$, where $\Delta(u) = \sum u_{(1)} \otimes u_{(2)}$. Similarly, for $\mathbf{\Delta}$, denote $\mathbf{\Delta}(u) = \sum u_{((1))} \otimes u_{((2))}$.



Example (cont.)

Theorem

For $A = \mathbb{Z}[a, b, c]/(X^3 - aX^2 - bX - c)$ with θ values as above, the map $\mathbf{\Delta} : A \to A \otimes A$ satisfies the following identities:

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\begin{aligned}
(\mathbf{m} \otimes \mathrm{id})(\mathrm{id} \otimes \mathbf{\Delta}) &= \Delta(1)(\epsilon\mu) - \tau, \\
(\ (\mathbf{m} \otimes \mathrm{id})(\mathrm{id} \otimes \mathbf{\Delta})\ )^2 &= \mathrm{id} + \Delta(1)(\epsilon\mu), \\
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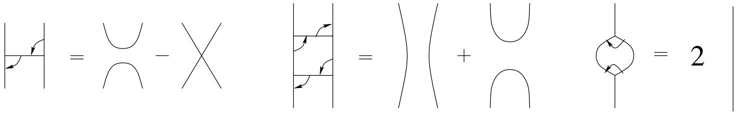

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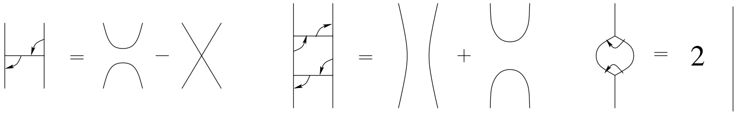

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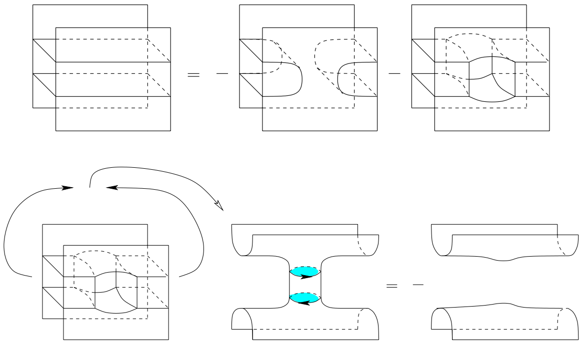
Example (cont.)

Relations to foam skeins:



Lie alg. $\xrightarrow{\text{Rep. of q-gp}}$ quantum knot inv.

Categorify by foams $\longrightarrow$ KhoHo $\xrightarrow{\text{TQFT and foams}}$ Lie alg.

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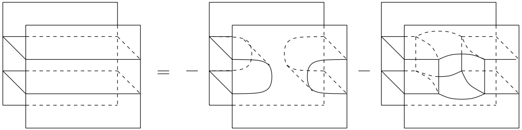

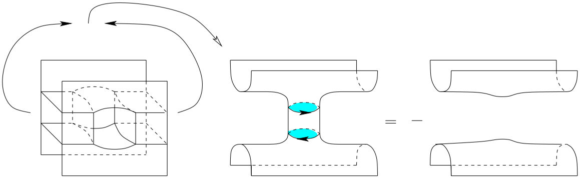

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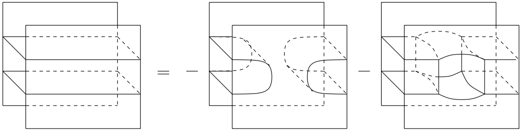

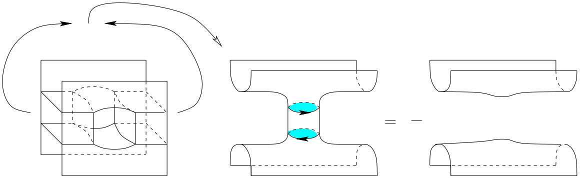

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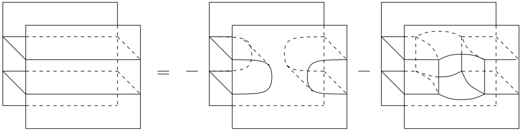

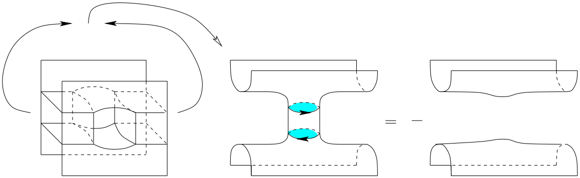

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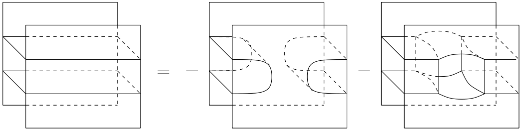

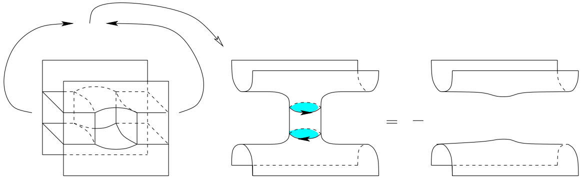

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Outline

Lie algebras

Are there other TQFTs and theta foam values that produce Lie algebras?

Theorem

For any positive odd integer $N > 1$, there exists a TQFT that induces a Lie algebra structure along branch circles.

Sketch proof. Let $A = R[X]/(X^N)$ for an odd integer $N > 1$. For $N > 3$, define

$$\theta(X^a, X^b, X^c) = \begin{cases} 1 & \text{if } a = 0, b + c = N, 1 < b < c, \\ -1 & \text{if } a = 0, b + c = N, 1 < c < b, \\ 0 & \text{otherwise} \end{cases}$$

and the same values for cyclic permutations.

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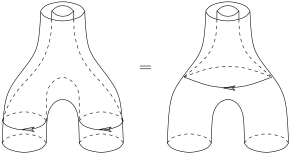
Let $A = R[G]$ be the group ring
with a commutative unital ring R.
$\boldsymbol{\Delta}(x) = x \otimes x$: bialgebra comultiplication
$\Delta(x) = \sum_{x=yz} y \otimes z$: Frobenius algebra comultiplication

In this case, foams admit both structures
(saddle $\leftrightarrow$ Frobenius, branch lines $\leftrightarrow$ bialgebra)
only when every element of G is of order 2.

Compatibility $\Delta(xy) = \Delta(x)\Delta(y)$ 

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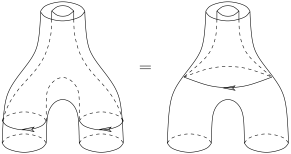

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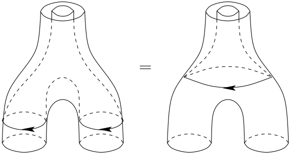
Compatibility $\Delta(xy) = \Delta(x)\Delta(y)$ 

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Skein modules

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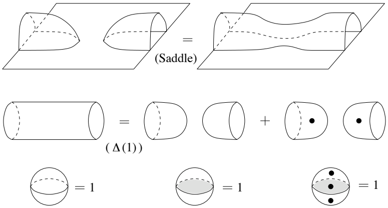
M : a compact 3-manifold

$\mathbf{FF}(M)$: free module on foams in M

$\mathbf{FS}(M)$: submodule generated by skein relations in the figure, where a dot represents $x \in A = R[x]/(x^2 - 1)$

$\mathbf{F}(M) = \mathbf{FF}(M)/\mathbf{FS}(M)$: the *surface skein module* with A



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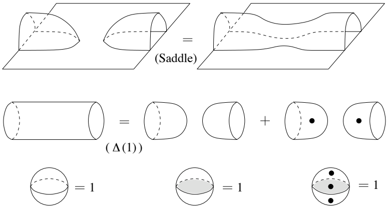

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We proposed an algebraic structure, commutative Frobenius pairs, that describes essential cobordisms in thickened surfaces and cobordisms formed by virtual circles with poles.

Constructions and new examples are provided, that may be useful in further generalizing Khovanov homology for thickened surfaces and virtual knots.

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We studied Lie algebra and bialgebra structures along branch circles.

Skein modules and more general foams need more study.

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