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Threshold Functions for Random Graphs on a Line Segment

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We look at a model of random graphs suggested by Gilbert: given an integer n and $\delta > 0$, scatter n vertices independently and uniformly on a metric space, and then add edges connecting pairs of vertices of distance less than δ apart.

We consider the asymptotics when the metric space is the interval $[0, 1]$, and $\delta = \delta(n)$ is a function of n, for $n \to \infty$. We prove that every upwards closed property of (ordered) graphs has at least a weak threshold in this model on this metric space. (But we do find a metric space on which some upwards closed properties do not even have weak thresholds in this model.) We also prove that every upwards closed property with a threshold much above Connectivity's threshold has a strong threshold. (But we also find a sequence of upwards closed properties with lower thresholds that are strictly weak.)

1. Introduction

We investigate strong and weak thresholds on a one-dimensional version of one of the oldest models of random graphs.

E. Gilbert's "random plane networks" ([11]) were not explored much (as such) until the early 1990s, although there was a lot of closely related work on "coverage processes" (see, e.g., [18]), "poisson point processes" (see, e.g., [7]), and even "close pairs" (see, e.g., [24]), when work on "disk graphs" (see, e.g., [6], [15]), "interval graphs" (see, e.g., [13]), "sphere-of-influence graphs" (see, e.g., [8], [20], [28]) and "random graphs on Euclidean space" (see, e.g., [27], [22]) started to appear. In the 1990s, E. Godehardt and B. Harris ([13]) started carrying out the Erdős-Rényi programme for Gilbert's Random Plane Networks, motivated by problems arising in cluster analysis [12, 14]; more logical considerations motivated [22], or for the more general situation, [29]. M. Penrose ([23]) has written a book on these networks.

In this article, we will look at weak and strong thresholds on random networks on line segments, and we will find:

- If the random network is on a line segment, then all upwards closed properties have at least weak thresholds.
- There is a metric space on which the upwards closed property "there are no isolated vertices" does not have even a weak threshold.
- If the random network is on a line segment, and if a given upwards closed property has a large enough (edge probability much bigger than $(\ln n)/n$) threshold, the threshold will be strong. (Compare this to the Erdős-Rényi Model, where Friedgut ([10]) shows that "big enough" is something like $1/\ln n$ (and conjectured to be as low as $1/(\ln n)^2$ or so), and where, by [4], for each rational $r > 0$, there exist strictly weak thresholds as low as n^{-r}.)
- However, there is a hierarchy of properties with low and strictly weak thresholds, similar to the evolution of very sparse graphs in the Erdős-Rényi Model.

Here is an outline of the paper. In Section 2, we describe the nomenclature and introduce the model of random graphs we are dealing with. In Section 3, we give precise characterizations of the models of random networks that we will look at. In Section 4, we prove that random networks over a line segment admit at least weak thresholds for all upwards closed properties, but that there is a space consisting of many line segments over which random networks do not admit a weak threshold for the property "there are no isolated vertices." In Section 5, we prove that for each k, the property "there is a k-vertex component" does not have a strong threshold; we also prove that any property whose threshold is much greater than the threshold of Connectivity has a strong threshold (in fact, if $\delta(n)$ is much higher than the threshold for Connectivity, then almost surely a random network of cutoff $\delta(n)$ is a subgraph of an (independently selected) random network of cutoff $(1+\varepsilon)\delta(n)$.

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2. Preliminaries

We presume familiarity with the basic notions of probability, graph theory, and thresholds. Our primary probability reference is [25].

Here are a few definitions and facts that we will need. For example, using [25, IV.5.1], one can show that:

Remark 2.1. *If γ is gamma distributed with parameter λ and t degrees of freedom, then for any ε, $0 < \varepsilon < 1/2$, $\mathbb{P}\left[|\gamma - t/\lambda| > \varepsilon t/\lambda\right] < 2e^{-\varepsilon^2 t/4}$.*

Denote the *order* of a graph $\mathfrak{G} = \langle V, E\rangle$, number of vertices in V, by $\|\mathfrak{G}\|$. If $<$ linearly orders V, call $\langle V, E, <\rangle$ an *ordered graph*; sometimes as a convenience, we will produce a linear ordering of a graph (and then put the ordering away when we are finished with it).

Definition 2.1. *If $\mathfrak{G}$ and $\mathfrak{H}$ are graphs such that $\mathfrak{G}$ and $\mathfrak{H}$ share the same vertices, and the edge set of $\mathfrak{G}$ is a subset of the edge set of $\mathfrak{H}$, write "$\mathfrak{G} \leq \mathfrak{H}$." Letting "$\cong$" mean isomorphism, if $\mathfrak{G}_1 \cong \mathfrak{G}_2 \leq \mathfrak{H}$, write $\mathfrak{G}_1 \leq \mathfrak{H}$. Write "$\mathfrak{G} < \mathfrak{H}$" if $\mathfrak{G} \leq \mathfrak{H}$ and $\mathfrak{G} \ncong \mathfrak{H}$.*

Let Θ be a property of graphs or of ordered graphs. Call Θ upwards-closed *if, for any graphs $\mathfrak{G}$ and $\mathfrak{H}$, $\mathfrak{G} \leq \mathfrak{H}$ implies that $\mathfrak{G} \in \Theta \Longrightarrow \mathfrak{H} \in \Theta$.*

The graphs we are interested in are defined on some kind of metric space. In this article, the metric space will be an interval of real numbers (or a union of intervals), with the one-dimensional Euclidean metric. The order will be induced by the standard ordering of the reals. We will need a space of possible tuples of vertices.

Definition 2.2. *For each n, let $\Omega_n = \{(x_1, \dots, x_n) \in \mathbb{R}\colon 0 < x_1 < \cdots < x_n\}$.*

Then for a given probability measure $\mathbb{P}_n$ on Ω_n with a probability density function f, we will select a tuple $\vec{x} = (x_1, \dots, x_n)$ of n points on $\mathbb{R}$ to be the vertices of the graph. The probability that the chosen tuple is an element of $A \subseteq \Omega_n$ is thus $\mathbb{P}_n[A] = \int \cdots \int_A f(\vec{x})\, d\vec{x}$.

Definition 2.3. *Fix a positive integer n, a positive real δ, and a probability measure $\mathbb{P}_n$ on Ω_n. Select $\vec{v} = (v_1, \dots, v_n) \in \Omega_n$ according to the measure $\mathbb{P}_n$, and let $V_{\vec{v}} = \{v_1, \dots, v_n\}$ and $E_{\vec{v},\delta} = \{\{v_i, v_j\} \subseteq V_{\vec{v}}\colon i, j \in [n]\ \&\ 0 < |v_i - v_j| < \delta\}$. Then the* random network *$G^{\mathbb{P}_n}_{n,\delta}$ is the network $\langle V_{\vec{v}}, E_{\vec{v},\delta}, <_{\vec{v}}\rangle$, where $<_{\vec{v}}$ is the ordering induced by the ordering of $\mathbb{R}$, and δ is its* cutoff.

Thus $G^{\mathbb{P}_n}_{n,\delta}$ is a graph-valued r.v., and

$$\mathbb{P}[G^{\mathbb{P}_n}_{n,\delta} \cong \mathfrak{G}] = \mathbb{P}_n\left[\{\vec{v} \in \Omega_n\colon \langle V_{\vec{v}}, E_{\vec{v},\delta}, <_{\vec{v}}\rangle \cong \mathfrak{G}\}\right]$$

and so

$$\mathbb{P}[G^{\mathbb{P}_n}_{n,\delta} \in \Theta] = \mathbb{P}_n\left[\bigcup_{\mathfrak{G}\in\Theta} \{\vec{v} \in \Omega_n\colon \langle V_{\vec{v}}, E_{\vec{v},\delta}, <_{\vec{v}}\rangle \cong \mathfrak{G}\}\right].$$

We need another 'obvious' fact (recalling the notation of Definitions 2.3 and 2.1): if $\delta < \delta'$, then $\langle V_{\vec{v}}, E_{\vec{v},\delta}, <_{\vec{v}}\rangle \leq \langle V_{\vec{v}}, E_{\vec{v},\delta'}, <_{\vec{v}}\rangle$. Thus if Θ is an upwards closed property and $\delta < \delta'$, then $\langle V_{\vec{v}}, E_{\vec{v},\delta}, <_{\vec{v}}\rangle \in \Theta \Longrightarrow \langle V_{\vec{v}}, E_{\vec{v},\delta'}, <_{\vec{v}}\rangle \in \Theta$. Thus:

Lemma 2.1. *If Θ is an upwards closed property, and if $0 < \delta < \delta'$, then for any n, $\mathbb{P}[G^{\mathbb{P}_n}_{n,\delta} \in \Theta] \leq \mathbb{P}[G^{\mathbb{P}_n}_{n,\delta'} \in \Theta]$.*

Before we turn to thresholds, we should note a difference between the random network models, in which the increasing parameter is the cutoff δ, and the Erdős-Rényi Model, in which the increasing parameter is the number of edges. In the random network models, the number of edges closely tracks the cutoff as follows: if we scatter n vertices uniformly and independently on the interval $[0, 1]$ (this is what we will call *the Model RN*), and if the cutoff is δ, then the number of edges is very close to $(\delta - \delta^2)n^2$.

We now turn to thresholds.

Definition 2.4. *Fix a sequence* $\mathcal{S} = (\mathbb{P}_n \colon n \in \mathbb{Z}^+)$*, where, for each* n*, the measure* $\mathbb{P}_n$ *is a probability distribution on* Ω_n*.*

For each function $\delta \colon \mathbb{Z}^+ \to \mathbb{R}^+$*, and each* n*, let* $G^{\mathbb{P}_n}_{n,\delta(n)}$ *be the random variable ranging over the* n*-vertex networks from vectors of distribution* $\mathbb{P}_n$ *and cutoff* $\delta(n)$*.*

Fix an upwards closed property Θ*.*

A function $\delta_\Theta \colon \mathbb{Z}^+ \to \mathbb{R}^+$ *is a* weak threshold function *for* Θ *on* $\mathcal{S}$ *if the following is true for every function* $\delta \colon \mathbb{Z}^+ \to \mathbb{R}^+$*:*

- *If* $\delta(n) \ll \delta_\Theta(n)$ *as* $n \to \infty$*, then* $\mathbb{P}[G^{\mathbb{P}_n}_{n,\delta(n)} \in \Theta] = o(1)$*.*
- *If* $\delta(n) \gg \delta_\Theta(n)$ *as* $n \to \infty$*, then* $\mathbb{P}[G^{\mathbb{P}_n}_{n,\delta(n)} \in \Theta] = 1 - o(1)$*.*

A function $\delta_\Theta \colon \mathbb{Z}^+ \to \mathbb{R}^+$ *is a* strong threshold function *for* Θ *on* $\mathcal{S}$ *if the following is true for every fixed* $\varepsilon > 0$:

- *First,* $\mathbb{P}[G^{\mathbb{P}_n}_{n,(1-\varepsilon)\delta(n)} \in \Theta] = o(1)$*.*
- *Second,* $\mathbb{P}[G^{\mathbb{P}_n}_{n,(1+\varepsilon)\delta(n)} \in \Theta] = 1 - o(1)$*.*

For example, by [22], $\delta(n) = (\ln n)/n$ is a strong threshold for Connectivity in Gilbert's random networks on $[0, 1]$ (Model RN (Model 3.1) below). We will make use of this fact below.

3. The Models

Here is the primary model that we are interested in. We define it with additional generality that we will find useful later.

Model 3.1 (Model RN). *Fix* $T, \delta > 0$*. If* ν *is the Lebesgue measure, let* $\mathbb{P}[A] = \nu[A]/T$ *for each measurable* $A \subseteq [0, T]$*. Independently select* $\rho_1, \ldots, \rho_n \in [0, T]$ *according to the (uniform) probability measure* $\mathbb{P}$*. Let* $G_{n,\delta,T}$ *be the network of vertex set* $\{\rho_1, \ldots, \rho_n\}$*, cutoff* δ*, and with* $<$ *being the usual ordering on the reals. Let* $G_{n,\delta} = G_{n,\delta,1}$*.*

Given $\rho_1, \ldots, \rho_n$*, let* $\vec{\xi} = (\xi_1, \ldots, \xi_n)$ *be the tuple of the numbers* $\rho_1, \ldots, \rho_n$ *listed in increasing order. Then if* $A \subseteq \Omega_n$*,* $\mathbb{P}_n[A]$ *is the probability that* $\vec{\xi} \in A$*.*

Note that $G_{n,\delta}$ has the same distribution as $G^{\mathbb{P}_n}_{n,\delta(n)}$.

For any permutation π: $[n] \to [n]$, $\Omega_{n,\pi,T} = \{\vec{x} \in [0,T]^n\colon x_{\pi(1)} < x_{\pi(2)} < \cdots < x_{\pi(n)}\}$, and the spaces $\Omega_{n,\pi,T}$ partition all but a null set of $[0,T]^n$ into $n!$ pieces of equal measure $T^n/n!$. Thus:

Proposition 3.1. *Fix $T > 0$ and a positive integer n. In* Model RN*, the probability density function for $\mathbb{P}_n$ is $f(x_1,\ldots,x_n) = n!/T^n$ for all $x_1,\ldots,x_n$ such that $0 \le x_1 < \cdots < x_n \le T$.*

We will need a gadget for Model RN. Given a measurable set $O \subseteq \Omega_n$, and given $c > 0$, let $O_c = \{(cx_1,\ldots,cx_n)\colon (x_1,\ldots,x_n) \in O\}$. The following is an exercise in changing variables, which we leave to the reader.

Lemma 3.1. *For any $n \in \mathbb{Z}^+$, $\delta, T, c > 0$ and any property Θ, $\mathbb{P}[G_{n,c\delta,cT} \in \Theta] = \mathbb{P}[G_{n,\delta,T} \in \Theta]$.*

3.1. Another Model

The Model RN is often difficult to deal with directly, so we will construct a similar if less natural model that approximates Model RN. This model will satisfy the threshold results we are after, and it is a close enough approximation of Model RN so that Model RN must also satisfy these threshold results.

Model 3.2 (Model RN*). *Fix a positive integer n, and a real number $\delta > 0$. Set $\zeta_0 = 0$. Let $\zeta_1,\ldots,\zeta_n$ be an independent set of exponential random variables of parameter n, and for each $t \in [n]$, let*

$$\xi_t = \sum_{l=1}^{t} \zeta_l.$$

Choose a random graph $G^\star_{n,\delta}$ as follows. Independently select real values for $\zeta_1,\ldots,\zeta_n$, and then compute $\xi_1,\ldots,\xi_n$. Let $V^\star_n = \{\xi_1,\ldots,\xi_n\}$ (and $\vec{\xi}^\star_n = (\xi_1,\ldots,\xi_n)$). Then $G^\star_{n,\delta}$ is the network of vertex set $V^\star_n$ and cutoff δ, and thus of edge set $\{\{\xi_i,\xi_j\}\colon i \ne j \;\&\; |\xi_i - \xi_j| < \delta\}$.

We should make three observations that will become important later.

Lemma 3.2. *The p.d.f. of the random variable* $\vec{\xi}_n^\star = (\xi_1, \ldots, \xi_n)$ *from Model RN** *above is* $f^\star(x_1, \ldots, x_n) = n^n e^{-nx_n}$ *if* $(x_1, \ldots, x_n) \in \Omega_n$.

Remark 3.1. *Notice that for each* $t \in [n]$, ξ_t *is a gamma variable of parameter* n *and degree* t.

We will obtain threshold results for Model RN*, and translate these to threshold results for Model RN. We will need an alternate version of Model RN*.

Proposition 3.2. *Suppose that* $\xi_1, \ldots, \xi_n$ *are selected are selected as follows:*

- *Let* γ_{n+1} *be gamma distributed with parameter* n *and* $n+1$ *degrees of freedom.*
- *Then independently and uniformly select* $\rho_1, \ldots, \rho_n \in [0, \gamma_{n+1}]$, *and let* $\xi_1, \ldots, \xi_n$ *be* $\rho_1, \ldots, \rho_n$ *in increasing order.*
- *Let* $V = \{\xi_1, \ldots, \xi_n\}$ *and* $E = \{\{\xi_j, \xi_k\} \colon |\xi_j - \xi_k| < \delta\}$.

Then the p.d.f. of the r.v.s $\xi_1, \ldots, \xi_n$ *is the p.d.f. of* Model RN*.

We leave this computation of p.d.f.s to the reader.

3.2. Comparing Models RN and RN*

We want to use Model RN* to approximate Model RN. Observant readers should notice a similarity between this result and [3, VII.3, Thm. 8].

Here is the basic lemma.

Lemma 3.3. *Fix* n, $\delta > 0$, *and* $T > 0$. *Let* $G_{n,\delta,T}$ *be the* Model RN *r.v. of these parameters, and recall that* $G_{n,\delta} = G_{n,\delta,1}$. *Choose* $\varepsilon, \alpha > 0$, $\varepsilon < 1/2$, *so that if* $\gamma_{Tn,n+1}$ *is gamma-distributed with parameter* Tn *and* $n+1$ *degrees of freedom, then*

$$\mathbb{P}\left[T - \frac{\varepsilon}{2} < \gamma_{Tn,n+1} < T + \frac{\varepsilon}{2}\right] > 1 - \alpha.$$

Then:

1 If $\mathbb{P}[G^\star_{n,\delta} \in \Theta] < \alpha$, *then* $\mathbb{P}[G_{n,(1-\varepsilon)\delta} \in \Theta] < \alpha/(1-\alpha)$.

2 *If* $\mathbb{P}[G_{n,\delta} \in \Theta] < \alpha$*, then* $\mathbb{P}[G^\star_{n,(1-\varepsilon)\delta} \in \Theta] < 2\alpha$*.*

3 *If* $\mathbb{P}[G^\star_{n,\delta} \in \Theta] > 1-\alpha$*, then* $\mathbb{P}[G_{n,(1+\varepsilon)\delta} \in \Theta] > (1-2\alpha)/(1-\alpha)$*.*

4 *If* $\mathbb{P}[G_{n,\delta} \in \Theta] > 1-\alpha$*, then* $\mathbb{P}[G^\star_{n,(1+\varepsilon)\delta} \in \Theta] > 1-2\alpha$*.*

Proof. We will prove (1) and (2); the proofs of (3) and (4) are similar. For simplicity, suppose that $T = 1$.

We start with the proof of (1). Let $g_{n,n+1}$ be the probability distribution function of $\gamma_{n,n+1}$. As we have $\mathbb{P}[G^\star_{n,\delta} \in \Theta] < \alpha$, by Proposition 3.2, it follows that

$$\int_{1-\varepsilon/2}^{1+\varepsilon/2} \mathbb{P}[G_{n,\delta,t} \in \Theta] g_{n,n+1}(t)\, dt < \mathbb{P}[G^\star_{n,\delta} \in \Theta] < \alpha.$$

As $\int_{1-\varepsilon/2}^{1+\varepsilon/2} g_{n,n+1}(t)\, dt > 1-\alpha$, and as $\mathbb{P}[G_{n,\delta,t} \in \Theta]$ decreases as t increases (by Lemmas 2.1 and 3.1), $\mathbb{P}[G_{n,\delta,1+\varepsilon/2} \in \Theta] < \alpha/(1-\alpha)$, hence $\mathbb{P}[G_{n,(1-\varepsilon)\delta} \in \Theta] = \mathbb{P}[G_{n,\delta,(1-\varepsilon)^{-1}} \in \Theta] \leq \mathbb{P}[G_{n,\delta,1+\varepsilon/2} \in \Theta] < \alpha/(1-\alpha)$.

Now for the proof of (2). If $\mathbb{P}[G_{n,\delta} \in \Theta] < \alpha$, then by Lemma 3.1, $\mathbb{P}[G_{n,(1-\varepsilon/2)\delta,1-\varepsilon/2} \in \Theta] < \alpha$, so, by Lemma 2.1, $\mathbb{P}[G_{n,(1-\varepsilon)\delta,1-\varepsilon/2} \in \Theta] < \alpha$. Compute:

$$\begin{aligned}
\mathbb{P}[G^\star_{n,(1-\varepsilon)\delta} \in \Theta] &= \int_0^\infty \mathbb{P}[G_{n,(1-\varepsilon)\delta,t} \in \Theta] g_{n,n+1}(t)\, dt \\
&< \alpha + \int_{1-\varepsilon/2}^{1+\varepsilon/2} \mathbb{P}[G_{n,(1-\varepsilon)\delta,t} \in \Theta] g_{n,n+1}(t)\, dt \\
&< \alpha + (1-\alpha)\mathbb{P}[G_{n,(1-\varepsilon)\delta,1-\varepsilon/2} \in \Theta],
\end{aligned}$$

the last inequality also by Lemma 2.1. But $\mathbb{P}[G_{n,(1-\varepsilon)\delta,1-\varepsilon/2} \in \Theta] = \mathbb{P}[G_{n,\beta\delta} \in \Theta]$, where $\beta = (1-\varepsilon)/(1-\varepsilon/2) < 1$, so $\mathbb{P}[G_{n,\beta\delta} \in \Theta] \leq \mathbb{P}[G_{n,\delta} \in \Theta] < \alpha$, and thus $\mathbb{P}[G_{n,(1-\varepsilon)\delta} \in \Theta] < \alpha + (1-\alpha)\alpha < 2\alpha$, and we have proven (2). □

Theorem 3.1. *Models* RN *and* RN* *have the same strong and weak thresholds.*

Proof. In Models RN and RN*, fix $T = 1$. Recall that the expected value of $\gamma_{n,n+1}$ is $n/(n+1) \sim 1$, and that the variance of $\gamma_{n,n+1}$ is $n/(n+1)^2 \sim n^{-1}$. Thus the standard deviation of $\gamma_{n,n+1}$ is about $1/\sqrt{n}$ — and for any $\varepsilon, \alpha > 0$, we can choose N so large that $n > N$ implies that $\mathbb{P}[1 - \frac{\varepsilon}{2} < \gamma_{n,n+1} < 1 + \frac{\varepsilon}{2}] > 1 - \alpha$.

Suppose that δ_Θ is a strong threshold function for Θ in Model RN. Fix $\alpha, \varepsilon > 0$, and choose N as in the previous paragraph. From Lemma 3.3, if $\mathbb{P}[G_{n,(1-\varepsilon)\delta_\Theta(n)} \in \Theta] < \alpha$, then $\mathbb{P}[G^\star_{n,(1-2\varepsilon)\delta_\Theta(n)} \in \Theta] < 2\alpha$. Similarly, if $\mathbb{P}[G_{n,(1+\varepsilon)\delta_\Theta(n)} \in \Theta] > 1-\alpha$, then $\mathbb{P}[G^\star_{n,(1+2\varepsilon)\delta_\Theta(n)} \in \Theta] > 1-2\alpha$. Thus as ε and α were arbitrary, Model RN* also has a strong threshold for Θ at δ_Θ.

Similarly, strong thresholds for Model RN* are strong thresholds for Model RN.

The proof for weak thresholds is similar, and we omit it, too. □

4. Weak Thresholds

We turn to the broader subject of weak thresholds.

First, we claim that on the interval $[0,1]$, in Model RN, all upwards closed properties admit weak thresholds. Second, we will construct a space on which Model RN does not admit a weak threshold for the upwards closed property "there are no isolated vertices."

4.1. On the Existence of Thresholds

First of all, all upwards closed properties admit weak thresholds in Model RN*. The proof of this is based on the following lemma.

Lemma 4.1. *Fix n, $\delta > 0$. Let Θ be an upwards closed property of ordered graphs. Then for every integer n and every $\delta > 0$, $\mathbb{P}[G^\star_{n,\delta} \notin \Theta]^2 \geq \mathbb{P}[G^\star_{n,2\delta} \notin \Theta]$.*

To prove Lemma 4.1, we need a variant of Model RN*, in which we choose two random networks independently, and then, in a sense, (deterministically) find their "meet."

Model 4.1 (Model RN*sqrd). *For fixed n and δ, use the distribution of Model 3.2 to independently choose the random networks $G^{\star,1}_{n,\delta}$ of vertices $\xi_1 < \cdots < \xi_n$, and $G^{\star,2}_{n,\delta}$ of vertices $\xi'_1 < \cdots < \xi'_n$. For each $k \in [n]$, let $\Delta_k = \min\{\xi_k - \xi_{k-1}, \xi'_k - \xi'_{k-1}\}$, letting $\xi_0 = \xi'_0 = 0$. Let $G^{2\star}_{n,\delta}$ be the random network of vertex set $\{\eta_1, \ldots, \eta_n\}$, where, for each k, $\eta_k = \sum_{l=1}^{k} \Delta_l$.*

And now for a trick.

Lemma 4.2. *For each n, δ, the distributions of $G^{2\star}_{n,\delta}$ and $G^{\star}_{n,2\delta}$ are identical.*

Proof. Let $\zeta_1, \ldots, \zeta_n$ be the successive exponential r.v.s (of parameter n) for constructing $G^{\star,1}_{n,\delta}$, and let $\zeta'_1, \ldots, \zeta'_n$ be the successive exponential r.v.s (of parameter n) for selecting $G^{\star,2}_{n,\delta}$. (Thus the r.v.s $\xi_k = \sum_{i=1}^{k} \zeta_i$, $k \in [n]$, give the vertices of $G^{\star,1}_{n,\delta}$ while the r.v.s $\xi'_k = \sum_{i=1}^{k} \zeta'_i$, $k \in [n]$, give the vertices of $G^{\star,2}_{n,\delta}$.) Then $G^{\star,1}_{n,\delta} \vee G^{\star,2}_{n,\delta}$ can be constructed by successively choosing $\Delta_1, \ldots, \Delta_n$, where $\Delta_k = \min\{\zeta_k, \zeta'_k\}$ for each $k \in [n]$, and then letting $\eta_k = \sum_{l=1}^{k} \Delta_l$ for each $k \in [n]$. To prove the lemma, it suffices to observe that each random variable Δ_k is exponential of parameter $2n$, which we leave to the reader. □

We can now prove Lemma 4.1.

Proof. For each n, $\delta > 0$, $\mathbb{P}[G^{\star}_{n,\delta} \notin \Theta]^2$ is the probability that in independently choosing two networks $G^{\star,1}_{n,\delta}$ and $G^{\star,2}_{n,\delta}$, of meet $G^{2\star}_{n,\delta}$, neither admits the property Θ. As Θ is upwards closed, $\mathbb{P}[G^{2\star}_{n,\delta} \notin \Theta] \leq \mathbb{P}[G^{\star,1}_{n,\delta} \notin \Theta \;\&\; G^{\star,2}_{n,\delta} \notin \Theta] = \mathbb{P}[G^{\star,1}_{n,\delta} \notin \Theta] \cdot \mathbb{P}[G^{\star,2}_{n,\delta} \notin \Theta]$, and thus $\mathbb{P}[G^{\star}_{n,\delta} \notin \Theta]^2 \geq \mathbb{P}[G^{2\star}_{n,\delta} \notin \Theta]$. As $\mathbb{P}[G^{2\star}_{n,\delta} \notin \Theta] = \mathbb{P}[G^{\star}_{n,2\delta} \notin \Theta]$ (by Lemma 4.2), we have $\mathbb{P}[G^{\star}_{n,\delta} \notin \Theta]^2 \geq \mathbb{P}[G^{\star}_{n,2\delta} \notin \Theta]$, and we are done. □

We can now prove that Model RN admits weak thresholds for upwards closed properties.

Theorem 4.1. *For each upwards closed property Θ of ordered graphs, there is a function $\delta_\Theta\colon \mathbb{N} \to \mathbb{R}^{\geq 0}$ such that for any function $\delta\colon \mathbb{N} \to \mathbb{R}^{\geq 0}$:*

- *If $\delta(n) \ll \delta_\Theta(n)$, then $\mathbb{P}[G_{n,\delta(n)} \in \Theta] = o(1)$.*
- *If $\delta(n) \gg \delta_\Theta(n)$, then $\mathbb{P}[G_{n,\delta(n)} \in \Theta] = 1 - o(1)$.*

The following is essentially the same proof as that of [21, Theorem 0.1].

Proof. By Theorem 3.1, it suffices to prove this theorem for Model RN*. Let Θ be upwards closed, and let $\delta_\Theta(n) = \sup_{\delta \geq 0} \left\{ \mathbb{P}[G^{\star}_{n,\delta} \in \Theta] \leq \frac{1}{2} \right\}$. By induction on k, as

$\mathbb{P}[G^\star_{n,\delta} \notin \Theta]^2 \geq \mathbb{P}[G^\star_{n,2\delta} \notin \Theta]$ by Lemma 4.1, $\mathbb{P}[G^\star_{n,2^k\delta_\Theta(n)} \in \Theta] = 1 - \mathbb{P}[G^\star_{n,2^k\delta_\Theta(n)} \notin \Theta] \geq 1 - (\mathbb{P}[G^\star_{n,2^k\delta_\Theta(n)} \notin \Theta])^{2^k} \geq 1 - 2^{-2^k}$, and so if $\delta(n) \gg \delta_\Theta(n)$, $\mathbb{P}[G^\star_{n,\delta(n)} \in \Theta] = 1 - o(1)$. Similarly, if $\delta(n) \ll \delta_\Theta(n)$, $\mathbb{P}[G^\star_{n,\delta(n)} \in \Theta] = o(1)$, and we are done. □

4.2. Sometimes there are no thresholds at all ...

There is a certain complacency about weak thresholds. Lacking melodrama, they are ignored or taken for granted. So we now note that it is not particularly difficult to construct models of random graphs that do not admit even weak thresholds for some upwards closed properties. The point of this section is to suggest that somewhere there is a very basic but very important theorem that says, "when the following natural conditions are satisfied, then the model admits at least weak thresholds for all upwards closed properties." And that this as yet unformulated theorem is not trivial.

We need a generalization of Model RN.

Model 4.2 (Generalized RN). *Fix a Lebesgue measurable $I \subseteq \mathbb{R}$ of finite measure. If ν is the Lebesgue measure, let $\mathbb{P}[A] = \nu[A]/\nu[I]$ for each measurable $A \subseteq I$. Independently select $\zeta_1, \ldots, \zeta_n \in I$ according to the (uniform) probability measure $\mathbb{P}$. Let $G^I_{n,\delta}$ be the network of vertex set $\{\zeta_1, \ldots, \zeta_n\}$, cutoff δ, and with $<$ being the usual ordering on the vertices.*

From now on, let's refer to the Generalized RN Model as Model GRN. And then:

Theorem 4.2. *There is a set $I \subseteq \mathbb{R}$, of Lebesgue measure 1, such that on I,* Model GRN *does not admit a weak threshold for the upwards closed property "there are no isolated vertices."*

Proof. Let Θ be the property of having no isolated vertices.

Let us first construct the set I. For each integer $k \geq 0$, let $n_k = 2^{2^k}$, and let $\alpha^{-1} = \sum_{k=0}^\infty n_k^{-1}$. For each integer k, let $i_k = \alpha/n_k$ and let I_k be the half-open interval $[k, k+i_k)$. Let $I = \bigcup_{k=0}^\infty I_k$, and note that if ν is the Lebesgue measure, then $\nu[I] = 1$. Let the metric on I be induced by the standard metric on $\mathbb{R}$, so that, for example, the distance between $7, 12 \in I$ is 5.

Next, we set up a comparison of Models RN and GRN. For each n, let $G_{n,\delta}$ be the random network (r.v.) generated by Model RN (only now for Model RN, we work on the half-open interval $[0, 1)$ instead of $[0, 1]$, which will simplify nomenclature but make no measurable difference), and let $G^I_{n,\delta}$ be the random network (r.v.) generated by Model GRN on I. There is a measure-preserving map from tuples of vertices of Model RN to tuples of vertices of Model GRN, which we can construct as follows. We first construct a Lebesgue measure preserving map from $[0, 1)$ onto I: for any $v \in [0, 1)$,

$$\text{if } \sum_{l=0}^{k} i_l \leq v < \sum_{l=0}^{k+1} i_l, \quad \text{then let } h(v) = k + \left(v - \sum_{l=0}^{k} i_l \right).$$

The effect of this is to chop $[0, 1)$ into a partition $J_0 \cup J_1 \cup \cdots$, where J_l and I_l are of the same length for each l, and where h maps J_l onto I_l by a shift operation.

This suggests an Alternative Method for generating the random networks of Model GRN on I: generate the vertices $\xi_1, \ldots, \xi_n$ in Model RN, then let $\xi^*_l = h(\xi_l)$ for each l, and finally connect pairs of vertices ξ^*_j, ξ^*_l such that $|\xi^*_j - \xi^*_l| < \delta$. If $G_{n,\delta}$ is the Model RN graph, let $h(G_{n,\delta})$ be the Alternative Method graph obtained from $G_{n,\delta}$. Note that the distribution of $h(G_{n,\delta})$ is identical to that of $G^I_{n,\delta}$. But the Alternative Method tells us that for any j, l, $|\xi_j - \xi_l| \leq |\xi^*_j - \xi^*_l|$, and thus if ξ_l is isolated in $G_{n,\delta}$, then ξ^*_l is isolated in $h(G_{n,\delta})$. Thus $\mathbb{P}[G_{n,\delta} \in \Theta] \geq \mathbb{P}[h(G_{n,\delta}) \in \Theta] = \mathbb{P}[G^I_{n,\delta} \in \Theta]$.

Recall from [22] that a strong threshold for Connectivity in Model RN is $(\ln n)/n$. (This is easily verified for Model RN*: just look at the probability that there are or are not any gaps of length $(1 \pm \varepsilon)(\ln n)/n$ between any successive vertices.)

Continuing our investigation of the Alternative Method, the only ways that $h(G_{n,\delta})$, obtained from vertices that would generate $G_{n,\delta}$, can have isolated vertices is:

1 If $G_{n,\delta}$ has isolated vertices (which is a.s. untrue if n is large and $\delta \geq 2(\ln n)/n$), or

2 If there exists l and m such that $\xi_l \in J_m$ while $\xi_{l-1}, \xi_{l+1} \notin J_m$ and $\delta < 1 - 2^{-2^{m-1}}$.

Fix $\delta(n) = 2(\ln n)/n$. We will prove that

$$\overline{\lim}_{n\to\infty}\mathbb{P}[G^I_{n,\delta(n)} \in \Theta] = \overline{\lim}_{n\to\infty}\mathbb{P}[h(G_{n,\delta(n)}) \in \Theta] \geq e^{-\alpha}$$

and that $\underline{\lim}_{n\to\infty}\mathbb{P}[G^I_{n,0.9} \in \Theta] \leq 1 - \alpha e^{-\alpha}$. As $1 < \alpha < 2$, this will prevent Θ from having a weak threshold in Model GRN.

First, we claim that $\overline{\lim}_{n\to\infty}\mathbb{P}[G^I_{n,\delta(n)} \in \Theta] = \overline{\lim}_{n\to\infty}\mathbb{P}[h(G_{n,\delta(n)}) \in \Theta] \geq e^{-\alpha}$. This follows from the following observations.

- If $n = n_k$, we claim that the probability that there are at least two vertices of $G^I_{n,\delta}$ in each I_l, $l < k$, is $1 - o(1)$. To see this, first let $\eta_{k,l}$ be the number of vertices of $G^I_{n,\delta}$ in I_l, and we claim that

$$\mathbb{P}\left[\bigvee_{l<k} \eta_{k,l} \leq 1\right] \leq \sum_{l=0}^{k-1} \mathbb{P}[\eta_{k,l} \leq 1] = o(1).$$

  For each k, l, $\eta_{k,l}$ is a binomial random variable of mean $\mathbb{E}[\eta_{k,l}] = n_k i_l = \alpha 2^{2^k - 2^l}$ and variance $\mathbb{V}[\eta_{k,l}] = n_k i_l(1 - i_l) < n_k i_l = \alpha 2^{2^k - 2^l}$. By Chebyshev's Inequality, and noting that $\alpha > 1$,

$$\begin{aligned}\sum_{l=0}^{k-1} \mathbb{P}[\eta_{k,l} \leq 1] &\leq \sum_{l=0}^{k-1} \mathbb{P}[|\eta_{k,l} - n_k i_l| > \sqrt[4]{n_k i_l}\sqrt{n_k i_l}] \\ &< \sum_{l=0}^{k-1} (n_k i_l)^{-1/2} = \frac{1}{\sqrt{\alpha}} \sum_{l=0}^{k-1} 2^{(-2^k + 2^l)/2} < k 2^{-2^{k-2}} \to 0,\end{aligned}$$

  as $k \to \infty$.

- If $n = n_k$, we claim that the probability that there are no vertices of $G^I_{n,\delta}$ in $\bigcup_{l=k}^{\infty} I_l$ is asymptotically $e^{-\alpha}$ as $k \to \infty$. To see this, compute the probability

$$\left(1 - \sum_{l=k}^{\infty} i_l\right)^{n_k} = \left(1 - \alpha \sum_{l=k}^{\infty} 2^{-2^l}\right)^{n_k} = \left(1 - \alpha(1 + \beta_k) 2^{-2^k}\right)^{2^{2^k}}$$

  where $\lim_{k\to\infty} \beta_k = 0$. Thus we have

$$\left(1 - \alpha(1 + \beta_k) 2^{-2^k}\right)^{2^{2^k}} \sim e^{-\alpha(1+\beta_k)} \to e^{-\alpha}.$$

Combining these two observations, $h(G_{n,\delta(n)})$ has no isolated vertices if $G_{n,\delta(n)}$ does not (which it a.s. does not as $\delta(n) = 2(\ln n)/n$), and with probability of at least $e^{-\alpha}$ if $n = n_k$,

k large, there are no vertices in $\bigcup_{l=k}^{\infty} I_l$. Thus $\overline{\lim}_{n\to\infty}\mathbb{P}[G^I_{n,\delta} \in \Theta] = \overline{\lim}_{n\to\infty}\mathbb{P}[h(G_{n,\delta}) \in \Theta] \geq e^{-\alpha}$.

Second, we claim that $\underline{\lim}_{n\to\infty}\mathbb{P}[G_{n,0.9} \in \Theta] \leq 1-\alpha e^{-\alpha}$. For this, it suffices to observe that the probability of having precisely one vertex of $G^I_{n_k,0.9}$ in $\bigcup_{l=k}^{\infty} I_l$ is

$$\binom{n_k}{1}\left(\sum_{l=k}^{\infty} i_l\right)\left(1-\sum_{l=k}^{\infty} i_l\right)^{n_k-1} = 2^{2^k}\left((1+o(1))\alpha 2^{-2^k}\right)\left(1-\alpha(1+\beta_k)n_k^{-1}\right)^{n_k-1}$$

which approaches $\alpha e^{-\alpha}$ as $\beta_k \to 0$. □

5. Strong versus Weak Thresholds

We now turn to strong thresholds. In Subsection 5.1, we find a sequence of upwards closed queries whose thresholds are not strong. In Subsection 5.2, we prove that if an upwards closed property's threshold is sufficiently high, then it's threshold must be strong.

5.1. Weakness Amidst Sparsity

First, some upwards closed properties with low thresholds have strictly weak thresholds.

Theorem 5.1. *Fix an integer* $k > 1$. *In* Model RN, *the threshold of*

$$\Theta_k \equiv \text{there is a connected component of} \geq k \text{ vertices}$$

is not strong.

Proof. We will prove that the probability of such a component is approximately Poisson, and the theorem will follow from an examination of the Poisson parameter. First, we need a formula for the cutoffs: for any $\alpha > 0$, and any integer $k > 0$, set $\delta = \delta_{k,\alpha}(n) = \alpha n^{-k/(k-1)}$ for each n.

In model RN, let $\xi_1, \ldots, \xi_n$ be the n vertices, selected independently, and given k, $\delta > 0$ and $i_1, \ldots, i_k \in [n]$, where $i_1 < \cdots < i_k$, let $g_k(\xi_{i_1}, \ldots, \xi_{i_k}; \alpha) = 1$ if the points $\xi_{i_1}, \ldots, \xi_{i_k}$ are vertices of a connected component of a network of cutoff δ, and let $g_k(\xi_{i_1}, \ldots, \xi_{i_k}; \alpha) = 0$ otherwise.

We will use Silverman & Brown's Theorem A of [26] (see also [23, Cor. 3.6]; there are a number of similar results going back to, say, [17], and expanded on by, say, [2] and [19]).

We will need: for any k, let β'_k be the probability that if $k-1$ points are independently and uniformly chosen in $[-(k-1)\delta, (k-1)\delta]$, then they will become the vertices of a connected network of cutoff δ, with a vertex at a real within distance δ of 0; thus the probability that k points independently and uniformly chosen in $[0,1]$ are the vertices of a connected network of cutoff δ is approximately $[(2k-2)\delta]^{k-1}\beta'_k$. Let $\beta_k = (2k-2)^{k-1}\beta'_k$.

Now for the parameters for Silverman & Brown Theorem A. Let $\lambda = \lambda_{k,\alpha} = \lim_{n\to\infty} \binom{n}{k}$ $\mathbf{E}[g_k(\xi_1, \ldots, \xi_k; \alpha)] = \lim_{n\to\infty} \binom{n}{k} \delta^{k-1}\beta_k = \alpha^{k-1}\beta_k/k!$. Then observe that if $1 \le p \le k-1$, then for any $i_1 < \cdots < i_{2k-p}$, $g_k(\xi_{i_1}, \xi_{i_2}, \ldots, \xi_{i_k}; \alpha) \cdot g_k(\xi_{i_{k+1-p}}, \ldots, \xi_{i_{2k-p}}; \alpha) \le g_{2k-p}(\xi_{i_1}, \ldots, \xi_{i_{2k-p}}; \alpha)$. Let $T_{k,\alpha} = \sum_{1\le i_1<\cdots<i_k\le n} g_k(\xi_{i_1}, \ldots, \xi_{i_k}; \alpha)$.

By the comment on [26, p. 819] after the proof of Theorem A, $T_{k,\alpha}$ is asymptotically Poisson distributed (with parameter $\lambda_{k,\alpha}$) if, for each $p \in [k-1]$ and each $\xi_{i_1}, \ldots, \xi_{i_{2k-p}}$, where $i_1 < \cdots < i_{2k-p}$, the expected value of $g_k(\xi_{i_1}, \ldots, \xi_{i_k}; \alpha) \cdot g_k(\xi_{i_{k-p+1}}, \ldots, \xi_{i_{2k-p}}; \alpha)$ is $o(n^{p-2k})$. But note that as $1 \le p \le k-1$, $g_k(\xi_{i_1}, \ldots, \xi_{i_k}; \alpha) \cdot g_k(\xi_{i_{k-p+1}}, \ldots, \xi_{i_{2k-p}}; \alpha) \le g_k(\xi_{i_1}, \ldots, \xi_{i_{2k-p}}; \alpha)$, and the expected value of $g_k(\xi_{i_1}, \ldots, \xi_{i_{2k-p}}; \alpha)$ is $\delta^{2k-p-1}\beta_{2k-p} \approx \text{constant} \cdot n^{-2k+(p-1)k/(k-1)} = o(n^{p-2k})$, so that T_k is indeed approximately Poisson of parameter λ.

Thus $\mathbb{P}[\Theta_k|\text{cutoff} = \delta_{k,\alpha}] = \mathbb{P}[T_{k,\alpha} > 0] \to (\exp(\lambda_{k,\alpha})-1)\exp(-\lambda_{k,\alpha}) = 1-\exp(-\lambda_{k,\alpha})$ as $n \to \infty$, and similarly, $\mathbb{P}[\Theta_k|\text{cutoff} = 2\delta_{k,\alpha}] = \mathbb{P}[\Theta_k|\text{cutoff} = \delta_{k,2\alpha}] \to 1-\exp(-\lambda_{k,2\alpha}) = 1-\exp(-2^{k-1}\lambda_{k,\alpha})$. Thus if the cutoff is near $\delta_{k,\alpha}$, the probabilities are bounded away from both 0 and 1, and the threshold of Θ_k must be strictly weak. □

5.2. Strength Amidst Density

We now get some sharp thresholds. Recall from Definition 2.1 that if $\mathfrak{G}$ and $\mathfrak{H}$ are linearly ordered graphs of the same number of vertices, and if the order-preserving map from $\mathfrak{G}$ to $\mathfrak{H}$ preserves all edges of $\mathfrak{G}$ (but not vice versa), we write "$\mathfrak{G} < \mathfrak{H}$."

Recall that by [22], $\delta(n) = (\ln n)/n$ is the sharp threshold function for connectivity in Model RN. We claim that any upwards closed property whose threshold much higher than this has a strong threshold. Indeed, we claim something a bit stronger: if $\varepsilon > 0$ and $\delta(n) \gg (\ln n)/n$, then a.s. $G_{n,\delta(n)} < G_{n,(1+\varepsilon)\delta}$, which is *not* true in the Erdős-Rényi

Model: if $G_{n,m(n)}$ ranged over random n-vertex, $m(n)$-edge graphs, then by [4], there is a strictly weak threshold (much above that of Connectivity) for "there is a 4-clique," and so if $m(n)$ was a threshold for this property in the Erdős-Rényi Model, there exists $\varepsilon > 0$ such that $P[K_4 < G_{n,(1-\varepsilon)m(n)}]$ is bounded from 0 while $P[K_4 < G_{n,(1+\varepsilon)m(n)}]$ is bounded from 1, and hence it is *not* true that a.s. $G_{n,(1-\varepsilon)m(n)} < G_{n,(1+\varepsilon)m(n)}$.

We will use the following graphs.

Definition 5.1. *For each n, $\delta > 0$, let $\mathfrak{H}_{n,\delta}$ be the following network. The vertices are the real numbers $\{(2k-1)/(2n) \colon k \in [n]\}$. The edges are assigned as follows. For each $i, j \in [n]$, $(2i-1)/(2n)$ and $(2j-1)/(2n)$ are joined by an edge iff $\left|\frac{2i-1}{2n} - \frac{2j-1}{2n}\right| < \delta$, i.e., $|i-j| < \delta n$.*

The main idea is captured by the following technical lemma on Model RN*.

Lemma 5.1. *Fix $\varepsilon > 0$. If $1-\varepsilon > \delta(n) \gg (\ln n)/n$, then a.s. $G^\star_{n,(1-\varepsilon)\delta(n)} < \mathfrak{H}_{n,\delta(n)} < G^\star_{n,(1+\varepsilon)\delta(n)}$ as $n \to \infty$.*

Proof. Let $\kappa(n) = \delta(n)n/\ln n$, and note that $\kappa(n) \gg 1$ and that $\kappa(n)(\ln n)/n = \delta(n)$.

We claim that it suffices to prove that if $t = \lfloor \kappa(n) \ln n \rfloor = \lfloor n\delta(n) \rfloor$, then the following is a.s. true. If we used the exponential distribution of parameter n to independently generate $\zeta_{n,1}, \ldots, \zeta_{n,n}$ (as in Model RN*), and we let $\eta_{n,t,k} = \sum_{i=k+1}^{k+t} \zeta_{n,i}$ for any $k \in [n-t+1]$, then we get: for each k,

$$(1-\varepsilon)\delta(n) < \eta_{n,t,k} < (1+\varepsilon)\delta(n). \tag{1}$$

To see that this suffices, suppose that the Inequalities 1 hold and let $\xi_{n,k} = \sum_{i=1}^{k} \zeta_{n,i}$ for each n, k. Then a.s. an edge of $G^\star_{n,(1-\varepsilon)\delta(n)}$ will connect a pair $\xi_{n,i}$, $\xi_{n,j}$, $j > i$, only if $j - i < t$, for otherwise a.s., $\xi_{n,j} - \xi_{n,i} = (\xi_{n,j} - \xi_{n,i+t}) + (\xi_{n,i+t} - \xi_{n,i}) \geq 0 + \eta_{n,t,i} > (1-\varepsilon)\delta(n)$. But every pair $(2i-1)/(2n)$, $(2j-1)/(2n)$, $j > i$, such that $j - i < t$ is connected by an edge in $\mathfrak{H}_{n,\delta(n)}$, so that any edge of $G^\star_{n,(1-\varepsilon)\delta(n)}$ corresponds to an edge of $\mathfrak{H}_{n,\delta(n)}$. Thus a.s. $G^\star_{n,(1-\varepsilon)\delta(n)} < \mathfrak{H}_{n,\delta(n)}$. The argument that a.s. $\mathfrak{H}_{n,\delta(n)} < G^\star_{n,(1+\varepsilon)\delta(n)}$ is similar.

To prove that Inequalities 1 holds, it suffices to prove that for each k,

$$\mathbb{P}[(1-\varepsilon)\delta(n) < \eta_{n,t,k} < (1+\varepsilon)\delta(n)] > 1 - o(n^{-1}), \tag{2}$$

for then

$$\mathbb{P}[\forall k \in [n-t+1], (1-\varepsilon)\delta(n) < \eta_{n,t,k} < (1+\varepsilon)\delta(n)] > 1 - o(1).$$

Simplifying and recalling that $\eta_{n,t,k}$ is gamma distributed with parameter n and t degrees of freedom, we prove Inequality 2 as follows. By Remark 2.1, as $\kappa(n) \gg 4/\varepsilon^2$, we have

$$\begin{aligned}
\mathbb{P}[(1-\varepsilon)\delta(n) < \eta_{n,t,k} < (1+\varepsilon)\delta(n)] &= \mathbb{P}\left[|\eta_{n,t,k} - \delta(n)| < \varepsilon\delta(n)\right] \\
&\approx \mathbb{P}\left[\left|\eta_{n,t,k} - \frac{t}{n}\right| < \varepsilon\frac{t}{n}\right] \\
&> 1 - 2\exp[-\varepsilon^2 t/4] \\
&\approx 1 - 2\exp[-\varepsilon^2\kappa(n)(\ln n)/4] \\
&= 1 - 2n^{-\varepsilon^2\kappa(n)/4} = 1 - o\left(\frac{1}{n}\right),
\end{aligned}$$

and we are done. □

And thus:

Theorem 5.2. *Let Θ be an upwards closed property with a threshold function $\delta_\Theta = \delta_\Theta(n)$ such that $\delta_\Theta(n) \gg (\ln n)/n$. Then Θ's threshold is sharp in Model* RN.

Proof. By Theorem 3.1, it suffices to prove this for Model RN*.

We claim that

$$\delta_\Theta(n) = \sup\left\{\delta \colon \mathbb{P}[G^\star_{n,\delta} \in \Theta] < \frac{1}{2}\right\} \gg \frac{\ln n}{n}$$

is a sharp threshold function for Θ. Choose any $\varepsilon > 0$, and we claim that $\mathbb{P}[G^\star_{n,(1-\varepsilon)\delta_\Theta(n)} \in \Theta] = o(1)$, and that $\mathbb{P}[G^\star_{n,(1+\varepsilon)\delta_\Theta(n)} \in \Theta] = 1 - o(1)$.

As $(1-\varepsilon/3)\delta_\Theta(n) < \delta_\Theta(n)$, $\mathbb{P}[G^\star_{n,(1-\varepsilon/3)\delta_\Theta(n)} \in \Theta] < \frac{1}{2}$. And by Lemma 5.1, for any $\varepsilon > 0$, if n is sufficiently large, a.s. $\mathfrak{H}_{n,(1-2\varepsilon/3)\delta_\Theta(n)} < G^\star_{n,(1-\varepsilon/3)\delta_\Theta(n)}$. As Θ is upwards closed, and as there is a good probability of choosing a graph $G^\star_{n,(1-\varepsilon/3)\delta_\Theta(n)} > \mathfrak{H}_{n,(1-2\varepsilon/3)\delta_\Theta(n)}$ where $G^\star_{n,(1-\varepsilon/3)\delta_\Theta(n)} \notin \Theta$, $\mathfrak{H}_{n,(1-2\varepsilon/3)\delta_\Theta(n)} \notin \Theta$. Again, by Lemma 5.1, if n is sufficiently large, a.s. $G^\star_{n,(1-\varepsilon)\delta_\Theta(n)} < \mathfrak{H}_{n,(1-2\varepsilon/3)\delta_\Theta(n)}$, so again as Θ is upwards closed, a.s. $G_{n,(1-\varepsilon)\delta_\Theta(n)} \notin \Theta$. The argument that a.s. $G_{n,(1+\varepsilon)\delta_\Theta(n)} \in \Theta$ is similar. □

And we conclude this subsection by coming full circle up our spiral stair: we started with Lemma 5.1 for Model RN*, and we conclude with the corresponding result for Model RN.

Theorem 5.3. *Let $\varepsilon > 0$. If $1 - \varepsilon > \delta(n) \gg (\ln n)/n$, then a.s. $G_{n,\delta(n)} < G_{n,(1+\varepsilon)\delta(n)}$.*

Proof. Let $\Theta_<$ be the property that, for a graph $\mathfrak{G}$ of n vertices,

$$\mathfrak{G} \in \Theta_< \qquad \equiv \qquad \mathfrak{H}_{n,(1+\varepsilon/2)\delta(n)} < \mathfrak{G}.$$

Then $\Theta_<$ is upwards closed. Similarly, let $\Theta_{\not>}$ be the property, for a graph $\mathfrak{G}$ of n vertices,

$$\mathfrak{G} \in \Theta_{\not>} \qquad \equiv \qquad \mathfrak{H}_{n,(1+\varepsilon/2)\delta(n)} \not> \mathfrak{G}.$$

And $\Theta_{\not>}$ is upwards closed.

By Lemma 5.1, $\Theta_<$ and $\Theta_{\not>}$ share their strong threshold in Model RN*. Then by Theorem 3.1, $\Theta_<$ and $\Theta_{\not>}$ share their strong threshold $(1 + \varepsilon/2)\delta$ in Model RN. Thus $\mathbb{P}[G_{n,\delta(n)} < \mathfrak{H}_{n,(1+\varepsilon/2)\delta(n)}] = 1 - o(1)$ while $\mathbb{P}[\mathfrak{H}_{n,(1+\varepsilon/2)\delta(n)} < G_{n,(1+\varepsilon)\delta(n)}] = 1 - o(1)$, as $n \to \infty$, and the theorem follows. □

6. The General Problem

We offer three conjectures, towards the goal of generalizing these results to higher dimensions, along the lines of [1].

Conjecture 6.1. *Let I be a compact, convex subspace of $\mathbb{R}^n$ for some n. Then in* Model GRN*, all upwards closed attributes have at least weak threshold functions.*

Sometimes we are dropping points onto some manifold, like a torus. We expect weak thresholds here, too. Unfortunately, spacing arrangements of the metric space may make counterexamples resembling that of Proposition 4.2 possible. So we expect that something like the following is true.

Conjecture 6.2. *Let I be a compact, convex subspace of $\mathbb{R}^n$ for some n, and let $\varphi\colon I \mapsto \mathbb{R}^m$ (for some m) be diffeomorphic and whose derivative is bounded away from 0 on I. Then in* Model GRN *on $\varphi(I)$, all upwards closed attributes have at least weak threshold functions.*

And for strong thresholds:

Conjecture 6.3. *Let I be a compact, convex subspace of $\mathbb{R}^n$ for some n, and let $\varphi\colon I \mapsto \mathbb{R}^m$ (for some m) be diffeomorphic and whose derivative is bounded away from 0 on I. Let Θ be any upwards closed property such that in* Model RN *on $\varphi(I)$, the threshold of Θ is much greater than the threshold for Connectivity. Then in* Model GRN *on $\varphi(I)$, Θ has a strong threshold.*

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