On the Structure of Random Unlabelled Acyclic Graphs

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Abstract

One can use Poisson approximation techniques to get results about the asymptotics of graphical properties on random unlabelled acyclic graphs – i.e., on random unlabelled free (rootless) trees. We will use some "colored" partitions to get some rough descriptions of the structure of "most" unlabelled acyclic graphs. In particular, we will prove that for any fixed rooted tree $\mathfrak{T}$, almost every sufficiently large acyclic graph has a "subtree" isomorphic to $\mathfrak{T}$.

We can use this result to get a zero-one law for Monadic Second Order queries on random unlabelled acyclic graphs.

Key words: Colored partitions of integers, Monadic second order logic, Random trees, Subtrees of random trees, Unlabelled zero-one laws.

1 Introduction

In this article, we will describe the anatomy of almost all unlabelled trees. Our primary goal is to prove that for any unlabelled rooted tree $\mathfrak{T}$, almost all sufficiently large unlabelled free trees have $\mathfrak{T}$ as a subtree. We will use some relatively elementary combinatorial and probabilistic methods, although the details will be technically complicated.

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The motivation for this problem comes from logic, although this article will have very little logic in it (and assumes no logical background of the reader). In [15], it was proven that for every r, there exists a subtree $\mathfrak{T}_r$ such that if the free trees $\mathfrak{A}$ and $\mathfrak{B}$ both have copies of $\mathfrak{T}_r$ as subtrees, then $\mathfrak{A}$ and $\mathfrak{B}$ agree on all Monadic Second Order (MSO) sentences of quantifier depth $\leq r$. Thus the main result of this paper implies that for any MSO sentence θ, either θ is true of almost all sufficiently large unlabelled free trees, or θ is false of almost all sufficiently large free trees. (This result was proven in [15] for labelled trees.)

This settles a question closely related to a question of Compton ([5]): is it true that for any MSO sentence θ, there is a fixed probability p_θ such that among sufficiently large unlabelled rooted trees, the probability of choosing one satisfying θ is (approximately) p_θ? This question was answered affirmatively by Woods ([22]), who used sophisticated methods involving generating functions to get the result. These methods could also be used to prove the main result of this paper, but instead we have developed a more elementary (albeit no less trying) approach.

(Compton asked and Woods answered this question for labelled trees as well.)

The basic notion is that a tree has a sort of skeleton of vertices with many "dependents," and one can describe what the skeleton of a random unlabelled free tree will look like by looking at the space of all unlabelled free trees sharing that particular skeleton.

This paper is in two sections.

First, we outline the basic material needed for the proof of the main result. In Subsection 2.1, we present the basic material on trees and state the Main Theorem 2.1. In Subsection 2.2, we describe Monadic Second Order logic and its "zero-one" laws, for those readers interested in that aspect of this problem. In Subsection 2.3, we introduce a device that we will find useful: partitions in which the blocks are colored. In Subsection 2.4, we describe the skeleton of a tree, and also the non-skeletal leaves and ... twigs.

Second, we prove that for any rooted tree $\mathfrak{T}$, almost all unlabelled free trees have $\mathfrak{T}$ as a subtree. In Subsection 3.1, we introduce a technical device for dealing with chains of skeletal vertices. In Subsection 3.2, we introduce a technical device for making global descriptions of skeletons. And in Subsection 3.3, we finish the proof by proving that (a) almost every unlabelled free tree has either many vertices of large degree or at least one vertex of very large degree, and (b) it follows that almost surely $\mathfrak{T}$ will be a subtree attached to such a vertex.

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2 Setting the Stage

This section will be devoted to a more general view of the situation, with an eye towards developing a toolbox for Section 3.

Before going further, let's recount some notation that we will use.

First, $\mathbb{N}$ is the set of natural numbers and $[n] = \{1, 2, \dots, n\}$; and if S is a set, let $|S|$ be the cardinality of S. And a graph $\mathfrak{G}$ will be regarded as an ordered pair $\langle G, E\rangle$, where G is the set of vertices and E is the anti-reflexive, symmetric edge relation. If $\mathfrak{G} = \langle G, E\rangle$ and $g \in G$, let $\mathfrak{G} - g = \langle G - \{g\}, E - \{\{g, g'\} \colon g' \in G\}\rangle$. We will denote $|\mathfrak{G}| = G$ as the set of vertices, and hence $\|\mathfrak{G}\| = |G|$ as the order of the graph. Let $\lceil\cdot\rceil$ be the integer ceiling function, and let $\lfloor\cdot\rfloor$ be the integer floor function.

More, if $\lim_{n\to\infty}[f(n)/g(n)] = 0$, write "$f(n) = o(g(n))$." In addition, if $\mathcal{B}_n$ is a set for each $n \in \mathbb{N}$, and if $\mathcal{A}_n \subseteq \mathcal{B}_n$ is such that $\lim_{n\to\infty}[|\mathcal{A}_n|/|\mathcal{B}_n|] = 1$, say that for large n, a randomly selected element of $\mathcal{B}_n$ is *almost surely* (a.s.) an element of $\mathcal{A}_n$.

2.1 Trees

In this paper, we will look at structures sometimes called 'free trees' and sometimes called 'acyclic graphs'. These are different from 'rooted trees' (or just 'finite trees', if you are a set theorist).

- A *rooted tree* is often regarded as a finite partial order with a single minimal element (the 'root') and such that for any x in the partial order, the set of all predecessors of x is linearly ordered.
- Graph theorists often regard a rooted tree as a connected acyclic graph with a distinguished vertex, the 'root'. A connected acyclic graph with no distinguished root is thus a 'non-rooted' or *free* tree. (An acyclic graph, not necessarily connected, is often called a 'forest.')

These two definitions are essentially equivalent in Monadic Second Order logic, so we can use one or the other interchangeably. In this paper, we go with the graph theorists: trees are graphs, etc.

It will turn out that Monadic Second Order queries on trees are very dependent on what "subtrees" a given tree has. So here is the basic notion of this paper.

Definition 2.1 *Given a free tree $\mathfrak{A}$ and a rooted tree $\mathfrak{T}$ with root $\mathfrak{t}$, $\mathfrak{A}$ has $\mathfrak{T}$ as a* subtree *if there is a graph embedding $\pi \colon |\mathfrak{T}| \mapsto |\mathfrak{A}|$ such that $\mathfrak{A} - \pi[\mathfrak{T} - \mathfrak{t}]$*

is connected.

We will sometimes imagine a vertex of a tree as being the root of many disjoint subtrees, disjoint except for their common root. Furthermore, in each of these subtrees, we will imagine the root to be of degree 1. Imagine a bunch of grapes: at a branching of the stem, many substems extend to subbranches.

Definition 2.2 *A rooted tree whose root is of degree* 1 *is called* planted.

In this article, we are not interested in the set of all free trees of vertex set $[n]$: we are interested in the isomorphism classes of these trees.

Definition 2.3 *An* unlabelled *tree is an isomorphism class of labelled trees.*

Let $\mathcal{U}_n$ be the set of all n-vertex unlabelled free trees. We can regard these sets as probability spaces, with each unlabelled tree of $\mathcal{U}_n$ being equally likely of being chosen. So if $\mathcal{B}_n \subseteq \mathcal{U}_n$ for each n, we can let $\nu_n[\mathcal{B}_n] = |\mathcal{B}_n|/|\mathcal{U}_n|$ be the probability that a tree in $\mathcal{B}_n$ is chosen. So if $\mathcal{B}$ is a property of unlabelled free trees, and $\mathcal{B}_n$ is the set of all n-vertex unlabelled free trees of property $\mathcal{B}$, then we are interested in the *asymptotic probability* of choosing an unlabelled free tree of property $\mathcal{B}$, i.e., in $\lim_{n\to\infty} \nu_n[\mathcal{B}_n]$.

In this paper, we will prove:

Theorem 2.1 (Main Theorem) *For any fixed rooted tree $\mathfrak{T}$, if $\mathcal{T}_n$ is the set of n-vertex unlabelled free trees having subtrees isomorphic to $\mathfrak{T}$, then*

$$\lim_{n\to\infty} \nu_n[\mathcal{T}_n] = 1.$$

From this we will derive the Main Corollary 2.1: If $\mathcal{B}$ is "Monadic Second Order Expressible", then $\lim_{n\to\infty} \nu_n[\mathcal{B}_n] \in \{0, 1\}$.

2.2 Zero-One Laws for Monadic Second Order Logic

As mentioned in the introduction, MSO logic provided the original motivation for this research. Nevertheless, it is not necessary to understand MSO for this paper: the reader can safely omit this subsection. However, for those readers interested in MSO, we will mention here some implications of these results. (For more on Monadic Second Order logic, see [6]; for MSO on trees, see [16].)

We will be working with graphs, i.e., with structures $\mathfrak{G} = \langle G, \mathrm{Edge} \rangle$, where $G = |\mathfrak{G}|$ is the *domain* of $\mathfrak{G}$ and Edge is the edge relation of $\mathfrak{G}$. (And $\mathfrak{G}$ has $\|\mathfrak{G}\| = |G|$ elements.)

Before we go into the logic, we should note what kind of variables we are using. The *first order* variables range over vertices of a graph, while the *second order* variables range over the relations. In this paper, all second order variables range over unary relations (equivalently, over subsets): they are thus *monadic.*

We will be working with MSO logic, in the language of graph theory, which we characterize as follows. Note that in the language of graph theory, there is one binary relation, Edge. We will need a notion of (first order) quantifier *depth*, which we will characterize as follows.

- The *atomic* formulas $\mathrm{Edge}(x, y)$ and $x = y$ are of depth 0; if S is a monadic second order variable, then $S(x)$ is of depth 0. Here, x could be a first order variable or a constant symbol.
- The *boolean combinations* are the negations, conjunctions, and disjunctions. If φ and ψ are of depth at most r, then the same is true of $\neg\varphi$, $\varphi \wedge \psi$, and $\varphi \vee \psi$.
- The *first order quantifications* are $\exists x\varphi$ and $\forall x\varphi$, where x is a first order variable. If φ is of depth r, then $\exists x\varphi(x)$ and $\forall x\varphi(x)$ are of depth $r + 1$. In first order quantification, we define free and bound first order variables in the usual way: in $\exists x\mathrm{Edge}(x, y)$, x is *bound* by the existential quantification, while y is *free.*
- Let $\theta(\vec{S})$ be a formula with no second order quantifications, no free first order variables, and of first order quantifier depth l. Let $\vec{S}$ be a list of k monadic second order variables. Then for any string $\mathbf{Q}$ of k quantifications, $\mathbf{Q}\vec{S}\theta(\vec{S}) \equiv Q_1S_1 \cdots Q_kS_k\theta(\vec{S})$ is a MSO sentence of *joint depth* (k, l).

Thus the MSO sentences we look at will be of the form $\mathbf{Q}\vec{S}\theta(\vec{S})$, where θ has no second order quantifications. This is just a fragment of MSO logic, but it is known that every MSO sentence is equivalent to a MSO sentence in this fragment.

Incidentally, the *First Order* (FO) sentences are those MSO sentences with no second order variables, and hence no second order quantifications.

Definition 2.4 *Fix an* MSO *sentence θ and a pair of graphs $\mathfrak{A}$ and $\mathfrak{B}$. Suppose that for every θ of joint depth (k, l), $\mathfrak{A}$ satisfies θ iff $\mathfrak{B}$ satisfies θ. Then write "$\mathfrak{A} \equiv_{k,l} \mathfrak{B}$."*

(For more on the equivalence relations $\equiv_{k,l}$, see [6].)

The following was proven in [15].

Theorem 2.2 *For each r, there exists a rooted tree $\mathfrak{T}_r$ such that if $\mathfrak{A}$ and $\mathfrak{B}$ are each free trees with $\mathfrak{T}_r$ as a subtree, then $\mathfrak{A} \equiv_{r,r} \mathfrak{B}$.*

Recall from Subsection 2.1 that if $\mathcal{U}_n$ is the set of all unlabelled n-vertex free trees, and $\mathcal{B}$ is a set of unlabelled free trees, then $\nu_n[\mathcal{B} \cap \mathcal{U}_n] = |\mathcal{B} \cap \mathcal{U}_n|/|\mathcal{U}_n|$.

Definition 2.5 *Let $\mathcal{B}$ be a set of free unlabelled trees; let $\mathcal{B}_n = \mathcal{B} \cap \mathcal{U}_n$ for each positive integer n. If $\lim_{n\to\infty} \nu_n[\mathcal{B}_n] \in \{0,1\}$, we say that $\mathcal{B}$ admits the* unlabelled zero-one law *on free trees. If B is a property of free trees, and $\mathcal{B}$ is the set of unlabelled free trees with Property B, then if $\mathcal{B}$ admits the unlabelled zero-one law, we say that Property B admits the* unlabelled zero-one law *on free trees. If a logic $\mathcal{L}$ is such that every $\mathcal{L}$-definable property admits an unlabelled zero-one law on free trees, we say that $\mathcal{L}$ admits an* unlabelled zero-one law *on free trees.*

In this article, we prove that MSO logic admits an unlabelled zero-one law on free trees.

Corollary 2.1 (Main Corollary) *If Property B of Definition 2.5 is "Monadic Second Order Expressible", then $\lim_{n\to\infty} \nu_n[\mathcal{B}_n] \in \{0,1\}$.*

Proof. By the Main Theorem 2.1, for each r, almost all sufficiently large unlabelled free trees have copies of $\mathfrak{T}_r$ as subtrees. By Theorem 2.2, all unlabelled free trees having copies of $\mathfrak{T}_r$ as subtrees agree on all MSO sentences of joint depth less than or equal to (r, r): let Θ_r be the set of sentences that all such trees satisfy. Thus all other MSO sentences of joint depth $\leq (r, r)$ are not satisfied by any of these trees: let $\bar{\Theta}_r$ be these unsatisfied sentences. Let $\Theta = \bigcup_r \Theta_r$ and $\bar{\Theta} = \bigcup_r \bar{\Theta}_r$, and note that all MSO sentences are in Θ (and are thus satisfied by almost all sufficiently large free unlabelled trees) or in $\bar{\Theta}$ (and are thus *not* satisfied by almost all sufficiently large free unlabelled trees). Thus the free unlabelled trees admit a MSO zero-one law. ■

It remains to prove Theorem 2.1.

2.3 Colored Partitions

We are going to dissect trees, and count subtrees, somewhat as was done in [15]. But we will need some additional enumerative apparatus; here we use a notion of a "colored partition" similar to that of [21]. The idea is this. Suppose that for each integer $i > 0$, we could color i any one of c_i colors. Then if we were given a partition of an integer $m = m_1 + m_2 + \ldots + m_h$, we could color m_1 one of c_{m_1} colors, m_2 one of c_{m_2} colors, and so on. Assuming that the order of the terms $m_1, \ldots, m_h$ is unimportant, how many "colored" partitions are there?

First, let's formally define colored partitions.

Definition 2.6 *For each positive integer i, let c_i be a positive integer. Let m*

be a positive integer. A colored partition *of m with respect to the sequence* $(c_i\colon i \in \mathbb{N})$ *is an expression*

$$m = m_1 + m_2 + \ldots + m_h,$$

together with a coloring function $\chi\colon [h] \to \mathbb{N}$ *such that:*

- *For each j and j', if $j < j'$, then $m_j \le m_{j'}$.*
- *For each $j \in [h]$, $\chi(j) \in [c_{m_j}]$.*
- *For each j and j', if $j < j'$ and $m_j = m_{j'}$, then $\chi(j) \le \chi(j')$.*

These are "unordered" colored partitions, as opposed to the "ordered" ones in [21]. As with ordinary partitions, we are chopping numbers into blocks; notice that if $c_m = 1$ for each m, we get the ordinary unordered partitions of Euler, Hardy, etc.

Theorem 2.3 *Suppose that for each positive integer j, we have c_j colors for j-element blocks. For each positive integer k, let $e_k = c_k + \cdots + c_1 - 1$. Then for each k, there exists a real $\zeta_k > 0$ such that as $m \to \infty$, the number of colored partitions of m into blocks of at most k elements each is $P_k(m) = (\zeta_k \pm o(1))m^{e_k}$.*

Proof. We will prove this theorem by induction on k. For $k = 1$, we get $P_1(m) = \binom{m+c_1-1}{c_1-1} = \left(\frac{1}{(c_1-1)!} + o(1)\right) m^{c_1-1}$, and we get $\zeta_1 = \frac{1}{(c_1-1)!}$ and $e_1 = c_1 - 1$.

Now, if $k > 1$, then since choosing a colored partition of blocks of up to k elements can consist of choosing and coloring some k-element blocks and then choosing a colored partition of the rest as blocks of up to $k-1$ elements,

$$P_k(m) = \sum_{j=0}^{\lfloor \frac{m}{k} \rfloor} \binom{j + c_k - 1}{c_k - 1} P_{k-1}(m - jk).$$

And by induction, we already know that for some ζ_{k-1}, if $e_{k-1} = c_{k-1} + \cdots + c_1 - 1$, $P_{k-1}(m) = (\zeta_{k-1} + o(1))m^{e_{k-1}}$. So if m is large, a little work will give us

$$P_k(m) = \frac{\zeta_{k-1} + o(1)}{(c_k - 1)!} \sum_{j=0}^{\lfloor \frac{m}{k} \rfloor} j^{c_k-1}(m - jk)^{e_{k-1}},$$

We will approximate this with an integral. Let $f_k(x) = x^{c_k-1}(1 - kx)^{e_{k-1}}$ on $[0, \frac{1}{k}]$, and since f_k is continuous,

$$\int_0^{\frac{1}{k}} f_k(x)\,dx = \lim_{m\to\infty} \sum_{j=0}^{\lfloor\frac{m}{k}\rfloor-1} f_k\left(\frac{j}{m}\right)\frac{1}{m}$$

$$= \lim_{m\to\infty} \frac{1}{m} \sum_{j=0}^{\lfloor\frac{m}{k}\rfloor-1} \left(\frac{j}{m}\right)^{c_k-1} \left(1-k\frac{j}{m}\right)^{e_{k-1}}$$

$$= \lim_{m\to\infty} \frac{1}{m^{c_k+e_{k-1}}} \sum_{j=0}^{\lfloor\frac{m}{k}\rfloor-1} j^{c_k-1}(m-jk)^{e_{k-1}}.$$

So if m is large

$$(\zeta_{k-1}+o(1))m^{c_k+e_{k-1}}\left(\int_0^{\frac{1}{k}} f_k(x)\,dx+o(1)\right) = (c_k-1)!P_k(m),$$

so setting $\zeta_k = \zeta_{k-1}\int_0^{\frac{1}{k}} f_k(x)\,dx/(c_k-1)!$ and $e_k = c_k+e_{k-1} = c_k+\cdots+c_1-1$, we have $P_k(m) = (\zeta_k+o(1))m^{e_k}$. ■

From this, we get the following blunt instrument.

Corollary 2.2 *For each k, there exists C_k such that for all m, $P_k(m) \le C_k m^{e_k}$.*

2.4 The Architecture of Trees

Now we turn to the structure of trees.

The idea is that all trees have a sort of “skeleton,” of vertices with many dependents. Such a skeleton is itself a tree, and it could be considered as the boughs of the tree. The vertices of the skeleton may have many “direct” (we'll define this term below) nonskeletal dependents: it might have many “twigs” and leaves sprouting out of it. Or it might have only a few twigs and leaves sprouting out of it, or none at all: it might be a bare bough vertex, supporting other limbs, or foliage attached to the end. With this image in mind, we see a way to divide the description of a tree: we first have a description of its skeleton, and then we add a description of the lighter foliage attached to the skeleton. This attachment can be described using the colored partitions of Subsection 2.3: each skeletal vertex v is the root of planted subtrees that are lighter foliage, and the colored partitions will fix the isomorphism types of the planted subtrees rooted at v.

So much for the motivating image. Now for the technical details. We will need a technical definition: we will need the notion of a 'center' of a tree.

Definition 2.7 *Let $\mathfrak{T}$ be a tree. The* center *of $\mathfrak{T}$ is the unique vertex (or unique pair of adjacent vertices) such that removing that vertex (or vertices) from $\mathfrak{T}$ leaves a collection of components each having less than half the vertices of $\mathfrak{T}$.*

All trees have centers (see, e.g., [2, Section 3.2]).

Imagine a tree of n vertices. Each vertex can be regarded as bearing a certain "weight", which is computed as follows. Given a tree $\mathfrak{U}$ and a vertex v, recall that $\mathfrak{U} - v$ is the result of deleting v from $\mathfrak{U}$; the result is an acyclic graph. If $\mathfrak{U} - v$ is connected, v was a leaf (which we will say is of weight 0); otherwise, $\mathfrak{U} - v$ consists of several acyclic components. If $\mathfrak{U} - v$ has no components of order greater than $n/2$, v is a central vertex.

Definition 2.8 *Let v be a vertex of a tree $\mathfrak{U}$. Say that a vertex w* depends *on v if the path from w to the central vertice(s) of $\mathfrak{U}$ goes through v. The vertex w* immediately *depends on v if w depends on v and is also adjacent to v.*

If v is not central, $\mathfrak{U} - v$ will have a largest component of more than $n/2$ vertices, and other components that collectively have less than $\frac{n}{2} - 1$ vertices: the "weight" of v will be the number of vertices in these smaller components. Our architectural definitions will be built from this notion of weight.

Definition 2.9 *Let $\mathfrak{U}$ be a free tree, and let $v \in |\mathfrak{U}|$. Then removing v from $\mathfrak{U}$ leaves us with a forest $\mathfrak{U} - v$ of acyclic components, of orders $\lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_{d(v)-1}$, where $d(v)$ is the degree of v. Let $\lambda_1 + \lambda_2 + \ldots + \lambda_{d(v)-1}$ be the* weight *of v.*

Note that if the orders of the components of $\mathfrak{U} - v$ are $\lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_{d(v)-1}$, then the vertices adjacent to v could be labelled $v_0, \ldots, v_{d(v)-1}$. Note that v is central iff $\lambda_0 \leq n/2$, and that v is a leaf iff $\lambda_1 = 0$. Note that for each i, $1 \leq i \leq d(v) - 1$, the weight of v_i is $\lambda_i - 1$.

We want to develop the notion of a "skeleton" of a tree, rather like the skeleton of an elm tree in a stormy February, with leaves and smaller twigs missing. What we do is chop off the vertices of smaller weight.

Definition 2.10 *Fix an integer $k > 0$, and let $\mathfrak{U}$ be a free tree. The k-*skeleton *of $\mathfrak{U}$ is the restriction of $\mathfrak{U}$ to vertices of weight $\geq k$ — or which are central. Denote the k-skeleton by $\mathfrak{S}_k[\mathfrak{U}]$.*

For any free tree $\mathfrak{U}$, $\mathfrak{S}_k[\mathfrak{U}]$ is connected: if $v, w \in |\mathfrak{S}_k[\mathfrak{U}]|$ and u is on the path from v to w in $\mathfrak{U}$, then the $\mathfrak{U} - u$ has at least two components with at least k vertices, hence u is either central or has a dependent of weight at least k, so the weight of u itself is at least k: so $u \in |\mathfrak{S}_k[\mathfrak{U}]|$.

Definition 2.11 *Let v be a vertex of a tree's k-skeleton. A vertex $u \neq v$ is a* k-direct *dependent of v if u is a dependent of v and the path from v to u has no k-skeletal vertices other than v itself.*

And now we want a definition for the clumps of light foliage that sprout from skeletal vertices.

Definition 2.12 *Let $v \in \mathfrak{S}_k[\mathfrak{U}]$. Let $\mathfrak{U}_{(v)}$ be the subtree of $\mathfrak{U}$ consisting of v and the k-direct dependents of v. This is a* k-cluster *of v if $\|\mathfrak{U}_{(v)}\| > k$.*

We will tend to write $\mathfrak{S}_k$ for $\mathfrak{S}_k[\mathfrak{U}]$ if $\mathfrak{U}$ is understood from the context.

We now want to identify those vertices in the k-skeleton that have many k-direct dependents. First of all, since $\mathfrak{S}_k$ is itself a tree, its leaves are vertices with at least k k-direct dependents as they each have weight at least k. Call the leaves of the k-skeleton the *k-skeletal leaves*. But there may be k-skeletal non-leaves that have many k-direct dependents. So we need a formal definition of these vertices with many (non-k-skeletal) k-direct dependents.

Definition 2.13 *The* k-strength *of a vertex v is the number $\|\mathfrak{U}_{(v)}\| - 1$of its k-direct dependents. If a k-skeletal vertex is of k-strength at least k, call it* k-strong.

So v is k-strong iff $\mathfrak{U}_{(v)}$ is a k-cluster.

We conclude this subsection by showing that when one chooses a random unlabelled free n-tree, that tree's k-skeleton a.s. has lots of vertices. To do this, we will use the colored partition function. Imagine a subtree of root v: it can be regarded as the union of many planted subtrees, with the roots identified. The isomorphism types of the subtree are thus fixed by the isomorphism types of the planted subtrees that make it up.

Remark 2.1 *For each positive integer k, there is a $\zeta_k > 0$ and an e_k such that the following is true.*

On a very large tree $\mathfrak{U}$, if v is a skeletal vertex of k-strength m, where m is very large, there are $(\zeta_k + o(1))m^{e_k}$ isomorphism types of rooted trees to replace $\mathfrak{U}_{(v)}$ to get a different tree of the same order as $\mathfrak{U}$.

Proof. Suppose that a vertex v has m direct dependents. First, we want to estimate the number of isomorphism types these direct dependents can be in (i.e., how many isomorphism types are there for the unlabelled rooted tree $\mathfrak{U}_{(v)}$). If the immediate dependents of v were, say, $v_1, \ldots, v_h$, of weights $w_1, \ldots, w_h < k$ resp., we could imagine that for each i, v_i is the root of one of the (isomorphism types of) unlabelled rooted trees of w_i+1 vertices. And the total number of these direct dependents of v is thus $m = w_1 + \cdots + w_h + h$, so we could count the number of isomorphism types of subtree clusters $\mathfrak{G}_{(v)}$ by counting colored partitions of m as follows.

For each w, let c_{w+1} be the number of isomorphism types of rooted $(w+1)$-trees, and let $P_k(m)$ be the number of isomorphism types of rooted trees of $m+1$ vertices, where each maximal planted subtree off the root is at most $k+1$ vertices. (Hence if v is the root of this planted subtree and v' is the one vertex of the subtree adjacent to v, the subtree of v' and its dependents have at most k vertices.) Then letting $e_k = c_k + \ldots + c_1 - 1$, Theorem 2.3 says that there are $(\zeta_k + o(1))m^{e_k}$ such colored partitions, hence that many isomorphism types for $\mathfrak{U}_{(v)}$. As these give distinct trees $\mathfrak{G}$, this gives us the remark. ■

Lemma 2.1 *For any fixed k and S, and $\varepsilon > 0$, if n is sufficiently large, the probability that an unlabelled free n-tree will have at most S k-skeletal vertices is less than ε.*

Proof. Fix k and $s \le S$. Call an unlabelled free n-tree *good* if it has s k-skeletal vertices. Such a tree may be constructed by taking s vertices, making the (k-skeletal) tree out of them, and then adding the remaining $n-s$ vertices as direct dependents (i.e., of weight $< k$) to these. We claim that the number of good n-trees is much less than the total number of unlabelled free n-trees — proportionately less than ε/S of them — which will prove the lemma.

Let C_k and e_k be defined in Corollary 2.2 and Remark 2.1 resp., so that $P_k(m) \le C_k m^{e_k}$ for all m. Thus for an $(m+1)$-vertex k-cluster, the number of isomorphism types of $(m+1)$-trees consisting of a k-skeletal vertex (as the root) and its k-direct dependents is $P_k(m)$ if that vertex is of k-strength m. So if we had $s \le S$ distinguishable k-skeletal vertices (to get an overestimate of the number of trees with small skeletons), and we distributed as k-direct dependents m_1, m_2, ..., m_s vertices to them respectively (where $n - s = \sum_{i=1}^s m_i$), then by Corollary 2.2, there would be at most

$$p_k(m_1, \ldots, m_s) = \prod_{i=1}^{s} P_k(m_i) \le C_k^s \left(\prod_{i=1}^{s} m_i^{e_k}\right) = C_k^s \left(\prod_{i=1}^{s} m_i\right)^{e_k}$$

isomorphism types of good n-trees created by taking that k-skeleton and attaching the appropriate number of k-direct dependent vertices to each. We want an upper bound on $p_k(m_1, \ldots, m_s)$, subject to the constraint $n - s = \sum_{i=1}^s m_i$; this upper bound will turn out to be a polynomial in n.

As the function $(m_1, \ldots, m_s) \mapsto \prod_{i=1}^s m_i$ subject to $\sum_{i=1}^s m_i = n - s$ is maximized at $m_i = (n-s)/s$ for each i (just use Lagrange multipliers), we get

$$p_k(m_1, \ldots, m_s) \leq \left(C_k \left(\frac{n-s}{s}\right)^{e_k}\right)^s \leq (C_k n^{e_k})^S = C_k^S n^{e_k S}.$$

Thus the total number of isomorphism types of good n-trees such that

- its k-skeleton is a particular unlabelled free s-tree, and
- the subtree clusters are of orders $m_1, \ldots, m_s$,

is bounded above by $C_k^S n^{e_k S}$.

There are $\binom{n-1}{s-1}$ ordered partitions of $n-s$ into s numbers $m_1, \ldots, m_s$. Thus there are at most

$$\binom{n-1}{s-1} C_k^S n^{e_k S} \leq \frac{C_k^S n^{Se_k+s-1}}{(s-1)!} \leq C_k^S n^{S(e_k+1)-1}$$

n-trees with that s-vertex k-skeleton. As there are s^{s-2} labelled s-trees, the total number of good n-trees is bounded above by $s^{s-2} C_k^S n^{S(e_k+1)-1}$, and hence by $S^{S-2} C_k^S n^{S(e_k+1)-1}$. Thus the total number of unlabelled free n-trees of at most S vertices is bounded above by $S \cdot S^{S-2} C_k^S n^{S(e_k+1)-1}$, and hence by $S^{S-1} C_k^S n^{S(e_k+1)-1}$, which is a polynomial bound.

But there are exponentially (in n) many unlabelled free n-trees. In particular, for some sufficiently large n, the proportion of unlabelled free n-trees with over s k-skeletal vertices is at least $1-\varepsilon$, and we are done. ■

Indeed, we suspect that:

Conjecture 2.1 *There exists a κ_k such that as $n \to \infty$, the number of k-skeletal vertices in a random unlabelled free tree is a.s. $(1 \pm o(1))\kappa_k n$.*

From [1], [9], and [15], notice that this conjecture is true for labelled random trees.

3 The Proof

Now that we have our toolbox, let's prove the Main Theorem. Remember that we want to prove that for any given rooted tree $\mathfrak{T}$, as $n \to \infty$, then a.s. a random unlabelled free n-tree has a subtree isomorphic to $\mathfrak{T}$. For sanity's sake, fix $k = \|\mathfrak{T}\|$, and suppress k in subsequent notation (thus 'skeletons' are

k-skeletons, 'strength' is k-strength, 'direct' means k-direct, etc.). Also, trees are unlabelled and free unless otherwise indicated.

The proof will be in three parts (each in its own subsection):

(1) In Subsection 3.1, we will prove a very technical lemma which very roughly says that an unlabelled random tree with two roots a.s. either contains many strong vertices or contains a "very strong" vertex (in a sense we will define below) on the path connecting the two roots. This technical lemma is used to prove the next part.
(2) In Subsection 3.2, we will develop a descriptive device to classify skeletons by their "reducts," and we will prove that for any one reduct, relatively few n-trees have a skeleton of that reduct.
(3) In Subsection 3.3, we will show that a random n-tree a.s. either has many strong vertices, or at least one "very strong" vertex. Then we will show that if there are many strong vertices (or if there is a "very strong" vertex), then there will a.s. be at least one copy of $\mathfrak{T}$ among the dependents of at least one of these strong vertices.

This will complete the proof of the Main Theorem.

3.1 "Chain Trees"

This subsection is of purely technical interest, for verifying the results in Subsection 3.2.

Definition 3.1 *A* labelled chain tree *is a graph $\mathfrak{C} = \langle C, R, a, b\rangle$, where $\langle C, R\rangle$ is a (labelled) free tree, and where $a, b \in C$. We say that a and b are the* roots *of $\mathfrak{C}$, while the* spine *of $\mathfrak{C}$ is the set of vertices on the path connecting a and b: denote the spine by $[a, b]$. We permit $a = b$. Call a chain tree* planted *if $a \neq b$ and* degree(a) =degree(b) $= 1$.

We will be dealing with *unlabelled* chain trees, i.e., isomorphism classes of labelled chain trees (under isomorphisms that preserve a and b). From now on, a *chain tree* is an unlabelled chain tree.

We can define weights and strengths analogously with Subsection 2.4, with the spine in place of the skeleton.

Definition 3.2 *We have definitions analogous to those of Definitions 2.11 and 2.13.*

- *A non-spinal vertex y is a* (spinal) direct dependent *of the spinal vertex x if the path from x to y intersects the spine only at x.*

- *For each spinal vertex x, let* $\operatorname{sstrength}(x)$ *be the number of non-spinal vertices directly dependent on x; if* $\operatorname{sstrength}(x) \geq k$, *call x* (spinally) strong.

Let $\operatorname{Dep}(x)$ *be the set of non-spinal vertices directly dependent on x, so that* $\operatorname{sstrength}(x) = |\operatorname{Dep}(x)|$.

Note the distinction between skeletal and spinal strength: the skeleton is the minimal free tree connecting all the strong vertices, while the spinally strong vertices are those vertices on the spine with lots of spinal direct descendents.

And now for our technical gadget:

Definition 3.3 *Fix k and an integer $\omega > 0$. We chop much of a chain tree $\mathfrak{C}$ into ω* chain tree fragments *as in the following paragraph.*

Let the spine of $\mathfrak{C}$ be $[a, b] = \{a = x_0, x_1, \ldots, x_m = b\}$ (going along the path from a to b), and define the following tuples by induction on $i = 0, 1, \ldots, \omega$:

(1) Let $t_0 = 1$ and given t_i, let t_{i+1} be the least $t > t_i$ such that

$$\sum_{j=t_i}^{t-1} [\operatorname{sstrength}(x_j) + 1] \geq k + 1.$$

Of course, it may be that no such t exists: in this event, just define $t_{i+1} = m$ and stop; it will turn out that such failure is improbable.

(2) For each i, if $t_i < m$, let $d_i = \sum_{j=t_i}^{t_{i+1}-1} [\operatorname{sstrength}(x_j) + 1]$. Given $\mathfrak{C}$, ω, and k, if $t_\omega < m$ we get the tuple $\vec{d} = (d_0, \ldots, d_{\omega-1})$, which we denote $\operatorname{CTF}(\mathfrak{C}, \omega; k) = \vec{d}$.

(3) For each i, the ith *chain tree fragment* is the graph $\langle C_i, R_i, x_{t_i}, x_{t_{i+1}-1} \rangle$, where

$$C_i = \{x_{t_i}, \ldots, x_{t_{i+1}-1}\} \cup \bigcup_{j=t_i}^{t_{i+1}-1} \operatorname{Dep}(x_j),$$

and R_i is the restriction of R to C_i. Note that the chain tree fragment is itself a chain tree. Note that it is possible that $x_{t_i} = x_{t_{i+1}-1}$.

We get the tuples $\vec{t}$ and $\vec{d}$, giving the spinal positions and number of vertices, respectively, of the ω chain tree fragments obtained by chopping $\mathfrak{C}$'s spine $[a, b]$, starting from a's end; notice that given ω and $\mathfrak{C}$ (and k, which is fixed throughout this section), the ω chain tree fragments of $\mathfrak{C}$ are determined. Notice that if $t_\omega \leq m$, than a and b are not in any of the chain tree fragments.

Next, we want the family of all chain trees of n vertices that have the same sequence of orders $\vec{d}$ of chain tree fragments.

Definition 3.4 *Given z and $\vec{d}$, let $\Gamma(z, \vec{d})$ be the set of planted chain trees $\mathfrak{C} = \langle C, R, a, b\rangle$ such that:*

- *There are z vertices: $|C| = z$.*
- *We get* $\mathrm{CTF}(\mathfrak{C}, \omega; k) = \vec{d}$.

Note that if $\vec{d} \neq \vec{d'}$, then $\Gamma(z, \vec{d}) \cap \Gamma(z, \vec{d'}) = \varnothing$.

We will be using a bound M (to be determined below) in subsequent computations. We will prove that almost surely there are at least $M(k+1)$ vertices in k-clusters. To do this, we will show that for any M, if n is large, then most free n-trees either have more than M strong vertices or have vertices of strength at least $M(k+1)$:

Definition 3.5 *Fix M (and k). In a tree, a skeletal vertex is* very strong *(with respect to M (and k)) if it has at least $M(k+1) - 1$ nonskeletal direct dependents. Similarly, a spinal vertex is* (spinally) very strong *if it has at least $M(k+1) - 1$ non-spinal direct descendents.*

Again, note the distinction between very strong and spinally very strong vertices. Once M is fixed, we will want to fix other numbers as well for technical results.

Note that in the construction of Definition 3.3, in getting the ith chain tree fragment $\mathfrak{F}_i$ of distinguished vertices x_{t_i} and $x_{t_{i+1}-1}$, that

$$\sum_{j=t_i}^{t_{i+1}-2} [\mathrm{sstrength}(x_j) + 1] \leq k,$$

so that if $x_{t_{i+1}-1}$ is not very strong, then $d_i < k + M(k+1)$.

Definition 3.6 *Fix M (and k), and $\varepsilon > 0$. If $k < d < k + M(k+1)$, let β_d be the proportion of d-vertex chain trees with at least one strong vertex but no very strong vertices.*

Let $\beta = \max\{\beta_d, 1 - \beta_d \colon k < d < k + M(k+1)\}$, and note that $0 < \beta < 1$. Let ω be such that $\binom{\omega}{M}\beta^\omega < \frac{\varepsilon}{M}$, and thus for each $i \leq M$

$$\binom{\omega}{i}\beta^i\beta^{\omega-i} < \frac{\varepsilon}{M}. \tag{1}$$

If Inequality 1 holds, then

$$\sum_{i=0}^{M-1}\binom{\omega}{i}\beta^\omega < \varepsilon. \tag{2}$$

Here, the (conditional) probability that a chain tree fragment (which is, after all a chain tree) with no very strong vertices has a strong vertex is at most $\beta < 1$, and the (conditional) probability that it does not have any strong vertices is at most $\beta < 1$. Thus Inequality 2 gives an upper bound on the (conditional) probability that a sufficiently large random chain tree $\mathfrak{C}$ with no very strong vertices will have the following property.

> If $\mathfrak{C}$ is partitioned into chain tree fragments as in Definition 3.3, there will be at most $M-1$ chain tree fragments with (spinally) strong vertices.

Hence Inequality 2 also gives an upper bound for the probability that a sufficiently large random chain tree will have at most $M-1$ strong vertices while not having any (spinally) very strong vertices.

As M, β, and ε are fixed, it follows that there exists ω_0 such that $\omega > \omega_0$ implies that Inequality 1 — and hence Inequality 2 — holds.

Lemma 3.1 *Fix M. For each $\varepsilon > 0$, there exists an $Z = Z(M, \varepsilon)$ such that the following is true for any z. If $z \geq Z$, then the probability that a randomly selected chain tree of z vertices (with no non-spinal vertices of strength $\geq k$) has less than M strong vertices — none very strong — is at most ε.*

Proof. Let's look at the construction of chain trees with no vertices of strength as great as $M(k+1)-1$.

For each d, $k < d < k + M(k+1)$, there is at least one chain tree (fragment) of d vertices whose right-most spinal vertex is strong. (A chain tree fragment $\langle C_i, R_i, x_i, x_j \rangle$, with a strong vertex other than x_j, will *not* be one of the fragments enumerated in the process of Definition 3.3.)

We will use the value of ω from Definition 3.6 for k, M, and ε in this proof. Let $Z = (M+1)(k+1)^2\omega + 2$: we claim that this Z satisfies the theorem. Note that if a planted chain tree has $z > Z$ vertices, but the spine is of length less than $(k+1)\omega$, there will be at least one very strong vertex on the chain tree.

Fix $z > Z$ and $\vec{d} = (d_0, \ldots, d_{\omega-1})$ such that $d_i < k+M(k+1)$ for each i, and fix $\Gamma(z, \vec{d})$ as in Definition 3.4. If $\omega' \leq \omega$, let $\Gamma(z, \vec{d}, \omega')$ be the set of chain trees in $\Gamma(z, \vec{d})$ such that precisely ω' of the chain tree fragments contain strong vertices, and let $\Gamma^*(z, \vec{d}) = \bigcup_{\omega' < M} \Gamma(z, \vec{d}, \omega')$. By (2) after Definition 3.6, if a random chain tree is selected from $\Gamma(z, \vec{d})$, then with probability at least $1-\varepsilon$, it will be from $\Gamma(z, \vec{d}) - \Gamma^*(z, \vec{d}) = \bigcup_{\omega' \colon \omega' \geq M} \Gamma(z, \vec{d}, \omega')$, as follows. Selecting a random chain tree from $\Gamma(z, \vec{d})$ is equivalent to choosing the first ω chain tree fragments independently, of orders $d_0, d_1, \ldots, d_{\omega-1}$ respectively, and then independently selecting the rest of the chain tree. By the definition of ω, for each $\vec{d}$, the probability that there are at least M chain fragments with strong vertices is at least $1-\varepsilon$.

As $\vec{d} \neq \vec{d'} \implies \Gamma(z,\vec{d}) \cap \Gamma(z,\vec{d'}) = \varnothing$, we can repeat this argument for each such $\vec{d}$, and then as each chain tree with no very strong vertices falls into precisely one set $\Gamma(z,\vec{d})$, we get the following. The probability that a randomly selected chain tree chosen from $\bigcup_{\vec{d}\colon k<d_i<k+M(k+1)} \Gamma(z,\vec{d})$ has at least M strong vertices is at least $1-\varepsilon$. As the chain trees not in $\bigcup_{\vec{d}\colon k<d_i<k+M(k+1)} \Gamma(z,\vec{d})$ must have very strong vertices, we conclude with probability at least $1-\varepsilon$ that a randomly selected chain tree either has at least M strong vertices, or a very strong vertex. ■

We now apply this to chains of skeletal vertices within trees ... .

3.2 Reductions

From Lemma 2.1, we know that a random unlabelled free tree has many skeletal vertices. We want to use Lemma 2.1 to show that many of these vertices are strong (or that at least one of these vertices is very strong). To do this, we will need to dissect skeletons to get a way of classifying the trees by skeleton. We develop such a classification in this subsection.

We want to modify a notion from graph theory (see [2], p. 80).

Definition 3.7 *Let $\mathfrak{G}$ be a graph. A* subdivision *of a graph $\mathfrak{G}$ is a graph $\mathfrak{H}$ that results from replacing one or more edges of $\mathfrak{G}$ with chains.*

*Let $W \subseteq |G|$ be a set of vertices of $\mathfrak{G}$. The graph $\mathfrak{G}$ is W-*irreducible *if the only vertices of degree 2 are in W.*

Notice that by reversing the subdivision operation, i.e., collapsing chains into edges *without* collapsing any two elements of W into the same vertex, a graph $\mathfrak{G}$ is reduced to a unique W-irreducible graph:

Definition 3.8 *The W-*reduct *of $\mathfrak{G}$ is the the unique W-irreducible graph that can be transformed into $\mathfrak{G}$ (modulo W-preserving isomorphism) by repeated subdivisions. If $W=\varnothing$, call the W-reduct the* reduct *of $\mathfrak{G}$.*

Note that the W-reduct of a tree is a tree.

We already know from Lemma 3.1 that a random chain-tree will a.s. either have many (spinal) strong vertices or a very (spinally) strong vertex. In Lemma 3.2 (below), we prove that (roughly speaking) for any W-irreducible $\mathfrak{C}$, the following is true as $n \to \infty$. If $\mathcal{T}$ is the set of n-trees $\mathfrak{T}$ whose skeleton has W-reduct $\mathfrak{C}$, where W is the set of strong vertices of $\mathfrak{T}$, then a.s. a tree $\mathfrak{T} \in \mathcal{T}$ has either a very strong vertex or a skeleton of long chains of distinct lengths. We will then be ready for Section 3.3, where we will generalize Lemma 3.1 to prove that on large random trees, a.s. there are many strong vertices, or at least one very strong vertex.

Remark 3.1 *Recall from elementary combinatorics that the number of ordered partitions of $[m]$ into c blocks is $\binom{m+c-1}{c-1}$: thus the number of ordered partitions with c blocks, all of sizes greater than z, is $\binom{m-zc-1}{c-1}$. Fix c and z. The number of ordered partitions with c blocks of whose sizes satisfy:*

- *The smallest block has at least z elements, and*
- *The cardinalities of any two blocks differ by at least z*

is $(1-o(1))\binom{m-zc-1}{c-1}$ as $m \to \infty$.

From now on, in a partition, call two blocks *very distinct* if their cardinalities differ by at least z.

Proof. Using inclusion-exclusion, we find that if we take the number N_1 of ordered partitions of $[m]$ into c blocks (each greater than z), and subtract the sum N_2 of the numbers of such partitions in which two blocks have cardinalities within z of each other, we get a lower bound on the number of such partitions into c blocks (each greater than z) of differing cardinalities. We have $N_1 = \binom{m-zc-1}{c-1}$. We want an upper bound for N_2.

First observe that for any l, k, the number of partitions in which the first two blocks B_1 and B_2 are of cardinalities l and $l+k$ is $\binom{m-z(c-2)-(2l+k)-1}{c-3} \leq \binom{m-z(c-2)-1}{c-3}$. Thus the number of partitions in which the first two blocks B_1 and B_2 satisfy $||B_1|-|B_2|| \leq z$ is bounded above by

$$m(2z-1)\binom{m-z(c-2)-1}{c-3}.$$

Since we get the same number of partitions no matter which pair of blocks are set to be have cardinalities within z of each other, we can just multiply $\binom{c}{2}$ with this bound to get

$$N_2 < \binom{c}{2} m(2z-1)\binom{m-z(c-2)-1}{c-3}.$$

Thus

$$
\begin{aligned}
N_1 - N_2 &\geq \binom{m - zc - 1}{c - 1} - m(2z - 1)\binom{c}{2}\binom{m - z(c - 2) - 1}{c - 3} \\
&= (1 - o(1))\binom{m - zc - 1}{c - 1}
\end{aligned}
$$

for z, c fixed and $m \to \infty$. ■

Here is a bit of nomenclature.

Definition 3.9 *Let $\mathfrak{T}$ be a tree. A* $*$-skeletal chain *is a maximal connected set of skeletal vertices of $\mathfrak{T}$, each being of degree 2 in its skeleton $\mathfrak{S}(\mathfrak{T})$, and none of them strong in $\mathfrak{T}$.*

And the $*$-reduct *of $\mathfrak{T}$ is defined as follows. Let $\mathfrak{S}(\mathfrak{T})$ be the skeleton of $\mathfrak{T}$, and let W be set of the strong vertices of $\mathfrak{T}$, and note that $W \subseteq |\mathfrak{S}(\mathfrak{T})|$. The $*$-reduct of $\mathfrak{T}$ is the W-reduct of $\mathfrak{S}(\mathfrak{T})$.*

We should make some elementary observations.

Remark 3.2 *Suppose that an n-tree $\mathfrak{T}$ has l skeletal leaves, w strong vertices, and s skeletal vertices.*

(1) As all skeletal leaves are strong, $l \leq w$.
(2) As all strong vertices are skeletal, $w \leq s$.
(3) By an induction on l, a tree of l leaves has at most $l-1$ vertices of degree greater than 2. Thus, the $*$-reduct of $\mathfrak{S}(\mathfrak{T})$ has at most $w+(l-1)$ vertices, hence at most $2w-1$ vertices.
(4) Thus the $*$-reduct has at least $l-1$ edges (in fact, at least $w-1$ edges) and at most $2w-2$ edges.

Note that the number of $*$-reducts of skeletons of trees of at most w strong vertices depends on w.

And now our technical lemma.

Lemma 3.2 *Fix z, w, and $\varepsilon > 0$.*

The following is true for all sufficiently large s (with respect to z, w, ε).

The following is true for all sufficiently large n (with respect to z, w, ε, s).

With probability greater than $1 - \varepsilon$, the following is true.

A randomly selected n-tree, of w strong vertices, on a skeleton of s vertices, either has a very strong vertex (of strength at least $M(k+1)-1$) or all its $*$-skeletal chains are of very distinct lengths greater than z.

Proof. It suffices to prove that for sufficiently large s, for sufficiently large n, with probability greater than $1-\varepsilon$, the following is true. A randomly selected n-tree, whose strong vertices make up the set W, $|W| = w$, on a skeleton of s vertices, but with no vertex of strength at least $M(k+1)-1$, has all $*$-skeletal chains of very distinct lengths greater than z.

As w is fixed, there are a finite number of $*$-reducts of skeletons of trees of at most w strong vertices (as there are at most $2w-2$ vertices in such a $*$-reduct). Choose one $*$-reduct, call it $\mathfrak{C}$, with l leaves and c edges. For large n, if an n-tree has a skeleton of $*$-reduct $\mathfrak{C}$, but no very strong vertices, then it will have less than $wM(k+1)$ vertices in k-clusters about the w strong vertices. We will prove that for large enough s and large enough n, the (conditional) probability that a random n-tree of skeleton $\mathfrak{S}$ of $*$-reduct $\mathfrak{C}$ (and no very strong vertices) has chains of very distinct lengths (all greater than z) is at least $1-\varepsilon$. Repeating this for each $*$-reduct (of skeletons of free trees of w strong vertices) will give us the lemma.

Fix a number of $*$-skeletal chains c in the skeleton, and recall from Remark 3.2 that $w-1 \leq c \leq 2w-2$. Choose any n-tree $\mathfrak{U}$ whose skeleton $\mathfrak{S}$ has s vertices, c $*$-skeletal chains, and w strong vertices (including all skeletal leaves), and consider:

- Let $\mathcal{S}_-$ be the multiset of rooted trees $\mathfrak{U}_{(v)}$, $v \in |\mathfrak{S}|$, v *not* strong, these being the subtree clusters formed by taking the skeletal vertex v and its non-skeletal dependents.
- Let $\mathcal{S}_+$ be the multiset of rooted trees $\mathfrak{U}_{(v)}$, $v \in |\mathfrak{S}|$, v strong, these being the subtree clusters formed by taking the skeletal vertex v and its non-skeletal dependents. As there are no very strong vertices, there will be at most $wM(k+1)$ vertices in the rooted trees in this multiset.
- Let $\mathfrak{S}'$ be the graph union of the skeleton $\mathfrak{S}$ and the rooted trees of $\mathcal{S}_+$ formed by identifying the strong vertices in $\mathfrak{S}$ and the corresponding roots of subtrees $\mathfrak{U}_{(v)} \in \mathcal{S}_+$; call $\mathfrak{S}'$ the *extended skeleton* of the tree $\mathfrak{U}$.

We now want to deal with the direct dependents of non-strong skeletal vertices. Restrict attention to rooted trees of at most k vertices. For each isomorphism type τ for a rooted tree of at most k vertices, there are, say, t_τ nonstrong skeletal vertices v such that $\mathfrak{U}_{(v)} \in \mathcal{S}_-$ is of type τ: $\sum_\tau t_\tau = s-w$. Suppose that there are h isomorphism types of rooted trees of at most k vertices. For the extended skeleton $\mathfrak{S}'$, suppose that W is the set of strong vertices. Suppose that τ_i has m_i vertices for each i. Let $\mathcal{N}$ be the set of assignments $\tau \to t_\tau$ such that:

$$\sum_\tau t_\tau = s-w \quad \& \quad \sum_{i=1}^{h} m_i t_{\tau_i} = n - \sum_{v \in W} (\text{strength}(v)+1)\,.$$

Then there are at most

$$b(t_{\tau_1}, \ldots, t_{\tau_h}) = \binom{s-w}{t_{\tau_1}, \ldots, t_{\tau_h}}$$

trees $\mathfrak{U}^\star$ of extended skeleton $\mathfrak{S}'$ (including $\mathfrak{U}$ itself) such that for each $i \in [h]$, there are t_{τ_i} subtrees of type τ_i off of non-strong skeletal vertices.

(The 'at most' in the preceding sentence arises from the possibility that an extended skeleton might admit automorphisms that permute $*$-skeletal chains, so that $b(t_{\tau_1}, \ldots, t_{\tau_h})$ would count trees of that extended skeleton more than once. But if all of $\mathfrak{S}$'s chains are of distinct lengths, $\mathfrak{S}$ admits no non-trivial automorphisms, and hence all the automorphisms of trees of extended skeleton $\mathfrak{S}'$ merely permute nonskeletal vertices within subtree clusters.)

Anyway, there are thus at most

$$b_1 = \sum_{(t_{\tau_1}, \ldots, t_{\tau_h}) \in \mathcal{N}} b(t_{\tau_1}, \ldots, t_{\tau_h})$$

free n-trees of extended skeleton $\mathfrak{S}'$. Furthermore, suppose that there are b_2 multisets of $|W|$ rooted trees satisfying: there are between k and $M(k+1)$ vertices in each rooted tree, with $\sum_{v \in W}(\text{strength}(v) + 1)$ vertices altogether in each multiset (and hence no very strong vertices), producing an extended skeleton $\mathfrak{S}'$. Then there are $b_1 b_2$ free n-trees of skeleton $\mathfrak{S}$ in which $W \subseteq |\mathfrak{S}|$ is the set of strong vertices.

Let's return from our examination of the particular n-tree $\mathfrak{U}$, and start counting from $*$-reducts. For each $*$-reduct $\mathfrak{C}$ of c edges, there are at most $\binom{s-w+c-1}{c-1} b_2$ extended skeletons of W-reduct $\mathfrak{C}$ and s skeletal vertices. Thus there are at most $\binom{s-w+c-1}{c-1} b_1 b_2$ n-trees with skeleton $\mathfrak{S}$. (Again, if the chains are not necessarily of distinct lengths, it is possible to count the same tree modulo isomorphism twice, hence the 'at most'). By Remark 3.1, the number of ways to partition $s-w$ into c blocks of very distinct sizes greater than z is bounded below by:

$$\begin{aligned} B(s, w, z) &= (1 - o(1)) \binom{s - w - zc - 1}{c - 1} \\ &\sim \binom{s - w + c - 1}{c - 1} \end{aligned}$$

for c, w, z fixed and $s \to +\infty$. As there are no non-trivial automorphisms of the skeleton if all its chains are of distinct lengths, there are at least $B(s, w, z) b_1 b_2$ n-trees of s skeletal vertices and w strong vertices and $*$-reduct $\mathfrak{C}$, with the given multiset of subtrees $\mathfrak{U}_{(v)} \in \mathcal{S}_-$.

Choose n large enough so that by Lemma 2.1, with probability $> 1-\varepsilon/2$, we have a number s of skeletal vertices so large that:

$$B(s,w,z)/\binom{s-w+c-1}{c-1} > 1-\frac{\varepsilon}{2},$$

and the proportion of n-trees with:

- the $*$-reduct $\mathfrak{C}$,
- and all skeletal chains of different lengths greater than z

is at least $1-\varepsilon$.

Repeat for each $*$-reduct $\mathfrak{C}$ of w strong vertices, and $1-\varepsilon$ of the n-trees $\mathfrak{U}$ of $*$-reduct $\mathfrak{C}$ have $*$-skeletal chains of very distinct lengths greater than z, and we are done. ■

3.3 Strong vertices

We are now ready to prove that on a random n-tree, almost surely either there are lots of strong vertices, or there is at least one very strong vertex. This will allow us to prove the Main Theorem 2.1. We will need two notions.

Definition 3.10 *Let $\mathfrak{U}$ be a free tree of skeleton $\mathfrak{S} = \mathfrak{S}(\mathfrak{U})$. If $a, b \in |\mathfrak{U}|$, let $CONN(x, a; b)$ mean that the path from x to a does not go through b. If $a, b \in |\mathfrak{U}|$, let*

$$\mathfrak{U}[a,b] = \{x \in |\mathfrak{U}|\colon\ CONN(x,a;b) \wedge CONN(x,b;a)\}$$

be the connecting path in between, and dependents of the connecting path.

We could regard the (planted!) chain tree $\mathfrak{U}[a, b]$ as a sort of tree equivalent of an interval from a to b.

Definition 3.11 *If $\mathfrak{D} = \langle D, \mathrm{Edge}, a', b'\rangle$ is a planted chain tree, let* SUB$(\mathfrak{U}$, $\mathfrak{U}[a,b]/\mathfrak{D})$ *be the result of taking $\mathfrak{U}$ and replacing the planted chain tree $\mathfrak{U}[a, b]$ with the planted chain tree $\mathfrak{D}$, by identifying the two roots a' and b' of $\mathfrak{D}$ with a and b, respectively.*

We now generalize Lemma 3.2 to trees.

Lemma 3.3 *Fix M, and $\varepsilon > 0$. For sufficiently large n, the probability that a randomly selected unlabelled tree has less than M strong vertices, none very strong, is less than ε.*

Proof. For large enough n, we will do the following. We will construct a collection of sets Δ of n-trees, and prove that:

(1) These sets Δ are pairwise disjoint.
(2) The probability that a random n-tree of less than M strong vertices is not in one of these sets of n-trees is less than $\varepsilon/2$.
(3) In each of these sets Δ of n-trees, the proportion with less than than M strong vertices, and with no very strong vertices, is less than $\varepsilon/2$.

Thus the proportion of n-trees of at least M strong vertices (or at least one very strong one), either in one of the sets Δ or not, is greater than $1-\varepsilon$, giving us the lemma.

We will need a pair of technical notions about chain trees for the construction.

First, suppose we are given ω satisfying Inequality 1 in Definition 3.6 (and hence Inequality 2 afterwards), and let z be an integer such that $z \geq (k+1)\omega M + 2$, and let $\vec{d} = d_1, \ldots, d_\omega$ be such that $k < d_i < M(k+1)$ for each $i \in [\omega]$.

We should have a definition for the next technical notion.

Definition 3.12 *The* skeletal weight *of a skeletal vertex is the number of its skeletal dependents.*

Second, given a set Γ of z-vertex planted chain trees, call an n-tree Γ-*good* if:

- its $*$-skeletal chains are of very distinct lengths, and
- its shortest $*$-skeletal chain is of length greater than z, and
- choose the $*$-skeletal chain of minimum length in the skeleton, and letting a be the endpoint of lesser skeletal weight, there is another vertex b on that same shortest skeletal chain such that $\mathfrak{U}[a,b]$ is isomorphic to a chain tree in Γ (after the endpoints have pruned to be of degree 1).

Notice that no automorphism of a Γ-good n-tree permutes its $*$-skeletal chains, so that modulo automorphisms, a and b are fixed. In the rest of this proof, all (unlabelled) n-trees $\mathfrak{U}$ have a and b defined as above.

Here is the construction. Our goal is to compute n, construct the sets Δ of n-trees, and then prove that the sets Δ satisfy (1), (2) and (3) above. Let z be the number of vertices for random chain-trees to satisfy the requirements of Lemma 3.1 for M strong vertices (or one very strong vertex) and error $\varepsilon/2$, and let n be the number of vertices for random n-trees to have appropriate chain lengths with error $\varepsilon/2$ from Lemma 3.2. Then, recalling Definitions 3.3

and 3.4:

(A) for each tuple $\vec{d} = d_1, \ldots, d_\omega$, where $k < d_i < k + M(k+1)$ for each i, where $\sum_{i=1}^{\omega} d_i < z$, and

(B) for each $\Gamma(z, \vec{d})$-good n-tree $\mathfrak{U}$ of less than M strong vertices, and whose $*$-skeletal chains are of very distinct lengths greater than z, and letting $\mathfrak{U}[a, b]$ be $\mathfrak{U}$'s shortest chain, witnessing $\mathfrak{U}$ being $\Gamma(z, \vec{d})$-good,

we recall Definition 3.11 and define the sets $\Delta = \Delta(\mathfrak{U}, \vec{d})$ of n-trees to be:

$$\Delta(\mathfrak{U}, \vec{d}) = \{\mathrm{SUB}(\mathfrak{U}, \mathfrak{U}[a, b]/\mathfrak{D})\colon \mathfrak{D} \in \Gamma(z, \vec{d})\}.$$

We claim that these sets $\Delta(\mathfrak{U}, \vec{d})$ satisfy (1), (2), and (3) above. First (1).

Claim 3.1 *These sets Δ are pairwise disjoint.*

Proof. First, we claim that if $\mathfrak{B} \in \Delta(\mathfrak{U}_1, \vec{d_1}) \cap \Delta(\mathfrak{U}_2, \vec{d_2})$, then $\mathfrak{U}_1$ and $\mathfrak{U}_2$ agree on all but one end of their shortest skeletal chains. By (A), the lengths of the skeletal chains of $\mathfrak{U}_1$ and $\mathfrak{U}_2$ differ by more than $\sum_{i=0}^{\omega-1} d_i$. Yet as changes in (at most) that many edges can transform $\mathfrak{U}_1$ to $\mathfrak{U}_2$, all those changes must have been done on the shortest skeletal chain of each. Thus if $\vec{d_1} \neq \vec{d_2}$, then for any n-trees $\mathfrak{U}_1 \in \Gamma[z, \vec{d_1}]$, $\mathfrak{U}_2 \in \Gamma[z, \vec{d_2}]$, $\Delta(\mathfrak{U}_1, \vec{d_1}) \cap \Delta(\mathfrak{U}_2, \vec{d_2}) = \varnothing$.

So we now ask about intersections of the form $\Delta(\mathfrak{U}_1, \vec{d}) \cap \Delta(\mathfrak{U}_2, \vec{d})$. If $\mathfrak{U}_1$ and $\mathfrak{U}_2$ are n-trees of $M' < M$ strong vertices in which $\mathfrak{U}_1[a, b], \mathfrak{U}_2[a, b] \in \Gamma(z, \vec{d})$ for some z and $\vec{d}$, then then we claim that either $\Delta(\mathfrak{U}_1, \vec{d}) = \Delta(\mathfrak{U}_2, \vec{d})$ or $\Delta(\mathfrak{U}_1, \vec{d}) \cap \Delta(\mathfrak{U}_2, \vec{d}) = \varnothing$.

- First, if $\mathfrak{U}_1$ and $\mathfrak{U}_2$ differ elsewhere than on (the end of the chain starting at a of) $\mathfrak{U}_1[a, b]$ and $\mathfrak{U}_2[a, b]$, then there can be no isomorphism sending any $\mathfrak{U}_1' \in \Delta(\mathfrak{U}_1, \vec{d})$ onto any $\mathfrak{U}_2' \in \Delta(\mathfrak{U}_2, \vec{d})$: such an isomorphism cannot send $\mathfrak{U}_1'[a, b]$ to any chain fragment except $\mathfrak{U}_2'[a, b]$, and as the partial isomorphism $\mathfrak{U}_1'[a, b] \to \mathfrak{U}_2'[a, b]$ cannot be extended to an isomorphism $\mathfrak{U}_1' \to \mathfrak{U}_2'$, we get

$$\left[\mathfrak{U}_1' \in \Delta(\mathfrak{U}_1, \vec{d}) \quad \& \quad \mathfrak{U}_2' \in \Delta(\mathfrak{U}_2, \vec{d})\right] \Longrightarrow \mathfrak{U}_1' \neq \mathfrak{U}_2'.$$

  So we must have $\Delta(\mathfrak{U}_1, \vec{d}) \cap \Delta(\mathfrak{U}_2, \vec{d}) = \varnothing$.
- Second, if $\mathfrak{U}_1$ and $\mathfrak{U}_2$ differ only on (the end of the chain starting at a of) $\mathfrak{U}_1[a, b]$ and $\mathfrak{U}_2[a, b]$, then for any $\mathfrak{U}$ such that $\mathfrak{U}[a, b] \in \Gamma(z, \vec{d})$, $\mathfrak{U}$ is in (say) $\Delta(\mathfrak{U}_1, \vec{d})$ iff $\mathfrak{U} - (\mathfrak{U}[a, b] - \{a, b\})$ is (partially) isomorphic to $\mathfrak{U}_1 - (\mathfrak{U}_1[a, b] - \{a, b\})$. But $\mathfrak{A}_1 - (\mathfrak{A}_1[a, b] - \{a, b\})$ is (partially) isomorphic to to $\mathfrak{U}_2 - (\mathfrak{U}_2[a, b] - \{a, b\})$, so $\mathfrak{U} \in \Delta(\mathfrak{U}_2, \vec{d})$. Hence, $\Delta(\mathfrak{U}_1, \vec{d}) \subseteq \Delta(\mathfrak{U}_2, \vec{d})$. Similarly, $\Delta(\mathfrak{U}_2, \vec{d}) \subseteq \Delta(\mathfrak{U}_1, \vec{d})$.

■Claim 3.1

We now prove (2).

Claim 3.2 *The probability that a random n-tree $\mathfrak{U}$ of less than M strong vertices, none of them very strong, is not in one of these sets Δ of n-trees, is at most $\varepsilon/2$.*

Proof of Claim. The only way that a random n-tree $\mathfrak{U}$ of at most M strong vertices (none of them very strong) could fail to be in one of these sets Δ is if it failed to meet criterion (B) above requiring $\mathfrak{S}[\mathfrak{U}]$ to have $*$-skeletal chains of lengths greater than (and differing from each other by at least) z. But by using Lemma 3.2, we chose n so large that this failure occurs with probability less than $\varepsilon/2$. ■Claim 3.2

Now we prove (3):

Claim 3.3 *In each of these sets Δ of n-trees, proportionately less than $\varepsilon/2$ of them have less than M strong vertices, none very strong.*

Proof of Claim. By Lemma 3.1 as applied to z from (A) above (within error $\varepsilon/2$), more than $1 - \varepsilon/2$ of the relevant chain trees have at least M strong vertices by themselves (or a very strong vertex), thus forcing the trees they are part of to have at least M strong vertices (or a very strong vertex). ■Claim 3.3

Having proven (1), (2), and (3), we are done. ■

So now we see that for any M, and any $\varepsilon > 0$, there exists N such that if $n > N$, then with the probability $1 - \varepsilon$, either a random, free unlabelled n-tree has more than M k-strong vertices, or it has a vertex of strength at least $M(k+1)$.

At last, the Main Theorem:

Theorem 2.1 *Fix a rooted tree $\mathfrak{T}$ and fix $\varepsilon > 0$. There exists N such that if $n > N$, the probability that a random n-tree admits a subtree isomorphic to $\mathfrak{T}$ is at least $1 - \varepsilon$.*

Proof. We first claim that there exists a fixed γ, $0 < \gamma < 1$ such that for any $K > k$, of the K-vertex rooted trees with only one strong vertex (that being the root), the proportion of these trees with no copy of $\mathfrak{T}$ among its subtrees is at most γ.

To see this, note that if c_l is the number of isomorphism types of rooted trees of l vertices, and $e_k = \sum_{l=1}^{k} c_l - 1$, then by Remark 2.1, when K is large, there are $(\text{constant} + o(1))K^{e_k}$ (isomorphism types of) rooted trees (with only one strong vertex), at most $(\text{constant}' + o(1))K^{e_k - 1}$ of which do not have $\mathfrak{T}$ as a subtree. (More precisely, only $(\text{constant}' + o(1))K^{e_k-1}$ of these rooted K-trees do not admit vertices adjacent to the root that are also roots of subtrees isomorphic to $\mathfrak{T}$.) For each $K > k$, let γ_K be the proportion of these rooted K-trees — whose only strong vertex is its root — not admitting a copy of $\mathfrak{T}$. As $\lim_{K\to+\infty}[(\text{constant}' + o(1))K^{e_k-1}]/[(\text{constant} + o(1))K^{e_k}] = 0$, $\lim_{K\to+\infty} \gamma_K = 0$, so we can set $\gamma = \max_{K>k} \gamma_K$, and notice that $\gamma < 1$.

Fix M so large that:

- first, $\gamma^M < \varepsilon/3$, and
- second, $M' \geq M(k+1) \implies \gamma_{M'} < \varepsilon/3$.

Then choose n so large that by Lemma 3.3, the probability that there are at most M strong vertices, none very strong, on a free n-tree, is less than $\varepsilon/3$.

If $\mathfrak{U}$ is an n-tree chosen at random, the probability that it has less than M strong vertices, none very strong, is less than $\varepsilon/3$. If $\mathfrak{U}$ has at least M strong vertices, the probability that there is no copy of $\mathfrak{T}$ among their direct dependents is less than $\varepsilon/3$. If $\mathfrak{U}$ has a very strong vertex, i.e., a vertex being the root of a subtree of at least $M(k+1)$ vertices, the probability that there is no copy of $\mathfrak{T}$ among its dependents is less than $\varepsilon/3$. Putting this together: the probability that $\mathfrak{U}$ has no copy of $\mathfrak{T}$ is less than ε. ■

In the above theorem, if $\mathfrak{T}$ is the tree $\mathfrak{T}_r$ of Theorem 2.2, then for each r, the MSO formulas of depth $\leq r$ a.s. hold on free n-trees, and repeating for each r, we have proven Corollary 2.1.

4 Excelsior

Section 3 is a good example of an ugly proof. It is rather more elementary than the proof in [22], but not shorter, and in comparison is rather a contraption, even if an illuminating one. It is not clear how to use these methods to get the result of Woods that on unlabelled rooted trees, every MSO sentence has an asymptotic probability. However, we suspect that a more careful dissection of random unlabelled trees can give us that result.

(On the other hand, there is more work to be done in the generating function approach, e.g., finding a common generalization of the results of this paper and of [3].)

In addition, the dissection itself suggests a few questions, like Conjecture 2.1. We should also not forget that there are other models of random trees as well, and such dissection techniques might be useful in analyzing them (see [16] for a discussion of random trees).

Finally, remembering the automata-like behavior investigated in [4], [12], etc., we remark that in [17], it is proven that on labelled, free random trees, with a linear order, all order-invariant MSO queries are eventually periodic in the following sense. If θ is an order-invariant MSO sentence in the language of graphs with order, there exists a vector $\langle p_0, p_1, \ldots, p_{t-1}\rangle$ such that

$$\lim_{n\to\infty} \left|\Pr\left[\mathfrak{T} \models \theta \middle| \|\mathfrak{T}\| = n\right] - p_{n \bmod t}\right| = 0.$$

In that paper, we conjectured that this was true even for rooted labelled random trees, and in addition:

Conjecture 4.1 *On unlabelled, rooted random trees, with a linear order, all order-invariant MSO queries are eventually periodic.*

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