Sequences of Quandle Extensions and Cocycle Knot Invariants

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Abstract

Quandle 2-cocycles define invariants of classical and virtual knots, and extensions of quandles. We show that the quandle 2-cocycle invariant with respect to a non-trivial 2-cocycle is constant, or takes some restricted form, for classical knots when the corresponding extensions satisfy certain algebraic conditions. Specific examples are presented from the list of connected quandles of order less than 48.

Key words: quandles, quandle cocycle invariants, abelian extensions of quandles
MSC: 57M25

1 Introduction

Sets with certain self-distributive operations called *quandles* have been studied since the 1940s [18], and have been applied to knot theory since early 1980s [13, 14]. The number of colorings of knot diagrams by quandle elements, in particular, has been widely used as a knot invariant. Algebraic homology theories for quandles were defined [3, 12], and investigated. Knot invariants using cocycles have been defined [3] and applied to knots and knotted surfaces [6]. Extensions of quandles by cocycles have been studied, for example, in [1, 2, 11].

Computations using GAP [20] significantly expanded the list of quandles of small connected quandles. These quandles, called *Rig* quandles, may be found in the GAP package Rig [19]. Rig includes all connected quandles of order less than 48, at this time. Properties of some of Rig quandles, such as homology groups and cocycle invariants, are also found in [19]. We use the notation $Q(n, i)$ for the i-th quandle of order n in the list of Rig quandles.

It is observed that some Rig quandles have non-trivial second homology, yet have constant 2-cocycle invariants with non-trivial 2-cocycles, as much as computer calculations have been performed for the knot table (Remark 4.5, [9]). It does not seem to have been proved previously whether they actually have constant values for all classical knots. From Theorem 5.5 in [7] and using the Kronecker product, any non-trivial 2-cocycle has non-constant invariant values for some virtual links. Thus it is of interest if these quandles actually have constant values for all classical knots. More generally, possible values of the cocycle invariants are largely unknown, and are of interest. In this paper, we show that certain algebraic properties of quandles imply that the cocycle invariant is constant, or takes some restricted form, for classical knots. In particular, we prove that several specific Rig quandles, including some of those conjectured in [9], have constant cocycle invariant values for all classical knots for some non-trivial 2-cocycles, and that several Rig quandles take certain specific form as cocycle invariant values.

In Section 2, definitions, terminology and lemmas are presented. The main results and corollaries, and their proofs are given in Section 3.

2 Preliminaries

In this section we briefly review some definitions and examples. More details can be found, for example, in [6].

A *quandle* X is a non-empty set with a binary operation $(a, b) \mapsto a * b$ satisfying the following conditions.

$$\text{(Idempotency)} \quad \text{For any } a \in X,\ a * a = a. \tag{1}$$

$$\text{(Right invertibility)} \quad \text{For any } b, c \in X, \text{ there is a unique } a \in X \text{ such that } a * b = c. \tag{2}$$

$$\text{(Right self-distributivity)} \quad \text{For any } a, b, c \in X, \text{ we have } (a * b) * c = (a * c) * (b * c). \tag{3}$$

A *quandle homomorphism* between two quandles X, Y is a map $f : X \to Y$ such that $f(x *_X y) = f(x) *_Y f(y)$, where $*_X$ and $*_Y$ denote the quandle operations of X and Y, respectively. A *quandle isomorphism* is a bijective quandle homomorphism, and two quandles are *isomorphic* if there is a quandle isomorphism between them. A quandle epimorphism $f : X \to Y$ is a *covering* [11] if $f(x) = f(y)$ implies $a * x = a * y$ for any $a, x, y \in X$.

Let X be a quandle. The *right translation* $R_a : X \to X$, by $a \in X$, is defined by $R_a(x) = x * a$ for $x \in X$. Then R_a is an automorphism of X by Axioms (2) and (3). The subgroup of $\mathrm{Sym}(X)$ generated by the permutations R_a, $a \in X$, is called the *inner automorphism group* of X, and is denoted by $\mathrm{Inn}(X)$. The map $\varphi : X \to \varphi(X) \subset \mathrm{Inn}(X)$ defined by $\varphi(x) = R_x$ is called the *right-translation map*. A right-translation map is a covering.

A quandle is *connected* if $\mathrm{Inn}(X)$ acts transitively on X. A quandle is *faithful* if the mapping $a \mapsto R_a$ is an injection from X to $\mathrm{Inn}(X)$.

A *generalized Alexander quandle* is defined by a pair (G, f) where G is a group and $f \in \mathrm{Aut}(G)$, and the quandle operation is defined by $x*y = f(xy^{-1})y$. If G is abelian, this is called an *Alexander quandle*.

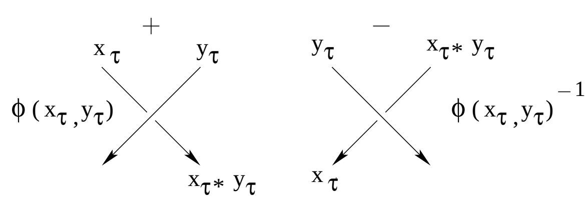

Figure 1: Colored crossings and cocycle weights

In this paper we denote by A a finite multiplicative abelian group, and the identity element is denoted by 1. A function $\phi : X \times X \to A$ for an abelian group A is called a *quandle* 2*-cocycle* [3] if it satisfies

$$\phi(x, y)\phi(x, z)^{-1}\phi(x * y, z)\phi(x * z, y * z)^{-1} = 1$$

for any $x, y, z \in X$ and $\phi(x, x) = 1$ for any $x \in X$. For a quandle 2-cocycle ϕ, $E = X \times A$ becomes a quandle by

$$(x, a) * (y, b) = (x * y, a\,\phi(x, y))$$

for $x, y \in X$, $a, b \in A$, denoted by $E(X, A, \phi)$ or simply $E(X, A)$, and it is called an *abelian extension* of X by A. Let $\pi : E(X, A) = X \times A \to X$ be the projection to the first factor. We also say that a quandle epimorphism $f : Y \to X$ is an *abelian extension* if there exists an isomorphism $\nu : E(X, A) \to Y$ such that $\pi = f\nu$. An abelian extension is a covering. See [2] for more information on abelian extensions of quandles and [3–5] for more on quandle cohomology.

Let X be a quandle, and ϕ be a 2-cocycle with coefficient group A, a finite abelian group. For a coloring of a knot diagram by a quandle X as depicted in Figure 1 at a positive (left) and negative (right) crossing, respectively, the pair (x_τ, y_τ) of colors assigned to a pair of nearby arcs is called the *source* colors. The third arc receives the color $x_\tau * y_\tau$.

The 2-cocycle (or cocycle, for short) invariant is an element of the group ring $\mathbb{Z}[A]$ defined by $\Phi_\phi(K) = \sum_{\mathcal{C}} \prod_\tau \phi(x_\tau, y_\tau)^{\epsilon(\tau)}$, where the product ranges over all crossings τ, the sum ranges over all colorings of a given knot diagram, (x_τ, y_τ) are source colors at the crossing τ, and $\epsilon(\tau)$ is the sign of τ as specified in Figure 1. For a given coloring C, the element $\prod_\tau \phi(x_\tau, y_\tau)^{\epsilon(\tau)} \in A$, in multiplicative notation, is denoted by $B_\phi(K, C)$. For an abelian group A, the cocycle invariant values take the form $\sum_{a \in A} n_a a$ where $n_a \in \mathbb{Z}$, and it is *constant* if $n_a = 0$ when a is not the identity of A.

A 1-tangle is a properly embedded arc in a 3-ball, and the equivalence of 1-tangles is defined by ambient isotopies of the 3-ball fixing the boundary (cf. [10]). A diagram of a 1-tangle is defined in a manner similar to a knot diagram, from a regular projection to a disk by specifying crossing information, see Figure 2(A). An orientation of a 1-tangle is specified by an arrow on a diagram as depicted. A knot diagram is obtained from a 1-tangle diagram by closing the end points by a trivial arc outside of a disk. This procedure is called the *closure* of a 1-tangle. If a 1-tangle is oriented, then the closure inherits the orientation. Two diagrams of the same 1-tangle are related by Reidemeister moves. There is a bijection between knots and 1-tangles, and invariants of 1-tangles give rise to invariants of knots, see [11], for example.

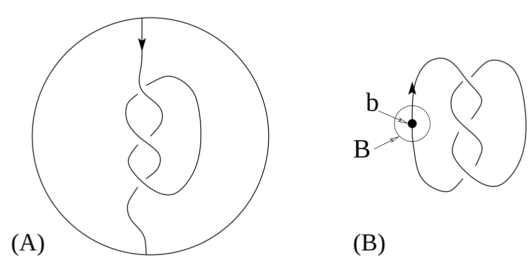

Figure 2: 1-tangles

A 1-tangle is obtained from a knot K as follows. Choose a base point $b \in K$ and a small open neighborhood B of b in the 3-sphere $\mathbb{S}^3$ such that $(B, K \cap B)$ is a trivial ball-arc pair (so that $K \cap B$ is unknotted in B, see Figure 2(B)). Then $(\mathbb{S}^3 \setminus \mathrm{Int}(B), K \cap (\mathbb{S}^3 \setminus \mathrm{Int}(B)))$ is a 1-tangle called the 1-tangle associated with K. The resulting 1-tangle does not depend on the choice of a base point. If a knot is oriented, then the corresponding 1-tangle inherits the orientation.

A quandle coloring of an oriented 1-tangle diagram is defined in a manner similar to those for knots. We do not require that the end points receive the same color for a quandle coloring of 1-tangle diagrams. We say that a quandle X is *end monochromatic* [9] for a tangle T if any coloring of T by X assigns the same color on the two end arcs. We use the same notations $\Phi_\phi(T) = \sum_{\mathcal{C}} \prod_\tau \phi(x_\tau, y_\tau)^{\epsilon(\tau)}$ and $B_\phi(T, C)$ for tangles T. Figures 1, 2, 3 are taken from [9].

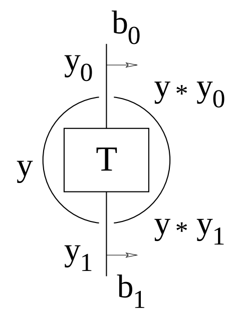

Figure 3: Colorings of a tangle

We recall the following two lemmas.

Lemma 2.1 ([11])**.** *Let $f : Y \to X$ be a covering, and $C_X : \mathcal{A}(T) \to X$ be a coloring of a 1-tangle T by X. Let b_0, b_1 be the top and bottom arcs as depicted in Figure 3. Then for any $y \in Y$ such that $f(y) = C_X(b_0)$, there exists a unique coloring $C_Y : \mathcal{A}(T) \to Y$ such that $fC_Y = C_X$ and $C_Y(b_0) = y$.*

Lemma 2.2 ([2,9])**.** *Let $f : X \to Q$ be an abelian extension with a 2-cocycle ϕ. Then X is end monochromatic for T if and only if $\Phi_\phi(K)$ is constant, where T is a 1-tangle for a knot K.*

Sketch proof of Lemmas 2.1 and 2.2. Lemma 2.1 is shown by traveling the tangle from top to bottom, and extending C_X with a given starting color $y \in Y$ at b_0 to C_Y through each under-crossing using the coloring and the covering conditions. At a positive crossing, as shown in Figure 4, if the color at the left under-arc is $x \in X$ and lifted to $\tilde{x} \in Y$, and the over-arc is colored by $y \in X$, then the color at the right under-arc is lifted to $\tilde{x} * \tilde{y} \in Y$, where the this lifting is independent of choice of $\tilde{y} \in Y$ such that $f(\tilde{y}) = y$ so that it depends only on y. The lifting is similarly done at negative crossings, and the unique lifting continues inductively.

Foe Lemma 2.2, let $\nu : X \to E = E(Q, A) = Q \times A$ be an isomorphism such that $\pi = f\nu$. Let b_0, b_1 be the top and bottom arcs as depicted in Figure 3, and let C_X be a coloring of T by X. From [2], for any $a \in A$, there is a coloring $C_E : \mathcal{A}(T) \to E$ such that $C_E(b_0) = (C_X(b_0), a)$ and $C_E(b_1) = (C_X(b_1), a\, B_\phi(T, C_X))$. This is also indicated in Figure 4, where the contribution of the cocycle value at a crossing appears at a crossing.

Suppose X is end monochromatic. Then Q is also end monochromatic. Thus for any coloring C_X, $C_X(b_1) = C_X(b_0)$ holds. Hence $\nu(C_X(b_1), a\ B_\phi(T, C_X)) = \nu(C_X(b_0), a)$, that is, $B_\phi(T, C_X) = 1$. Since $C_X(b_0) = C_X(b_1)$, C_X defines a coloring of K, and $B_\phi(K, C_X) = 1$, hence $\Phi_\phi(K)$ is constant. The converse also holds. □

Lemma 2.3. *Let $\varphi : Y \to X = \varphi(Y) \subset \mathrm{Inn}(Y)$ be the right-translation map, and $C : \mathcal{A}(T) \to Y$ be a coloring of a 1-tangle T by Y. For the top and bottom arcs b_0 and b_1 of T, respectively, let*

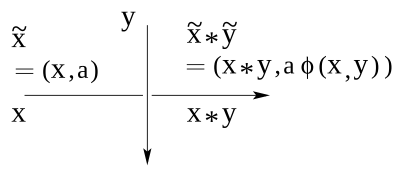

Figure 4: Lifting colorings

$y_0 = C(b_0)$ and $y_1 = C(b_1)$. Then it holds that $\varphi(y_0) = R_{y_0} = R_{y_1} = \varphi(y_1)$ and hence X is end monochromatic with T.

Proof. The proof in [9], based on corresponding statements in [15–17] on faithful quandles, applies in this situation. The idea of proof is seen in Figure 3. The large circle behind the tangle T in the figure can be pulled out of T if T corresponds to a classical knot. Hence any color y at the left of the large circle should extend to the color to the right, so that for the colors y_0 and y_1 for the top and bottom arcs b_0 and b_1 must satisfy $y * y_0 = y * y_1$, hence we obtain $R_{y_0} = R_{y_1}$. □

3 Cocycle invariants and sequences

In this section we state and prove the main result of the paper.

Theorem 3.1. *Let $Y \stackrel{\varphi}{\to} X \stackrel{\alpha}{\to} Q$ be a sequence of quandle homomorphisms where φ is the right-translation map and α is an abelian extension with respect a 2-cocycle ϕ. Then the quandle cocycle invariant $\Phi_\phi(K)$ is constant for any classical knot K.*

Proof. Let T be a 1-tangle of K, b_0, b_1 be the top and bottom arcs of T, respectively. Let C be a coloring of a diagram of K by Q, and use the same notation $C : \mathcal{A}(T) \to Q$ for a corresponding coloring of T such that $C(b_0) = C(b_1) = x \in Q$. Then C extends to a coloring $C_X : \mathcal{A}(T) \to X$.

Recall that the right-translation map is a covering. By assumption and Lemma 2.1, C_X extends to a coloring C_Y. Since α is an abelian extension, Lemma 2.2 and Lemma 2.3 imply that the cocycle invariant is constant. □

To apply the theorem to some Rig quandles, we observe the following.

Lemma 3.2. *If $\varphi : Y \to \varphi(Y) = X$, for connected quandles X and Y, satisfies $|Y|/|X| = 2$, then φ is an abelian extension.*

Proof. For $\varphi : Y \to \varphi(Y) = X$, where X and Y are connected quandles, it is proved in [1] that there is a quandle isomorphism $\nu : X \times S \to Y$ for a set S, such that $\pi = \varphi\nu$ for the projection $\pi : X \times S \to X$. The quandle operation on $X \times S$ is defined by

$$(x, s) * (y, t) = (x * y, \beta_{x,y}(s)) \quad \text{for} \quad (x, s), (y, t) \in X \times S,$$

for $\beta : X^2 \to \mathrm{Sym}(S)$. The proof of Theorem 7.1 of [9] shows that if the cardinality of S is 2, then we can assume $S = \mathbb{Z}_2$ and $\beta_{x,y}(a) = a\,\phi(x, y)$ where ϕ is a 2-cocycle with coefficient group $A = \mathbb{Z}_2$. Hence φ is an abelian extension. □

We say that an abelian extension $f : Y \to X$ is *of index* k if $|Y|/|X| = k$, and also the corresponding 2-cocycle ϕ is said to be *of index* k, with the coefficient group A of order k.

Corollary 3.3. *The following Rig quandles have non-trivial second cohomology groups with the coefficient group* $A = \mathbb{Z}_2$*, yet give rise to the constant quandle* 2*-cocycle invariant for any classical knot with the corresponding non-trivial* 2*-cocycles:*

$$\begin{array}{llllllll} Q(6,1), & Q(10,1), & Q(12,5), & Q(12,6), & Q(12,7), & Q(12,8), & Q(16,4), & Q(16,5), \\ Q(16,6), & Q(18,1), & Q(18,2), & Q(18,8), & Q(18,9), & Q(18,10), & Q(24,3), & Q(24,4), \\ Q(24,13), & Q(24,22), & Q(30,2), & Q(30,7), & Q(30,8), & Q(40,8), & Q(40,9), & Q(40,10), \\ Q(42,1), & Q(42,3), & Q(42,4), & Q(42,7), & Q(42,8). & & & \end{array}$$

Proof. Computer calculations show the following quandle sequences of Rig quandles:

$$\begin{array}{ccccc} Q(24,1) & \stackrel{\varphi}{\to} & Q(12,1) & \stackrel{\varphi}{\to} & Q(6,1), \\ Q(40,2) & \stackrel{\varphi}{\to} & Q(20,3) & \stackrel{\varphi}{\to} & Q(10,1). \end{array}$$

For the other quandles for which the sequences go out of bounds of Rig quandles, sequences of right-translation maps are listed in Tables 1 and 2. The result follows from Lemma 3.2 and Theorem 3.1 when the maps are of index 2. The quandles listed in the statement satisfy the condition of index 2. □

We remark that the above list contains all Rig quandles of order less than or equal to 16 that were conjectured in [9] to have constant quandle 2-cocycle invariants for any classical knot with non-trivial 2-cocycles except $Q(12,9)$, $Q(15,2)$, and $Q(15,7)$, for which the conjecture is still open. Those in the above list of order larger than 16 do not even appear in the conjectured list in [9]. The reason why we were able to prove Corollary 3.3 for the conjectured and more extensive Rig quandles is by the use of φ in Theorem 3.1.

Tables 1 and 2 contain sequences of connected quandles where all arrows represent right-translation maps. The quandle on the left of each sequence is a generalized Alexander quandle, but others in the sequence may or may not be generalized Alexander quandles. The right-most quandle in each sequence is faithful, so the sequences cannot be extended non-trivially to the right with right-translation maps. The notation $R(n, j)$ is used to indicate a quandle of order n when $n > 47$ and hence not a Rig quandle. The index j is simply to distinguish non-isomorphic quandles.

$$\begin{array}{ccccccc} R(192,2) & \to & R(48,3) & \to & Q(24,3) & \to & Q(12,6) \\ R(192,3) & \to & R(48,4) & \to & Q(24,4) & \to & Q(12,5) \end{array}$$

Table 1: Four-term sequences of right-translation maps for small connected quandles

Remark 3.4. Terminating sequences of φ,

$$X = X_0 \stackrel{\varphi}{\to} X_1 = \varphi(X_0) \stackrel{\varphi}{\to} \cdots \stackrel{\varphi}{\to} X_n = \varphi(X_{n-1}),$$

are discussed in [1], where X_n is faithful and X_j are not, for $j = 1, \ldots, n-1$. For the 790 Rig quandles of order less than 48, there are 66 non-faithful quandles X, and all but two have faithful images $\varphi(X)$. The two exceptions are the above first two Rig quandles $Q(24,1)$ and $Q(40,2)$.

On the other hand, Tables 1 and 2 include many quandles X with $\varphi(X)$ being non-faithful Rig quandles.

$R(64,1)$	$\rightarrow$	$Q(32,6)$	$\rightarrow$	$Q(16,4)$	$R(64,2)$	$\rightarrow$	$Q(32,7)$	$\rightarrow$	$Q(16,5)$
$R(64,3)$	$\rightarrow$	$Q(32,8)$	$\rightarrow$	$Q(16,6)$	$R(64,4)$	$\rightarrow$	$Q(32,5)$	$\rightarrow$	$Q(16,4)$
$R(64,5)$	$\rightarrow$	$Q(32,6)$	$\rightarrow$	$Q(16,4)$	$R(64,6)$	$\rightarrow$	$Q(32,5)$	$\rightarrow$	$Q(16,4)$
$R(72,1)$	$\rightarrow$	$Q(36,21)$	$\rightarrow$	$Q(18,10)$	$R(72,2)$	$\rightarrow$	$Q(36,17)$	$\rightarrow$	$Q(18,8)$
$R(72,3)$	$\rightarrow$	$Q(36,20)$	$\rightarrow$	$Q(18,9)$	$R(72,4)$	$\rightarrow$	$Q(36,4)$	$\rightarrow$	$Q(18,2)$
$R(72,5)$	$\rightarrow$	$Q(36,1)$	$\rightarrow$	$Q(18,1)$	$R(96,1)$	$\rightarrow$	$Q(24,4)$	$\rightarrow$	$Q(12,5)$
$R(96,2)$	$\rightarrow$	$Q(24,3)$	$\rightarrow$	$Q(12,6)$	$R(96,3)$	$\rightarrow$	$R(48,1)$	$\rightarrow$	$Q(24,22)$
$R(96,4)$	$\rightarrow$	$Q(24,6)$	$\rightarrow$	$Q(12,9)$	$R(96,5)$	$\rightarrow$	$Q(24,6)$	$\rightarrow$	$Q(12,9)$
$R(96,6)$	$\rightarrow$	$Q(24,6)$	$\rightarrow$	$Q(12,9)$	$R(96,7)$	$\rightarrow$	$Q(24,5)$	$\rightarrow$	$Q(12,8)$
$R(96,8)$	$\rightarrow$	$Q(24,5)$	$\rightarrow$	$Q(12,8)$	$R(96,9)$	$\rightarrow$	$R(48,2)$	$\rightarrow$	$Q(24,22)$
$R(120,1)$	$\rightarrow$	$Q(20,3)$	$\rightarrow$	$Q(10,1)$	$R(120,2)$	$\rightarrow$	$R(60,1)$	$\rightarrow$	$Q(30,8)$
$R(120,3)$	$\rightarrow$	$R(60,2)$	$\rightarrow$	$Q(30,7)$	$R(120,4)$	$\rightarrow$	$R(60,3)$	$\rightarrow$	$Q(30,2)$
$R(120,5)$	$\rightarrow$	$Q(30,1)$	$\rightarrow$	$Q(15,2)$	$R(160,1)$	$\rightarrow$	$R(80,1)$	$\rightarrow$	$Q(40,10)$
$R(160,2)$	$\rightarrow$	$Q(40,20)$	$\rightarrow$	$Q(20,5)$	$R(160,3)$	$\rightarrow$	$Q(40,19)$	$\rightarrow$	$Q(20,6)$
$R(160,4)$	$\rightarrow$	$R(80,2)$	$\rightarrow$	$Q(40,9)$	$R(168,1)$	$\rightarrow$	$R(84,1)$	$\rightarrow$	$Q(42,8)$
$R(168,2)$	$\rightarrow$	$R(84,2)$	$\rightarrow$	$Q(42,3)$	$R(168,3)$	$\rightarrow$	$R(84,3)$	$\rightarrow$	$Q(42,7)$
$R(168,4)$	$\rightarrow$	$R(84,4)$	$\rightarrow$	$Q(42,4)$	$R(168,5)$	$\rightarrow$	$R(84,5)$	$\rightarrow$	$Q(42,1)$
$R(192,1)$	$\rightarrow$	$Q(24,14)$	$\rightarrow$	$Q(12,7)$	$R(192,4)$	$\rightarrow$	$R(48,5)$	$\rightarrow$	$Q(24,13)$
$R(216,1)$	$\rightarrow$	$R(72,6)$	$\rightarrow$	$Q(24,21)$	$R(216,2)$	$\rightarrow$	$Q(36,17)$	$\rightarrow$	$Q(18,8)$
$R(216,3)$	$\rightarrow$	$Q(36,1)$	$\rightarrow$	$Q(18,1)$					

Table 2: Three-term equences of right-translation maps for small connected quandles

Let $X = Q(12,5)$ or $Q(12,6)$. Then the second quandle cohomology group $H^2_Q(X,\mathbb{Z}_4)$ is known [19] to be isomorphic to $\mathbb{Z}_4$. See [3, 6], for example, for details on quandle cohomology. Let $\psi : X \times X \to \mathbb{Z}_4$ be a 2-cocycle which represents a generator of $H^2_Q(X,\mathbb{Z}_4) \cong \mathbb{Z}_4$. Let u denote a multiplicative generator of $A = \mathbb{Z}_4$. Then recall that the cocycle invariant is written as $\Phi_\psi(K) = \sum_{j=0}^{3} a_j(K)\, u^j \in \mathbb{Z}[A]$ for any knot K. The cocycle invariants $\Phi_\psi(K)$ for $X = Q(12,5)$ or $Q(12,6)$ with respect to ψ, computed for some knots in the table in [19] up to 9 crossing knots, contain non-constant values, while for $A = \mathbb{Z}_2$ the invariant is stated to be constant in Corollary 3.3. This is explained by the following, which solves the conjecture stated in [9].

Corollary 3.5. *Let $X = Q(12,5)$ or $Q(12,6)$, and $\psi : X \times X \to A = \mathbb{Z}_4$ be a 2-cocycle which represents a generator of $H^2_Q(X,\mathbb{Z}_4) \cong \mathbb{Z}_4$. Let $\Phi_\psi(K) = \sum_{j=0}^{3} a_j(K)\, u^j \in \mathbb{Z}[A]$ be the cocycle invariant. Then $a_1(K) = a_3(K) = 0$ for any classical knot K.*

Proof. The 2-cocycles ϕ corresponding to the sequences used in Corollary 3.3 and found in Table 2 are of index 2 and is cohomologous to ψ^2. Then

$$\Phi_\phi(K) = \sum_{j=0}^{3} a_j(K)\, u^{2j} = [\, a_0(K) + a_2(K)\,]\, u^0 + [\, a_1(K) + a_3(K)\,]\, u^2$$

is constant by Corollary 3.3. Since $a_j(K)$ are non-negative integers, the result follows. ∎

A similar situation is found for $X = Q(18,1)$ or $Q(18,8)$, where $H^2_Q(X,\mathbb{Z}_6) \cong \mathbb{Z}_6$. Let u be a multiplicative generator of $A = \mathbb{Z}_6$. Then the invariant values are restricted to the following form.

Corollary 3.6. *Let $X = Q(18,1)$ or $Q(18,8)$, and $\psi : X \times X \to \mathbb{Z}_6$ be a 2-cocycle which represents a generator of $H^2_Q(X, \mathbb{Z}_6) \cong \mathbb{Z}_6$. Let $\Phi_\psi(K) = \sum_{j=0}^{5} a_j(K)\, u^j \in \mathbb{Z}[A]$ be the cocycle invariant. Then $a_k(K) = 0$ for $k = 1, 3, 5$ for any classical knot K.*

The cocycle invariant for connected quandles of order 18 are computed in [19] for up to 7 crossing knots at the time of writing. The invariant values for $Q(18,8)$ do contain non-constant values. For $Q(18,1)$, the invariant is constant, and we do not know whether this is an artifact of limited number of knots or it is constant for all classical knots. In [19], homology groups are computed for connected quandles of order less than 36, and we are not able to derive similar conclusions for larger quandles listed in Corollary 3.3 at this time.

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