Some families of Connected Quandles

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1 Non-abelian version of Alexander quandles

More information and references about the constructions below can be found at:
http://shell.cas.usf.edu/quandle.

Joyce and Matveev showed that, for a group G and its automorphism f, the operation $x * y = f(xy^{-1})y$ defines a quandle structure on G.

It is shown by Xiang-dong Hou that it gives a latin, non-Alexander quandle if f fixes only the identity element and G is not abelian. He also showed that two such automorphism f and g give isomorphic quandles if and only if they are conjugate in $\mathrm{Aut}(G)$.

Edwin Clark (with help from Michael Kenyon) found that the smallest non-abelian group G with such an automorphism has order 27, and is unique. He found that there are two non-conjugate such automorphisms, and they give rise to the rig quandles $C[27, 27]$ and $C[27, 28]$. He also computed that the (order, number) of such quandles as follows, for order 100 or less.

```
27, 2
48, 1
64, 21
75, 3
80, 1
81, 10
```

For all non-abelian groups G up to order 63 with a representative f of each conjugacy class of $\mathrm{Aut}(G)$, such that f is not fixed point free and the resulting quandle is connected.

(There are 319 groups of order up to 63, among which 213 are non-abelian. There are 63 quandles of the above type out of these, and 4688 quandles that are not connected.)

The following is a list of those connected non-latin such quandles with order < 36, with the rig quandle numbering and the groups they come from.

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Edwin's list and comments:

 M[ 1] = C[ 8, 1], connected = true,             from group = Q8.
 M[ 2] = C[12, 2], connected = true,             from group = A4.
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M[ 3] = C[12, 1], connected = true,             from group = A4.
M[ 4] = C[24, 2], connected = true,             from group = ‘SL(2,3)‘.
M[ 5] = C[24, 1], connected = true,             from group = ‘SL(2,3)‘.
M[ 6] = C[24, 8], connected = true,             from group = ‘C3 x Q8‘.
M[ 7] = C[27,14], connected = true,             from group = ‘(C3 x C3) : C3‘.
M[ 8] = C[27, 6], connected = true,             from group = ‘(C3 x C3) : C3‘.
M[ 9] = C[27, 1], connected = true,             from group = ‘(C3 x C3) : C3‘.
M[10] = C[32, 7], connected = true,             from group = ‘(C4 x C2) : C4‘.
M[11] = C[32, 9], connected = true,             from group = ‘(C4 x C2) : C4‘.
M[12] = C[32, 1], connected = true,             from group = ‘(C4 x C2) : C4‘.
M[13] = C[32, 8], connected = true,             from group = ‘(C4 x C2) : C4‘.
M[14] = C[32, 5], connected = true,             from group = ‘C2 x C2 x Q8‘.
M[15] = C[32, 2], connected = true,             from group = ‘C2 x C2 x Q8‘.
M[16] = C[32, 3], connected = true,             from group = ‘(C2 x D8) : C2‘.
M[17] = C[32, 6], connected = true,             from group = ‘(C2 x D8) : C2‘.
M[18] = C[32, 4], connected = true,             from group = ‘(C2 x Q8) : C2‘.
```

It is interesting that C[8,1] is non-Abelian Alexander. Aut(Q8) = S_4 so it should be possible to figure out what f gives C[8,1] if it hasn’t already been done.

This raises the question for what f are the quandles connected?

Here’s a list of all the groups for which such connected non-latin quandles exist.
I have eliminated those that as above have several.

```
[ "(C2 x C2 x C2 x C2) : C3", "(C2 x C2 x C2) : C7", "(C2 x C2) : C9",
  "(C2 x D8) : C2", "(C2 x Q8) : C2", "(C3 x C3) : C3", "(C4 x C2) : C4",
  "(C4 x C4) : C3", "A4", "A5", "C2 x C2 x A4", "C2 x C2 x Q8", "C3 x A4",
  "C3 x Q8", "C5 x A4", "C5 x Q8", "C7 x Q8", "Q8", "SL(2,3)" ]
```

All 63 of them with the orders of f are as follows.

```
1, Q8, automorphism order = 3
2, A4, automorphism order = 4
3, A4, automorphism order = 2
4, SL(2,3), automorphism order = 4
5, SL(2,3), automorphism order = 2
6, C3 x Q8, automorphism order = 6
7, (C3 x C3) : C3, automorphism order = 6
8, (C3 x C3) : C3, automorphism order = 4
9, (C3 x C3) : C3, automorphism order = 2
10, (C4 x C2) : C4, automorphism order = 6
11, (C4 x C2) : C4, automorphism order = 6
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12, (C4 x C2) : C4, automorphism order = 3
13, (C4 x C2) : C4, automorphism order = 6
14, C2 x C2 x Q8, automorphism order = 6
15, C2 x C2 x Q8, automorphism order = 3
16, (C2 x D8) : C2, automorphism order = 3
17, (C2 x D8) : C2, automorphism order = 6
18, (C2 x Q8) : C2, automorphism order = 5
19, (C2 x C2) : C9, automorphism order = 12
20, (C2 x C2) : C9, automorphism order = 6
21, (C2 x C2) : C9, automorphism order = 4
22, (C2 x C2) : C9, automorphism order = 2
23, (C2 x C2) : C9, automorphism order = 12
24, (C2 x C2) : C9, automorphism order = 6
25, C3 x A4, automorphism order = 12
26, C3 x A4, automorphism order = 6
27, C3 x A4, automorphism order = 4
28, C3 x A4, automorphism order = 2
29, C5 x Q8, automorphism order = 12
30, C5 x Q8, automorphism order = 12
31, C5 x Q8, automorphism order = 6
32, (C4 x C4) : C3, automorphism order = 4
33, (C4 x C4) : C3, automorphism order = 4
34, (C4 x C4) : C3, automorphism order = 8
35, (C4 x C4) : C3, automorphism order = 4
36, (C4 x C4) : C3, automorphism order = 2
37, C2 x C2 x A4, automorphism order = 12
38, C2 x C2 x A4, automorphism order = 6
39, (C2 x C2 x C2 x C2) : C3, automorphism order = 4
40, (C2 x C2 x C2 x C2) : C3, automorphism order = 2
41, (C2 x C2 x C2 x C2) : C3, automorphism order = 8
42, (C2 x C2 x C2 x C2) : C3, automorphism order = 4
43, C7 x Q8, automorphism order = 3
44, C7 x Q8, automorphism order = 6
45, C7 x Q8, automorphism order = 3
46, C7 x Q8, automorphism order = 6
47, C7 x Q8, automorphism order = 6
48, (C2 x C2 x C2) : C7, automorphism order = 3
49, (C2 x C2 x C2) : C7, automorphism order = 6
50, (C2 x C2 x C2) : C7, automorphism order = 3
51, (C2 x C2 x C2) : C7, automorphism order = 6
52, A5, automorphism order = 3
53, A5, automorphism order = 2
54, A5, automorphism order = 6

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55, A5, automorphism order = 2
56, A5, automorphism order = 4
57, A5, automorphism order = 5
58, C5 x A4, automorphism order = 4
59, C5 x A4, automorphism order = 4
60, C5 x A4, automorphism order = 4
61, C5 x A4, automorphism order = 4
62, C5 x A4, automorphism order = 4
63, C5 x A4, automorphism order = 2

There are additional 15 for groups of order 64:

1, C4 . (C4 x C4), automorphism order = 3
2, C4 . (C4 x C4), automorphism order = 6
3, C4 . (C4 x C4), automorphism order = 6
4, C4 . (C4 x C4), automorphism order = 6
5, ((C2 x Q8) : C2) : C2, automorphism order = 3
6, ((C2 x Q8) : C2) : C2, automorphism order = 6
7, ((C2 x Q8) : C2) : C2, automorphism order = 6
8, ((C2 x Q8) : C2) : C2, automorphism order = 6
9, Q8 x Q8, automorphism order = 3
10, Q8 x Q8, automorphism order = 6
11, ((C4 x C4) : C2) : C2, automorphism order = 3
12, ((C4 x C4) : C2) : C2, automorphism order = 6
13, (C2 x C2) . (C2 x C2 x C2 x C2), automorphism order = 5
14, C2 x C2 x C2 x Q8, automorphism order = 21
15, C2 x C2 x C2 x Q8, automorphism order = 21
```

2 Core of groups

If G is a group, Core(G) is the quandle $(G, *)$ with $x * y = yx^(-1)y$ (see, for example, Joyce's paper). These are always Keis.

If G is abelian this is an Alexander quandle with $t = -1$. So it suffices to look at non-abelian G only.

```
For |G| < 36 and nonabelian, GAP finds only the following connected Core(G)

 C [12, 8]  = Core(Alt(4))
 C [21, 8]  = Core(Z_7:Z_33)  (N:H is semidirect product of H acting on N)
 C [24,17] = Core(SL(2,3))
 C [27, 2]  = Core((Z_3 \times Z_3):Z_3) (also is Alexander)
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 C [27, 4]  = Core(Z_9 : Z_3) (also is Alexander)

For many groups G, Core(G) is not connected. I'm not sure
when Core(G) is connected. It is well-known that Core(G) is
Latin iff G has odd order. Note that two of the groups above
have even order.

For a brief review of Core(G) and some references see the introduction
to  http://www.karlin.mff.cuni.cz/~stanovsk/math/gop.pdf

Here are all examples of connected Core(G) where G is
not abelian and |G| < 2^7. GAP computes them easily.

order 12: Core(A4)
order 21: Core(C7 : C3)
order 24: Core(SL(2,3))
order 27: Core((C3 x C3) : C3)
order 27: Core(C9 : C3)
order 36: Core((C2 x C2) : C9)
order 36: Core(C3 x A4)
order 39: Core(C13 : C3)
order 48: Core((C4 x C4) : C3)
order 48: Core((C2 x C2 x C2 x C2) : C3)
order 55: Core(C11 : C5)
order 56: Core((C2 x C2 x C2) : C7)
order 57: Core(C19 : C3)
order 60: Core(A5)
order 60: Core(C5 x A4)
order 63: Core(C7 : C9)
order 63: Core(C3 x (C7 : C3))
order 72: Core(Q8 : C9)
order 72: Core(C3 x SL(2,3))
order 75: Core((C5 x C5) : C3)
order 80: Core((C2 x C2 x C2 x C2) : C5)
order 81: Core((C9 x C3) : C3)
order 81: Core(C9 : C9)
order 81: Core(C27 : C3)
order 81: Core((C3 x C3 x C3) : C3)
order 81: Core((C9 x C3) : C3)
order 81: Core((C9 x C3) : C3)
order 81: Core(C3 . ((C3 x C3) : C3) = (C3 x C3) . (C3 x C3))
order 81: Core(C3 x ((C3 x C3) : C3))
order 81: Core(C3 x (C9 : C3))
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order 81: Core((C9 x C3) : C3)
order 84: Core(C7 x A4)
order 84: Core((C14 x C2) : C3)
order 93: Core(C31 : C3)
order 96: Core(((C4 x C2) : C4) : C3)
order 96: Core((C2 x C2 x Q8) : C3)
order 96: Core(((C2 x D8) : C2) : C3)
order 105: Core(C5 x (C7 : C3))
order 108: Core((C2 x C2) : C27)
order 108: Core(C9 x A4)
order 108: Core((C18 x C2) : C3)
order 108: Core(C3 x ((C2 x C2) : C9))
order 108: Core(((C2 x C2) : C9) : C3)
order 108: Core((C6 x C6) : C3)
order 108: Core(C3 x C3 x A4)
order 111: Core(C37 : C3)
order 117: Core(C13 : C9)
order 117: Core(C3 x (C13 : C3))
order 120: Core(SL(2,5))
order 120: Core(C5 x SL(2,3))
order 125: Core((C5 x C5) : C5)
order 125: Core(C25 : C5)
```

3 Conjugation quandles in $PSL(2, q)$

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The following examples of conjugation quandles of order pq
were found by GAP, where p < q
are primes in the group PSL(2,q).

quandle order = 15 = 3*5 Group = PSL(2,5)
quandle order = 21 = 3*7 Group = PSL(2,7)
quandle order = 55 = 5*11 Group = PSL(2,11)
quandle order = 91 = 7*13 Group = PSL(2,13)
quandle order = 253 = 11*23 Group = PSL(2,23)
quandle order = 703 = 19*37 Group = PSL(2,37)
quandle order = 1081 = 23*47 Group = PSL(2,47)
quandle order = 1711 = 29*59 Group = PSL(2,59)
quandle order = 1891 = 31*61 Group = PSL(2,61)
quandle order = 2701 = 37*73 Group = PSL(2,73)
quandle order = 3403 = 41*83 Group = PSL(2,83)
quandle order = 5671 = 53*107 Group = PSL(2,107)
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quandle order = 12403 = 79*157 Group = PSL(2,157)
quandle order = 13861 = 83*167 Group = PSL(2,167)
quandle order = 15931 = 89*179 Group = PSL(2,179)
quandle order = 18721 = 97*193 Group = PSL(2,193)
quandle order = 25651 = 113*227 Group = PSL(2,227)

Moreover these are the only such example from PSL(2,q)
for the 48 primes q from 5 to 229. The primes p for which this happens
are  5, 7, 11, 13, 23, 37, 47, 59, 61, 73, 83, 107, 157, 167, 179, 193, 227
which is the OEIS sequence: http://oeis.org/A079149 --I omit 3 here: note
PSL(2,3) has conjugacy classes of sizes 1,4, and 3 , but not 3*2.

There must  be a theorem that PSL(2,q) has a conjugacy class of size pq if and only if
q is in A079149 and q > 3.   It seems that the  p in the order pq is a factor of either q-1 or
if fact it appears that q = 2p+1 or q = 2p-1.

Recall that there are no non-Alexander quandles of order 5*7.
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4 Families of quandles parametrized by group elements

Let G be a group. Let X be a right G-module, and on $G \times X$, the operation defined by $(a, g) * (b, h) = (h^{-1}gh, ah + b(1-h))$ gives a quandle structure. This construction was found by Inoue-Jang-Oshiro.

J. Przyticki gave the non-abelian version: Let X be a group, G be a subgroup of $\mathrm{Aut}(X)$ acting on the right ($(x, g) \mapsto x \cdot g = x^g$, $(x^g) \cdot h = x^{gh}$). Then the operation defined on $G \times X$ by $(g, a) * (h, b) = (h^{-1}gh, (ab^{-1})^h\ b)$ gives a quandle structure.

Motivated from these, W. Edwin Clark computed the quandle structure on $C \times X$ where C is a conjugacy class of the automorphism group of the group X, and the operation is given by $(g, a) * (h, b) = (h^{-1}gh, (ab^{-1})^h\ b)$ for $g, h \in C$, $a, b \in X$, such that $C \times X$ is connected, and the cardinality of C is larger than 1.

```
Q8 is the quaternion group of order 8.

Also N:H means the semidirect product of N and H  as in the example
Aut(X) = (((C2 x C2 x C2 x C2) : C3) : C2) : C2

X = C2 x C2, Aut(X) = S3 conj class size = 3.  order = 12.
X = Q8, Aut(X) = S4 conj class size = 6.  order = 48.
X = Q8, Aut(X) = S4 conj class size = 6.  order = 48.
X = C2 x C2 x C2, Aut(X) = PSL(3,2) conj class size = 42.  order = 336.
X = C2 x C2 x C2, Aut(X) = PSL(3,2) conj class size = 56.  order = 448.
X = C2 x C2 x C2, Aut(X) = PSL(3,2) conj class size = 24.  order = 192.
X = C2 x C2 x C2, Aut(X) = PSL(3,2) conj class size = 21.  order = 168.
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X = C2 x C2 x C2, Aut(X) = PSL(3,2) conj class size = 24.  order = 192.
X = C3 x C3, Aut(X) = GL(2,3) conj class size = 6.  order = 54.
X = C3 x C3, Aut(X) = GL(2,3) conj class size = 12.  order = 108.
X = C3 x C3, Aut(X) = GL(2,3) conj class size = 6.  order = 54.
X = A4, Aut(X) = S4 conj class size = 6.  order = 72.
X = A4, Aut(X) = S4 conj class size = 6.  order = 72.
X = C6 x C2, Aut(X) = D12 conj class size = 3.  order = 36.
X = C2 x Q8, Aut(X) = (((C2 x C2 x C2 x C2) : C3) : C2) : C2 conj class size = 12.
order = 192.
X = C2 x Q8, Aut(X) = (((C2 x C2 x C2 x C2) : C3) : C2) : C2 conj class size = 12.
order = 192.
X = C2 x Q8, Aut(X) = (((C2 x C2 x C2 x C2) : C3) : C2) : C2 conj class size = 12.
order = 192.
X = C2 x Q8, Aut(X) = (((C2 x C2 x C2 x C2) : C3) : C2) : C2 conj class size = 12.
order = 192.
X = C10 x C2, Aut(X) = C4 x S3 conj class size = 3.  order = 60.
X = C10 x C2, Aut(X) = C4 x S3 conj class size = 3.  order = 60.
X = C10 x C2, Aut(X) = C4 x S3 conj class size = 3.  order = 60.
X = SL(2,3), Aut(X) = S4 conj class size = 6.  order = 144.
X = SL(2,3), Aut(X) = S4 conj class size = 6.  order = 144.
X = C3 x Q8, Aut(X) = C2 x S4 conj class size = 6.  order = 144.
X = C3 x Q8, Aut(X) = C2 x S4 conj class size = 6.  order = 144.

The following are various properties of these 25 quandles in the same order as above:

M[ 1]: ord =  12, dual=M[ 1], Connected = true, LeftDist = false,
Latin = false, Faithful = false
M[ 2]: ord =  48, dual=M[ 2], Connected = true, LeftDist = false,
Latin = false, Faithful = false
M[ 3]: ord =  48, dual=M[ 3], Connected = true, LeftDist = false,
Latin = false, Faithful = false
M[ 4]: ord = 336, dual=M[ 4], Connected = true, LeftDist = false,
Latin = false, Faithful = false
M[ 5]: ord = 448, dual=M[ 5], Connected = true, LeftDist = false,
Latin = false, Faithful = false
M[ 6]: ord = 192, dual=M[ 8], Connected = true, LeftDist = false,
Latin = false, Faithful = true
M[ 7]: ord = 168, dual=M[ 7], Connected = true, LeftDist = false,
Latin = false, Faithful = false
M[ 8]: ord = 192, dual=M[ 6], Connected = true, LeftDist = false,
Latin = false, Faithful = true
M[ 9]: ord =  54, dual=M[11], Connected = true, LeftDist = false,
Latin = false, Faithful = true
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M[10]: ord = 108, dual=M[10], Connected = true, LeftDist = false,
Latin = false, Faithful = false
M[11]: ord =  54, dual=M[ 9], Connected = true, LeftDist = false,
Latin = false, Faithful = true
M[12]: ord =  72, dual=M[12], Connected = true, LeftDist = false,
Latin = false, Faithful = false
M[13]: ord =  72, dual=M[13], Connected = true, LeftDist = false,
Latin = false, Faithful = false
M[14]: ord =  36, dual=M[14], Connected = true, LeftDist = false,
Latin = false, Faithful = false
M[15]: ord = 192, dual=M[15], Connected = true, LeftDist = false,
Latin = false, Faithful = false
M[16]: ord = 192, dual=M[16], Connected = true, LeftDist = false,
Latin = false, Faithful = false
M[17]: ord = 192, dual=M[17], Connected = true, LeftDist = false,
Latin = false, Faithful = false
M[18]: ord = 192, dual=M[18], Connected = true, LeftDist = false,
Latin = false, Faithful = false
M[19]: ord =  60, dual=M[20], Connected = true, LeftDist = false,
Latin = false, Faithful = false
M[20]: ord =  60, dual=M[19], Connected = true, LeftDist = false,
Latin = false, Faithful = false
M[21]: ord =  60, dual=M[21], Connected = true, LeftDist = false,
Latin = false, Faithful = false
M[22]: ord = 144, dual=M[22], Connected = true, LeftDist = false,
Latin = false, Faithful = false
M[23]: ord = 144, dual=M[23], Connected = true, LeftDist = false,
Latin = false, Faithful = false
M[24]: ord = 144, dual=M[24], Connected = true, LeftDist = false,
Latin = false, Faithful = false
M[25]: ord = 144, dual=M[25], Connected = true, LeftDist = false,
Latin = false, Faithful = false
```

Note that these are not simple quandles and so they cannot be isomorphic to the large quandles found as conjugacy classes of simple non-abelian groups.