# Solving Q1 in the ABS Lattice Equations by Constructive Method

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# Outline

# I. ABS List and NSS of Q1

# Discretization

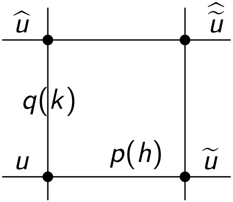
Discretization:

$$u(x,t):\ x = x_0 + nh,\ t = t_0 + mk, \quad u(x,t) \Rightarrow u_{n,m}$$

Notations:

$$u \equiv u_{n,m}, \quad \widetilde{u} \equiv u_{n+1,m}, \quad \underset{\sim}{u} \equiv u_{n-1,m}, \quad \widehat{u} \equiv u_{n,m+1}, \quad \widehat{\widetilde{u}} \equiv u_{n+1,m+1}$$

Map:



## Some Lattice Equations

LPKdV:

$$(u-\widehat{\widetilde{u}})(\widetilde{u}-\widehat{u})=p-q \tag{1}$$

LPmKdV:

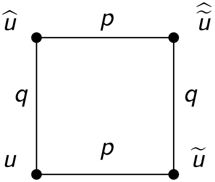
$$p(v\widehat{v}-\widetilde{v}\widehat{\widetilde{v}})=q(v\widetilde{v}-\widehat{v}\widehat{\widetilde{v}}) \tag{2}$$

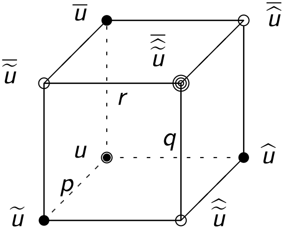
LSKdV:

$$\frac{(z-\widehat{\widetilde{z}})(\widetilde{z}-\widehat{z})}{(z-\widetilde{z})(\widehat{z}-\widehat{\widetilde{z}})}=\frac{p^2}{q^2} \tag{3}$$

# Multidimensional Consistency(MDC)/CAC[Nijhoff,Walker-01,GMJ]

$$Q(u,\widetilde{u},\widehat{u},\widehat{\widetilde{u}};p,q)=0$$



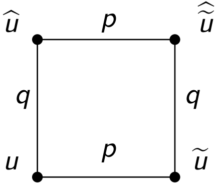

$$Q(u,\widetilde{u},\widehat{u},\widehat{\widetilde{u}};p,q)=0,\qquad Q(\overline{u},\overline{\widetilde{u}},\overline{\widehat{u}},\overline{\widehat{\widetilde{u}}};p,q)=0,$$

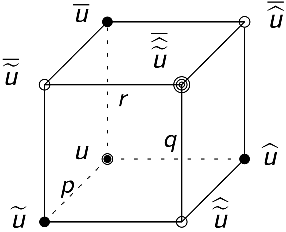
$$Q(u,\widetilde{u},\overline{u},\overline{\widetilde{u}};p,r)=0,\qquad Q(\widehat{u},\widehat{\widetilde{u}},\overline{\widehat{u}},\overline{\widehat{\widetilde{u}}};p,r)=0,$$

$$Q(u,\widehat{u},\overline{u},\overline{\widehat{u}};q,r)=0,\qquad Q(\widetilde{u},\widehat{\widetilde{u}},\overline{\widetilde{u}},\overline{\widehat{\widetilde{u}}};q,r)=0.$$

# Consistency Around the Cube (CAC) [ABS-03]

$$Q(u, \widetilde{u}, \widehat{u}, \widehat{\widetilde{u}}; p, q) = 0$$




[ABS-03]'s requirement: (CAC +D4+Tetrahedron)

- Linearity w.r.t. each $\{u, \widetilde{u}, \widehat{u}, \widehat{\widetilde{u}}\}$
- Symmetry: $Q$ invariant under group $D_4$
- Tetrahedron Condition: $\overline{\widehat{\widetilde{u}}} = f(\widetilde{u}, \widehat{u}, \overline{u}; p, q, r)$

# ABS List

ABS List: H1, H2, $H3^\delta$, $A1^\delta$, A2, $Q1^\delta$, Q2, $Q3^\delta$, Q4

- H1(LPKdV):

$$(u-\widehat{\widetilde{u}})(\widetilde{u}-\widehat{u})=p-q$$

- H2:

$$(u-\widehat{\widetilde{u}})(\widetilde{u}-\widehat{u})+(q-p)(u+\widetilde{u}+\widehat{u}+\widehat{\widetilde{u}})+q^2-p^2=0$$

- $H3^\delta$:

$$p(u\widetilde{u}+\widehat{u}\widehat{\widetilde{u}})-q(u\widehat{u}+\widetilde{u}\widehat{\widetilde{u}})+\delta(p^2-q^2)=0$$

# ABS List

ABS List: H1, H2, H3$^\delta$, A1$^\delta$, A2, Q1$^\delta$, Q2, Q3$^\delta$, Q4

- A1$^\delta$:

$$(p(u+\widehat{u})(\widetilde{u}+\widehat{\widetilde{u}}) - q(u+\widetilde{u})(\widehat{u}+\widehat{\widetilde{u}}) - \delta^2 pq(p-q) = 0$$

- A2:

$$(q^2-p^2)(u\widetilde{u}\widehat{u}\widehat{\widetilde{u}}+1)+q(p^2-1)(u\widehat{u}+\widetilde{u}\widehat{\widetilde{u}})-p(q^2-1)(u\widetilde{u}+\widehat{u}\widehat{\widetilde{u}}) = 0$$

# ABS List

ABS List: H1, H2, H3$^\delta$, A1$^\delta$, A2, Q1$^\delta$, Q2, Q3$^\delta$, Q4

- Q1$^\delta$:

$$p(u-\widehat{u})(\widetilde{u}-\widehat{\widetilde{u}})-q(u-\widetilde{u})(\widehat{u}-\widehat{\widetilde{u}})+\delta^2 pq(p-q)=0$$

- Q2:

$$\begin{aligned}&p(u-\widehat{u})(\widetilde{u}-\widehat{\widetilde{u}})-q(u-\widetilde{u})(\widehat{u}-\widehat{\widetilde{u}})\\&\quad+pq(p-q)(u+\widetilde{u}+\widehat{u}+\widehat{\widetilde{u}})-pq(p-q)(p^2-pq+q^2)=0\end{aligned}$$

- Q3$^\delta$:

$$\begin{aligned}&(q^2-p^2)(u\widehat{\widetilde{u}}+\widetilde{u}\widehat{u})+q(p^2-1)(u\widetilde{u}+\widehat{u}\widehat{\widetilde{u}})-p(q^2-1)(u\widehat{u}+\widetilde{u}\widehat{\widetilde{u}})\\&\quad-\delta^2(p^2-q^2)(p^2-1)(q^2-1)/(4pq)=0\end{aligned}$$

# ABS List

ABS List: H1, H2, $H3^\delta$, $A1^\delta$, A2, $Q1^\delta$, Q2, $Q3^\delta$, Q4

- Q4: [Adler/Hietarinta/Nijhoff/Krichever-Novikov/$\cdots$]

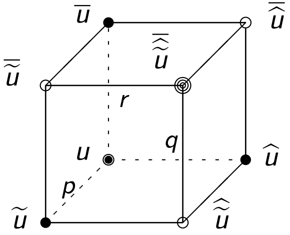
$$p(u\widetilde{u}+\widehat{u}\widehat{\widetilde{u}})-q(u\widehat{u}+\widetilde{u}\widehat{\widetilde{u}})-r(u\widehat{\widetilde{u}}+\widetilde{u}\widehat{u})+pqr(1+u\widetilde{u}\widehat{u}\widehat{\widetilde{u}})=0$$

$$(p,P)=(\sqrt{k}\,\mathrm{sn}(\alpha;k),\mathrm{sn}'(\alpha;k)),\quad (q,R)=(\sqrt{k}\,\mathrm{sn}(\beta;k),\mathrm{sn}'(\beta;k)$$
$$(r,R)=(\sqrt{k}\,\mathrm{sn}(\gamma;k),\mathrm{sn}'(\gamma;k)),\quad \gamma=\alpha-\beta$$

points on the elliptic curve:

$$\Gamma=\{(x,X):X^2=x^4+1-(k+1/k)x^2\}$$

# Benefit From CAC/MDC

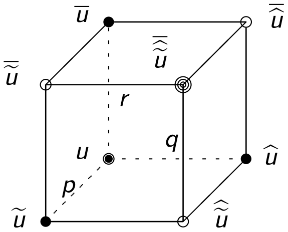



- Lax Pair
- BT/DT:

$$\begin{cases} Q(u,\widetilde{u},\overline{u},\widetilde{\overline{u}};p,r)=0 \\ Q(u,\widehat{u},\overline{u},\widehat{\overline{u}};r,q)=0 \end{cases}$$

- Miura Transformation [See more in [Atkinson-08]]

# Constructive approach[Hietarinta,Zh-09]




- 0SS: Fixed point idea [Atkinson,Hietarinta,Nijhoff-07]
- 1SS: DT/BT
- 2SS/3SS: Hirota's perturbation expansion
- Transformation/Casoratian/Bilinearization
- Casoratian proof

# Step 1: 0SS of Q1: $p(u-\widehat{u})(\widetilde{u}-\widehat{\widetilde{u}})-q(u-\widetilde{u})(\widehat{u}-\widehat{\widetilde{u}})-\delta^2pq(q-p)=0$

- 0SS: $\begin{cases} Q(u,\widetilde{u},\overline{u},\widetilde{\overline{u}};p,r)=0, \\ Q(u,\widehat{u},\overline{u},\widehat{\overline{u}};r,q)=0, \end{cases}$ (fixed point: $\overline{u}=u$)
- $\overline{u}\longrightarrow u+c$ :

$$u^{0SS}=\alpha n+\beta m+\gamma,$$

$$p=\frac{c^2/r-\delta^2 r}{a^2-\delta^2},\quad q=\frac{c^2/r-\delta^2 r}{b^2-\delta^2},\quad \alpha=pa,\quad \beta=qb.$$

- $\overline{u}\longrightarrow -u+c$ :

$$u^{0SS}=\tfrac{1}{2}c+A\alpha^n\beta^m+B\alpha^{-n}\beta^{-m},\quad AB=\delta^2r^2/16,$$

$$p=-\tfrac{1}{4}r(1-\alpha)^2/\alpha,\quad q=-\tfrac{1}{4}r(1-\beta)^2/\beta.$$

# Step 2: Q1: 1SS via BT/DT

- BT: $\begin{cases} Q(u,\widetilde{u},\overline{u},\widetilde{\overline{u}};p,r)=0, \\ Q(u,\widehat{u},\overline{u},\widehat{\overline{u}};r,q)=0, \end{cases}$
- 1SS (with linear 0SS):

$$u^{1SS}=\alpha n+\beta m+\gamma+\kappa\frac{1-\rho_{nm}}{1+\rho_{nm}},\quad \rho_{nm}=\left(\frac{a+k}{a-k}\right)^n\left(\frac{b+k}{b-k}\right)^n\rho_{00}$$

- 1SS (with power 0SS):

$$u^{1SS}=\frac{A'\alpha^n\beta^m(1+\kappa^{-2}\rho_{n,m})+B'\alpha^{-n}\beta^{-m}(1+\kappa^2\rho_{n,m})}{1+\rho_{n,m}}$$

# Step ...4: NSS/Bilinearization-I: Linear 0SS

- Trans: $u^{NSS}_{n,m} = \alpha n + \beta m + \gamma - (c^2/r - \delta^2 r)\frac{g}{f}$,
- Bilinearization-I:

$$
\begin{aligned}
\mathcal{Q}_1 &\equiv \bar{\widehat{\widetilde{f}}} f(b-\delta) + \widehat{\widetilde{f}}\bar{f}(a+\delta) - \bar{\widetilde{f}}\widehat{f}(a+b) = 0, \\
\mathcal{Q}_2 &\equiv \bar{\widehat{\widetilde{f}}} f(a-b) + \bar{\widetilde{f}}\widehat{f}(b+\delta) - \widetilde{f}\bar{\widehat{f}}(a+\delta) = 0, \\
\mathcal{Q}_3 &\equiv -\bar{\widetilde{f}}\widehat{f} + \bar{\widetilde{f}}\widehat{g}(-a+\delta) + \widetilde{f}\bar{\widehat{f}} + \bar{\widehat{f}}\widetilde{g}(b-\delta) + \bar{f}\widehat{\widetilde{g}}(a-b) = 0, \\
\mathcal{Q}_4 &\equiv \bar{\widehat{\widetilde{f}}} g(a-b) + \bar{\widetilde{f}}\widehat{g}(a+b) - \bar{\widehat{f}}\widetilde{g}(a+b) + \bar{f}\widehat{\widetilde{g}}(-a+b) = 0.
\end{aligned}
$$

- NSS: $f = |\widehat{N-1}|,\ g = |-1, \widetilde{N-1}|$

$\psi_i = \varrho_i^{+}(\delta + k_i)^l (a+k_i)^n (b+k_i)^m + \varrho_i^{-}(\delta - k_i)^l (a-k_i)^n (b-k_i)^m.$

# NSS/Bilinearization-II: Power 0SS

- Trans:

$$u^{NSS}_{n,m} = A\alpha^n\beta^m \frac{\bar{\bar{f}}}{f} + B\alpha^{-n}\beta^{-m}\frac{\underline{\underline{f}}}{f}, \quad AB = \delta^2 r^2/16,$$

- Bilinearization-II: (Bilinear H3):

$$\begin{aligned} \mathcal{B}_1 &\equiv 2cf\widetilde{f} + (a-c)\widetilde{\bar{f}}\underline{f} - (a+c)\bar{f}\underline{\widetilde{f}} = 0, \\ \mathcal{B}_2 &\equiv 2cf\widehat{f} + (b-c)\widehat{\bar{f}}\underline{f} - (b+c)\bar{f}\underline{\widehat{f}} = 0, \end{aligned}$$

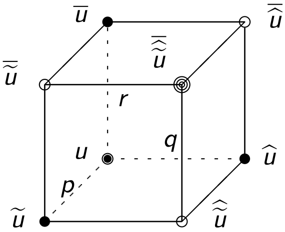
- NSS: $f = |\widehat{N-1}|$

$$\psi_i = \varrho_i^+(c+k_i)^l(a+k_i)^n(b+k_i)^m + \varrho_i^-(c-k_i)^l(a-k_i)^n(b-k_i)^m.$$

# II. New 1SS of Q1$^\delta$

# Miura Transformations[Atkinson-08]

- Idea:



# Miura Trans.

- $Q1^\delta$:

$$p(u-\widehat{u})(\widetilde{u}-\widehat{\widetilde{u}})-q(u-\widetilde{u})(\widehat{u}-\widehat{\widetilde{u}})-\delta^2pq(q-p)=0.$$

- $Q1^0$:

$$p(w-\widehat{w})(\widetilde{w}-\widehat{\widetilde{w}})-q(w-\widetilde{w})(\widehat{w}-\widehat{\widetilde{w}})=0.$$

- MT[Atkinson-08]:

$$(w-\widetilde{w})(u-\widetilde{u})=p(w\widetilde{w}-\delta^2),$$
$$(w-\widehat{w})(u-\widehat{u})=q(w\widehat{w}-\delta^2).$$

# Miura Trans.

- Q2:

$$p(u-\widehat{u})(\widetilde{u}-\widehat{\widetilde{u}})-q(u-\widetilde{u})(\widehat{u}-\widehat{\widetilde{u}}) = pq(q-p)(u+\widetilde{u}+\widehat{u}+\widehat{\widetilde{u}}-p^2+pq-q^2),$$

- $Q1^\delta$:

$$p(w-\widehat{w})(\widetilde{w}-\widehat{\widetilde{w}})-q(w-\widetilde{w})(\widehat{w}-\widehat{\widetilde{w}}) = \delta^2 pq(q-p).$$

- MT:

$$\delta(w-\widetilde{w})(u-\widetilde{u}) = -p[\delta^2(u+\widetilde{u})-2w\widetilde{w}]+\delta p^2(w+\widetilde{w}+\delta p),$$
$$\delta(w-\widehat{w})(u-\widehat{u}) = -q[\delta^2(u+\widehat{u})-2w\widehat{w}]+\delta q^2(w+\widehat{w}+\delta q).$$

# Closed MT for power 0SS

- $u_0^{0SS}$: $$\alpha^n\beta^m \quad \text{or} \quad \alpha^{-n}\beta^{-m}$$

$$\downarrow \quad MT$$

$u_\delta^{0SS}$: $$A\alpha^n\beta^m + B\alpha^{-n}\beta^{-m}, \quad 4AB=\delta^2$$

$$\downarrow \quad \delta = 0$$

$u_0^{0SS}$: $$\alpha^n\beta^m \quad \text{or} \quad \alpha^{-n}\beta^{-m}.$$

- Note: MT not for Linear($Q1^\delta$): $u^{0SS} = \alpha n + \beta m + \gamma$,

$$p = \frac{c^2/r - \delta^2 r}{a^2-\delta^2}, \quad q = \frac{c^2/r - \delta^2 r}{b^2-\delta^2}, \quad \alpha = pa, \quad \beta = qb.$$

(Different $p$ for $Q1^0$ and $Q1^\delta$)

# Open MT for linear 0SS

- $u_0^{0SS}$: $\quad an + bm + \gamma_1, \quad p = a^2, \; q = b^2$

$$\Big\downarrow \quad \text{MT/Keep same } p, q$$

New $u_\delta^{0SS}$: $\quad \frac{(an+bm+\gamma_1)^3}{3} - \frac{a^3 n + b^3 m + \gamma_2}{3} - \delta^2(an + bm + \gamma_3)$

$$\Big\downarrow \quad \delta = 0$$

- New $u_0^{0SS}$: $\quad \frac{(an+bm+\gamma_1)^3}{3} - \frac{a^3 n + b^3 m + \gamma_2}{3}.$

$$\Big\downarrow \quad \text{MT: } (w - \widetilde{w})(u - \widetilde{u}) = p(w\widetilde{w} - \delta^2)$$

- New $u_\delta^{0SS}$: $\quad$ 5th order $\cdots\cdots$.
- Polynomial 0SS chain for $Q1^\delta$.

# New 1SS of Q1$^\delta$

- New $u_\delta^{0SS}$:

$$\frac{(an+bm+\gamma_1)^3}{3}-\frac{a^3n+b^3m+\gamma_2}{3}-\delta^2(an+bm+\gamma_3)$$

- New $u_\delta^{1SS}$:

$$u_\delta^{1SS}=\frac{(an+bm+k+\gamma_1)^3}{3}-\frac{a^3n+b^3m+\lambda_2+k^3}{3}$$
$$-\delta^2(an+bm+k+\gamma_3)+\gamma_4+\frac{(-2k((an+bm+\gamma_1)^2-\delta^2)\rho_{n,m})}{1+\rho_{n,m}}$$
$$\rho_{n,m}=\left(\frac{a+k}{-a+k}\right)^n\left(\frac{b+k}{-b+k}\right)^m\rho_{0,0}$$

# III. 1SS of Q2

# Two 0SS of Q2: Linear background

- $u^{0SS}_{Q1^\delta}$: $\alpha n + \beta m + \gamma$

$$\downarrow \quad MT$$

$u^{0SS}_{Q2}$: $\left(\frac{\alpha n+\beta m}{\delta} + \gamma\right)^2 + \frac{1}{4}\left(\frac{c^2/r-\delta^2 r}{\delta^2}\right)^2$

- Parametrization:

$$p = \frac{c^2/r - \delta^2 r}{a^2 - \delta^2}, \quad q = \frac{c^2/r - \delta^2 r}{b^2 - \delta^2}, \quad \alpha = pa, \quad \beta = qb.$$

# Two 0SS of Q2: Power background

- $u^{0SS}_{Q1^\delta}$ $(\delta = 1)$:

$$\frac{1}{2}(A\alpha^n\beta^m + A^{-1}\alpha^{-n}\beta^{-m}), \quad p = \frac{(1-\alpha)^2}{2\alpha}, \quad q = \frac{(1-\beta)^2}{2\beta}$$

$$\downarrow \quad MT$$

- $u^{0SS}_{Q2}$:

$$\frac{1}{2}(A\alpha^n\beta^m + A^{-1}\alpha^{-n}\beta^{-m} + 4) + Z^{-1}_{n,m}(Pn + Qm + r)$$

$$Z_{n,m} = \frac{A\alpha^n\beta^m - 1}{A\alpha^n\beta^m + 1}, \quad P = \frac{(\alpha^2-1)(\alpha^2-4\alpha+1)}{4\alpha^2}, \quad Q = \frac{(\beta^2-1)(\beta^2-4\beta+1)}{4\beta^2}.$$

# 1SS from linear 0SS

- Linear background $u_{Q2}^{0SS}$:

$$\Big(\frac{\alpha n+\beta m}{\delta}+\gamma\Big)^2+\frac{1}{4}\Big(\frac{c^2/r-\delta^2 r}{\delta^2}\Big)^2$$

- $u_{Q2}^{1SS}$:

$$(\alpha n+\beta m+\gamma)^2+\frac{r^2}{4}+2s(\alpha n+\beta m+\gamma)\frac{1-\rho_{nm}}{1+\rho_{nm}}+s^2,$$

$$\rho_{n,m}=\left(\frac{\alpha z+sp}{\alpha z-sp}\right)^n\left(\frac{\beta z+sq}{\beta z-sq}\right)^m\rho_{0,0}$$

# Power 0SS

- $u_{Q2}^{0SS}$:

$$\frac{1}{2}(A\alpha^n\beta^m + A^{-1}\alpha^{-n}\beta^{-m} + 4) + Z_{n,m}^{-1}(Pn + Qm + r)$$

$$Z_{n,m} = \frac{A\alpha^n\beta^m - 1}{A\alpha^n\beta^m + 1},$$

$$P = \frac{(\alpha^2 - 1)(\alpha^2 - 4\alpha + 1)}{4\alpha^2},$$

$$Q = \frac{(\beta^2 - 1)(\beta^2 - 4\beta + 1)}{4\beta^2}.$$

$$s = (k+1) \pm \sqrt{k^2 + 2k}, \quad K = \frac{(s^2 - 1)(s^2 - 4s + 1)}{4s^2}.$$

# 1SS from power 0SS

- $u_{Q2}^{1SS}$:

$$\begin{aligned}&\tfrac{A}{2}\alpha^n\beta^m[1+s^{-1}(A^2\alpha^{2n}\beta^{2m}s^{-2}-1)\Gamma_{n,m}]\\&+\tfrac{A^{-1}}{2}\alpha^{-n}\beta^{-m}[1+s(A^2\alpha^{2n}\beta^{2m}s^{-2}-1)\Gamma_{n,m}]\\&+2[1-(A^2\alpha^{2n}\beta^{2m}s^{-2}-1)\Gamma_{n,m}]\\&+(Pn+Qm+r)\Big(1+\tfrac{2(1-s^2+s\rho_{n,m})}{(A\alpha^n\beta^m-1)(1-s^2)+(A\alpha^n\beta^m s^{-1}-s)\rho_{n,m}}\Big)\\&+\tfrac{A\alpha^n\beta^m+1}{A\alpha^n\beta^m-1}K,\end{aligned}$$

$$\Gamma_{n,m}=\frac{1}{A\alpha^n\beta^m-1}\cdot\frac{(1-s^2)\rho_{n,m}}{(A\alpha^n\beta^m-1)(1-s^2)+(A\alpha^n\beta^m s^{-1}-s)\rho_{n,m}},$$

$$\rho_{n,m}=\left(\frac{1-\alpha s}{\alpha-s}\right)^n\left(\frac{1-\beta s}{\beta-s}\right)^m\rho_{0,0}.$$

# Questions

- More 0SS of $Q1^\delta$?
- More NSS of $Q1^\delta$?
- Relation between them?
- 0SS and 1SS of $Q2$?

Based on:

J. Hietarinta, D.J. Zh, Soliton solutions for ABS lattice equations: II Casoratians and bilinearization, arXiv:0903.1717v1 [nlin.SI].

J. Hietarinta, D.J. Zh, Soliton solutions of Q2, in preparation, 2009.

Thank You!