Nonlinear evolution equation: Bäclund transformation and soliton solutions

Jun-Xiao Zhao
(cooperated Prof. X-B. Hu etc.)

School of Mathematical Sciences, Graduate University of Chinese Academy of Sciences, Beijing, P.R.China

E-mail: jxzhao@gucas.ac.cn

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Outline:

- Introduction
- Commutativity of pfaffianization and Bäcklund transformation: the 2-dimensional Toda lattice equation
  - ⋆ Pfaffian
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  - ⋆ Bilinear Bäcklund transformation formulae for the coupled 2-dimensional Toda lattice equations
- Conclusion and Discussion

1. Introduction

Bäclund transformation(BT) :

- One of the important integrable properties of non-linear evolution equations
- A effective tool to derive solutions

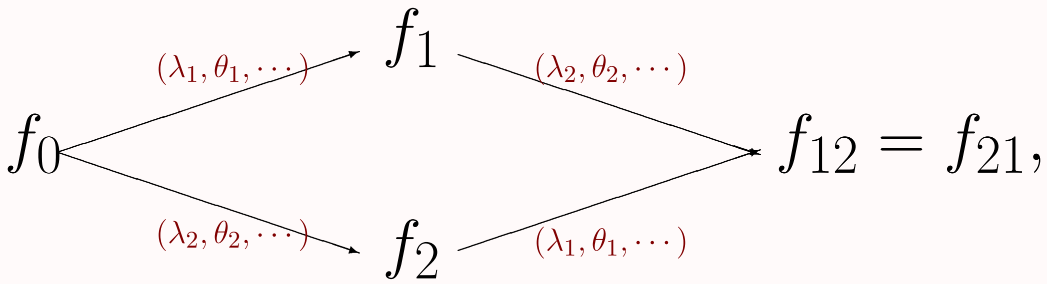
Difficulty: No universal algorithm for constructing

Recall:

- Compatibility of Lax pair

$$[L, A] \equiv LA - AL = 0, \qquad L, A \text{ are operators.}$$

- Commutativity of BT

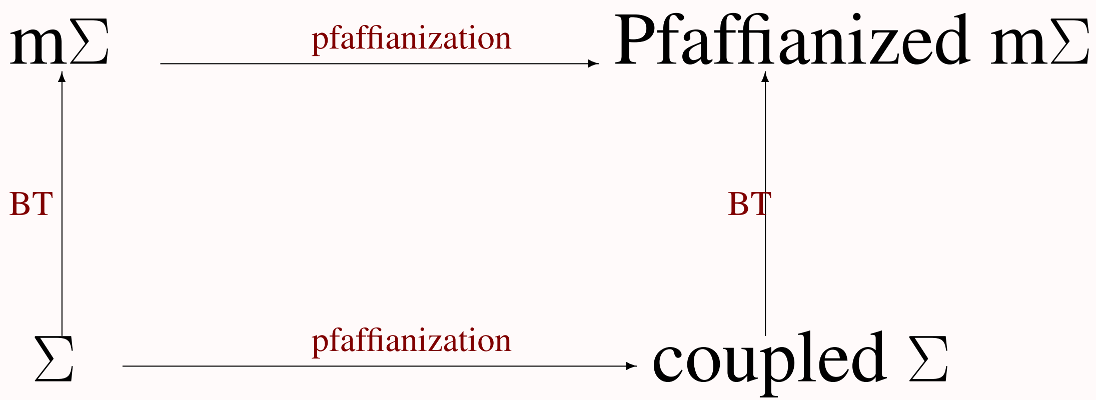


where $\lambda_i, \theta_i (i = 1, 2)$ are parameters in BT formulae.

- The Commuting symmetries to soliton equations,

$\cdots\cdots\cdots$

Our idea: Using the commutativity of pfaffianization and BT (CPBT)



- CPBT was proposed for the first time in 2005, and proved to be valid for the KP equation
  X-B. Hu and J-X. Zhao, Inverse Probl. 21(2005) 1461
- Here we show that CPBT is also valid for some differential-difference systems,such as the 2-D Toda lattice, the differential-difference KP equation and the Leznove lattice equaion, as a result, the BT formulae for the coupled systems of these equations are obtained.

2. CPBT: 2-D Toda lattice equation

Review:

- A **pfaffian** $P \equiv \mathrm{pf}(1, 2, \cdots, 2n)$ can be defined by an ***antisymmetric determinant***,

$$P^2 = \mathbf{det}(a_{ij})_{1 \le i,j \le 2n}, \quad \mathrm{pf}(j,k) \equiv a_{jk} \qquad (1)$$

where $a_{ij} = -a_{ji}, \;\; 1 \le i, j \le 2n.$

$*$ $\mathrm{pf}(j,k)$ is called pfaffian entry or element
$*$ $\mathrm{pf}(j,k) = -\mathrm{pf}(k,j)$
$*$ n: the order of pfaffian P

Example: when $n = 2$,

$$\begin{aligned}\det(a_{ij})_{1\le i,j\le 4} &= (a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23})^2 \\ &= \left[\mathrm{pf}(1,2)\mathrm{pf}(3,4) - \mathrm{pf}(1,3)\mathrm{pf}(2,4) + \mathrm{pf}(1,4)\mathrm{pf}(2,3)\right]^2 \\ &= [\mathrm{pf}(1,2,3,4)]^2\end{aligned}$$

Expansion formula:

$$P = \sum{}' \pm \mathrm{pf}(i_1, i_2)\mathrm{pf}(i_3, i_4)\cdots\mathrm{pf}(i_{2n-1}, i_{2n}), \tag{2}$$

where $\sum'$ denotes summation over all permutations $i_1, i_2, \cdots, i_{2n}$ of $(1, 2, \cdots, 2n)$ which satisfy the inequalities

$$i_1 < i_2, \quad i_3 < i_4, \cdots, i_{2n-1} < i_{2n}, \quad \text{and } i_1 < i_3 < \cdots < i_{2n-1},$$

and

$$\pm = \mathrm{sign}\begin{pmatrix} 1 & 2 & \cdots & 2n \\ i_1 & i_2 & \cdots & i_{2n} \end{pmatrix}$$

- Any determinant can be expressed by a pfaffian

$$\det(b_{jk})_{1\le j,k\le n} = \mathrm{pf}(1,2,\cdots,n,n',\cdots,2',1') \quad (3)$$

by defining

$$\mathrm{pf}(j,k)=0, \quad \mathrm{pf}(j',k')=0, \quad \mathrm{pf}(j,k')=b_{jk}. \quad (4)$$

Example: when n=2,

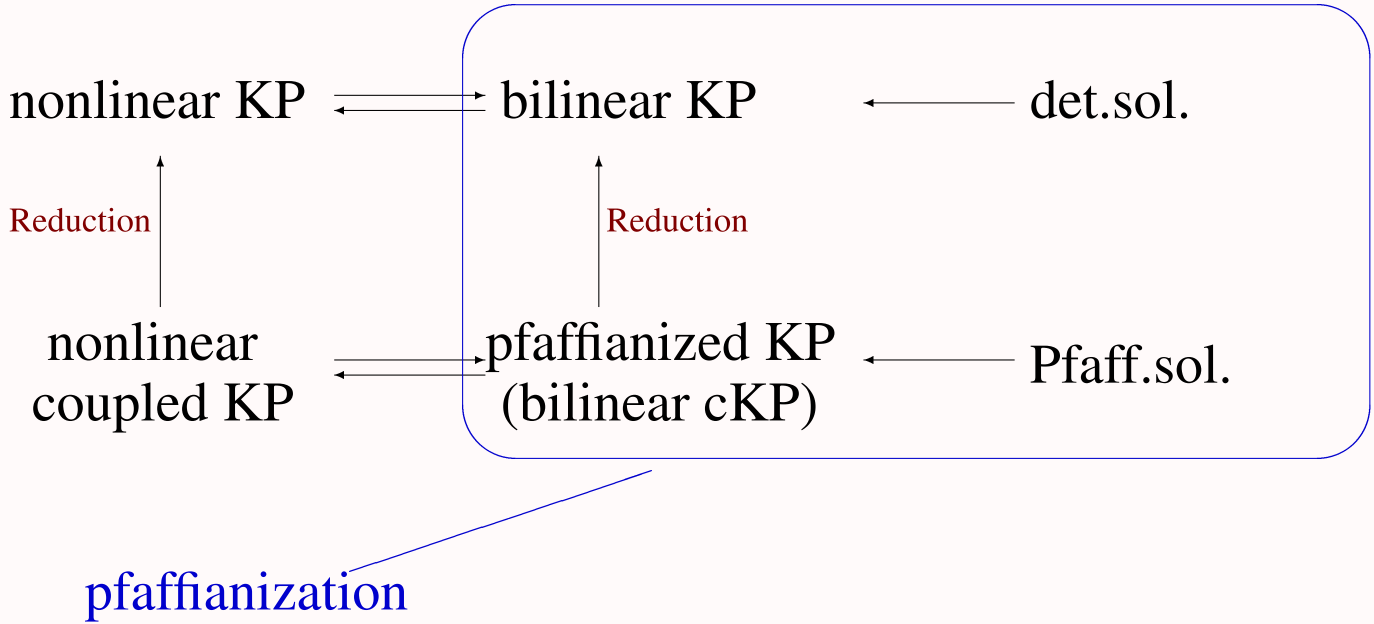
$$\begin{aligned}\mathrm{pf}(1,2,2',1') &= \mathrm{pf}(1,2)\mathrm{pf}(2',1') - \mathrm{pf}(1,2')\mathrm{pf}(2,1') + \mathrm{pf}(1,1')\mathrm{pf}(2,2') \\ &= -b_{12}b_{21} + b_{11}b_{22} \\ &= \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}.\end{aligned}$$

References:

- E.R.Cananiello, *Combinatorics and Renormalization in Quantum Field Theory*, (Benjamin Massachusetts), 1973.
- R.Hirota and Y.Ohta, J.Phys.Soc.Japan, Vol.60,(1991) 798-809.
- R. Hirota, *Direct method in soliton theory(In Japanese)*, (Iwanami Shoten, 1992);
  R.Hirota, *Direct method in soliton theory(In English)*, Edited and translated by A.Nagai,J.Nimmo and C.Gilson, Cambridge University Press, 2004.

  et.

• What is pfaffianization



⋆ R.Hirota and Y.Ohta, J.Phys.Soc.Japan,Vol.60(1991) 798-809.

- The discrete KP equation, C.Gilson, J.Nimmo and S.Tsujimoto, J.Phys.A: Math.Gen., 34(2001) 10569-10575.
- The Davey-Stewartson equations, C.Gilson and J.Nimmo, Theor.and Math.Phys., 128,(2001) 870.
- The 3D three-wave equation, J-X Zhao, Gegenhasi, H-W Tam and X-B Hu, J.Phys.A:Math.Gen.38(2005)1113-1118.
- The Differential-difference KP equation, J-X Zhao, C-X Li and X-B Hu, J.Phys.soc.Japan, Vol.73(2004)1159-1162.
- The 2D Toda lattice, X-B Hu, J-X Zhao and Hon-Wah Tam, J.Math.Anal.Appl. 296(2004)256-261.
- The semi-discrete Toda equation, C-X Li and X-B Hu, Phys.Lett.A, 329(2004)193-198.

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- **CPBT: 2-D Toda lattice equation**

The 2-D Toda lattice eq.

Nonlinear form: $$\frac{\partial^2}{\partial x \partial s}\ln(u_n+1) = u_{n+1}+u_{n-1}-2u_n. \qquad (5)$$

$$\Big\Downarrow u_n = \frac{\partial^2}{\partial x \partial s}\ln(f_n)$$

Nonlinear form: $$D_x D_s f_n \cdot f_n = 2(f_{n+1}f_{n-1} - f_n^2), \qquad (6)$$

$*$ **The bilinear operators**

$$D_y^m D_t^k a\cdot b \equiv \left(\frac{\partial}{\partial y}-\frac{\partial}{\partial y'}\right)^m \left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^k a(y,t)b(y',t')|_{y'=y,t'=t}. \qquad (7)$$

Aim: To obtain BT for the coupled 2-D Toda lattice equations:

$$\begin{aligned} &D_x D_s \tau_n \cdot \tau_n = 2(\tau_{n+1}\tau_{n-1} - \tau_n^2) - \sigma_n \widehat{\sigma}_n, \\ &D_x \sigma_n \cdot \tau_{n+1} = -D_s \sigma_{n+1} \cdot \tau_n, \\ &D_x \tau_n \cdot \widehat{\sigma}_{n+1} = -D_s \tau_{n+1} \cdot \widehat{\sigma}_n. \end{aligned} \tag{8}$$

X-B. Hu, J-X. Zhao, and H-W. Tam, J. Math. Anal. Appl., 296 (2004) 256.

Main steps:

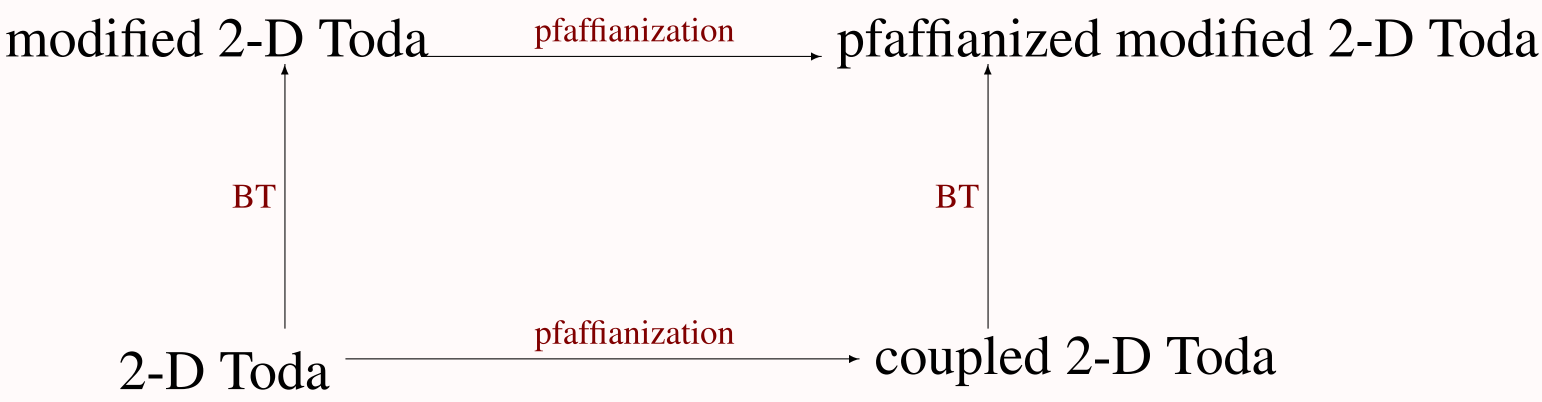


Fig.2

Step1: pfaffianize the modified 2-D Toda lattice equations

$$\begin{aligned} D_x f_{n+1}\cdot f'_n &= -\lambda^{-1} f_n f'_{n+1} + \nu f_{n+1} f'_n, \\ D_s f_n \cdot f'_n &= \lambda f_{n+1} f'_{n-1} - \mu f_n f'_n, \end{aligned} \tag{9}$$

Step2: Prove that the pfaffianized modified 2-D Toda lattice equations constitute a BT for the coupled 2-D Toda lattice (8)

For step1, we choose $\lambda = \mu = \nu = 1$ for convenience.

⋆ Wronskian solution to the modified 2-D Toda lattice equations (9):

$$f_n = \det(\phi_i(n+j-1)_{1\le i,j\le N}, \tag{10}$$

$$f'_n = \det(\widehat{\phi}_i(n+j-1))_{1\le i,j\le N}, \tag{11}$$

where $\phi_i(m)$ and $\widehat{\phi}_i(m)$ satisfy the following linear equations:

$$\widehat{\phi}_i(m) = \phi_i(m+1) - \phi_i(m), \tag{12}$$

$$\frac{\partial}{\partial x}\phi_i(m) = \phi_i(m+1), \tag{13}$$

$$\frac{\partial}{\partial s}\phi_i(m) = -\phi_i(m-1), \quad \text{for} \quad i = 1, 2, \cdots, N. \tag{14}$$

★ R. Hirota, : *Direct method in soliton theory (In English)*, (Edited and Translated by A. Nagai, J. Nimmo, and C. Gilson, Cambridge University Press, 2004.6).

⋆ Suitable pfaffian functions

$$\begin{aligned} f_n &= \text{pf}(1,2,\cdots,N)_n, \\ f'_n &= \text{pf}(1,2,\cdots,N,N+1,c)_n, \end{aligned} \qquad N \text{ is even}, \tag{15}$$

whose entries are defined by

$$\text{pf}(i,j)_n = \sum_{k=1}^{M} [\Phi_k(n+i)\Psi_k(n+j) - \Phi_k(n+j)\Psi_k(n+i)], \tag{16}$$

$$\text{pf}(i,c)_n = 1, \tag{17}$$

with M is positive integer, $\Phi_k(m)$ and $\Psi_k(m)$ satisfy equations (13) and (14).

⋆ The pfaffianized modified 2-D Toda lattice equations

$$
\begin{aligned}
&D_x f_{n+1}\cdot f'_n + f_n f'_{n+1} - f_{n+1} f'_n = g_{n+1}\widehat{g}'_n,\\
&D_s f_n\cdot f'_n - f_{n+1} f'_{n-1} + f_n f'_n = -g_n\widehat{g}'_n,\\
&D_x f_n\cdot \widehat{g}'_n - f_{n+1}\widehat{g}'_{n-1} - f_n\widehat{g}'_n = -\widehat{g}_n f'_n,\\
&D_s f_n\cdot \widehat{g}'_{n-1} + f_{n-1}\widehat{g}'_n + f_n\widehat{g}'_{n-1} = \widehat{g}_n f'_{n-1},\\
&D_x g_n\cdot f'_n - g_{n+1} f'_{n-1} - g_n f'_n = -f_n g'_n,\\
&D_s g_{n+1}\cdot f'_n + g_n f'_{n+1} + g_{n+1} f'_n = f_{n+1} g'_n.
\end{aligned}
\tag{18}
$$

For step2,

Proposition: The pfaffianized modified 2-D Toda equations (18) constitute a Bäcklund transformation of the bilinear coupled 2-D Toda equations (8)

$$D_x D_s \tau_n \cdot \tau_n = 2(\tau_{n+1}\tau_{n-1} - \tau_n^2) - \sigma_n \widehat{\sigma}_n,$$
$$D_x \sigma_n \cdot \tau_{n+1} = -D_s \sigma_{n+1} \cdot \tau_n,$$
$$D_x \tau_n \cdot \widehat{\sigma}_{n+1} = -D_s \tau_{n+1} \cdot \widehat{\sigma}_n.$$

X-B. Hu, J-X. Zhao, and H-W. Tam, J. Math. Anal. Appl., 296 (2004) 256.

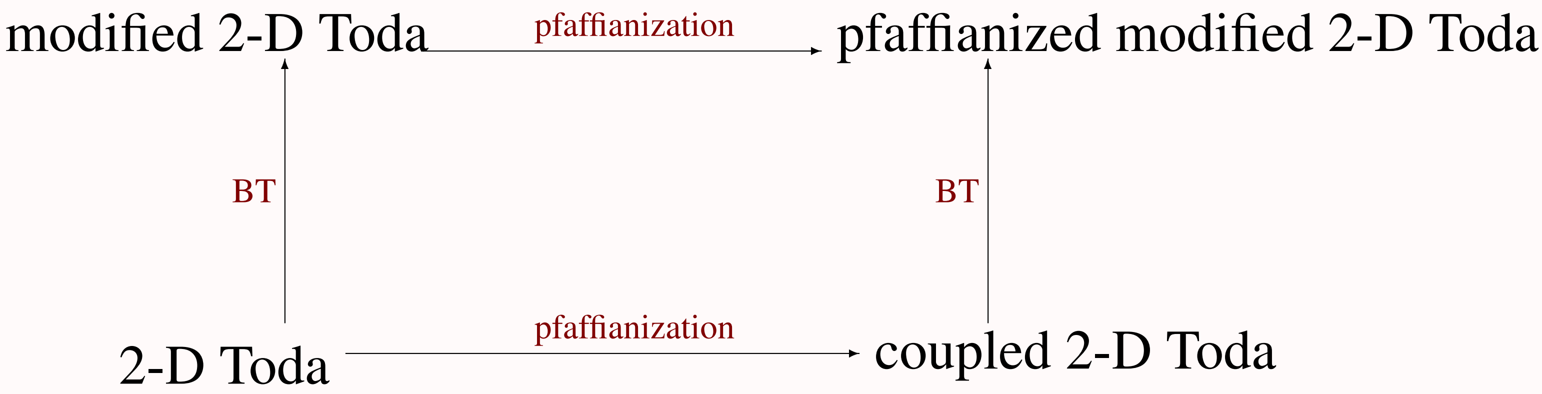
Key of proof: The pfaffian identities:

$$(\alpha_1,\alpha_2,\alpha_3,\alpha_4,1,\cdots,2n)(1,\cdots,2n) = (\alpha_1,\alpha_2,1,\cdots,2n)(\alpha_3,\alpha_4,1,\cdots,2n) - (\alpha_1,\alpha_3,1,\cdots,2n)(\alpha_2,\alpha_4,1,\cdots,2n) + (\alpha_1,\alpha_4,1,\cdots,2n)(\alpha_2,\alpha_3,1,\cdots,2n), \quad (19)$$

$$(\alpha_1,\alpha_2,\alpha_3,1,\cdots,2n-1)(\gamma,1,\cdots,2n-1) = (\alpha_1,1,\cdots,2n)(\gamma,\alpha_2,\alpha_3,1,\cdots,2n-1) - (\alpha_2,1,\cdots,2n-1)(\alpha_1,\gamma,\alpha_3,1,\cdots,2n-1) + (\alpha_3,1,\cdots,2n-1)(\alpha_1,\alpha_2,\gamma,1,\cdots,2n-1), \quad (20)$$

3. Conclusion and Discussion

- The CPBT is valid for the 2-D Toda lattice equations if $f_n,\ g_n$ and $\widehat{g}_n$ are solutions to equations (8)



- A BT for the couple 2-D Toda lattice equation (8)
- Pfaffian functions $f'_n,\ g'_n$ and $\widehat{g}'_n$ are new pfaffian solutions to the couple 2-D Toda lattice equations (8) if $f_n,\ g_n$ and $\widehat{g}_n$ are solutions to equations (8)

- We also prove that the CPBT is valid for the following equations:
  - ⋆ The D∆KP equation

$$\Delta(\frac{\partial u}{\partial t} + 2\frac{\partial u}{\partial y} - 2u\frac{\partial u}{\partial y}) = (2+\Delta)\frac{\partial^2 u}{\partial y^2},$$

J-X. Zhao, Gegenhasi and X-B. hu, J. Phys. Soc. Jpn., 78(2009)064005

  - ⋆ The Leznov lattice equations

$$\frac{\partial^2 Q_n}{\partial x \partial s} = V_{n+1} - 2V_n + V_{n-1}, \quad Q_n = \ln(1+V_n)$$

C-X. Li, J-X. Zhao and X-B.Hu, J. Nonlinear Math.Phys. 16(2009)1

  - ⋆ The semi-discrete Toda equation

$$\frac{d}{dt}\log\frac{v_n^{k+1}}{v_n^k} = v_{n+1}^{k+1} + v_{n-1}^k - v_n^k - v_n^{k+1}$$

J-X Zhao, Math. Comp. Simulation, 74 (2007)388

Thanks !