Dark soliton solution of Sasa-Satsuma equation

$$i u_t = u_{xx} + 2|u|^2 u + i\left(u_{xxx} + 6|u|^2 u_x + 3(|u|^2)_x u\right)$$

Sasa and Satsuma, J. Phys. Soc. Jpn. <u>60</u> ('91) 409.

$$\Big\Downarrow \begin{cases} \text{Gauge transformation} & u \longmapsto u e^{i(kx-\omega t)} \\ \text{Galilean transformation} & x \longmapsto x - v t \end{cases}$$

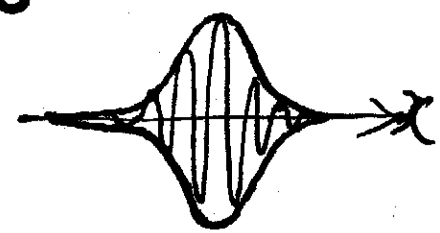
$$u_t = u_{xxx} + 6|u|^2 u_x + 3(|u|^2)_x u$$

NLS $\quad iu_t = u_{xx} \pm 2|u|^2 u$

Sign $\pm$ can not be changed by gauge, Galilean, scale transformations

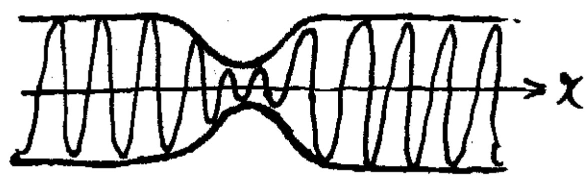
$+$: focusing equation

bright soliton $\quad u \to 0$ as $x \to \pm\infty$



$-$: defocusing equation

dark soliton $\quad u \to A_\pm e^{i(Kx - \Omega t)}$ as $x \to \pm\infty$



Bright soliton $\quad u = \dfrac{g}{f} \quad (f : \text{real})$

$\longleftarrow$ 2 component KP hierarchy

(Toda molecule equation)

Dark soliton $\quad u = \dfrac{g}{f} e^{i(k'x - \Omega' t)} \quad (f : \text{real})$

$\longleftarrow$ 1 component KP hierarchy

with negative weight time

(Toda lattice equation)

$\Uparrow$

<u>Regularity</u> $\quad f \neq 0$ for x, t : real

(physical solution)

- Without regularity condition,

both sol. for $\left\{\begin{array}{l}\text{focusing} \\ \text{defocusing}\end{array}\right\}$ eq. from $\left\{\begin{array}{l}\text{2-comp.} \\ \text{1-comp.}\end{array}\right\}$ KP

- Selection of sign $\pm$ by regularity condition

Motivation

Which kind of physical solution is possible for NLS type equation of sign $+$ or $-$?

coupled NLS, derivative NLS, Sasa-Satsuma, ...

Usually constructing bright soliton solution (regular) is easier because of more parameters in multi-component KP.

Dark soliton is more difficult.

Multi-dark soliton for coupled system

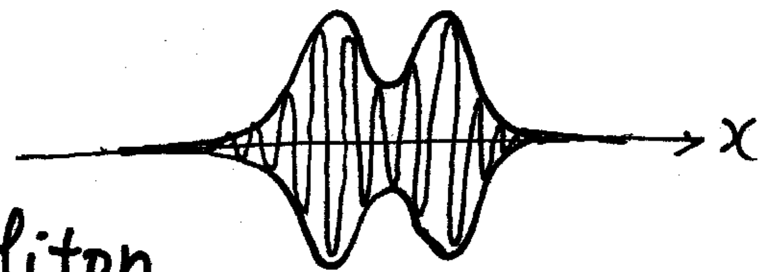
Hu, Chaos, Solitons & Fractals <u>7</u> ('96) 211.

Sasa-Satsuma eq.

$$u_t = u_{xxx} + 6|u|^2 u_x + 3(|u|^2)_x u$$

bright soliton with internal freedom

double hump soliton



oscillating double hump soliton

degeneration of coupled system

3-component CKP hierarchy

Dark soliton of Sasa-Satsuma eq.?

$$u_t = u_{xxx} + 6\varepsilon(|u|^2 - 1)u_x + 3\varepsilon(|u|^2)_x u$$

$\varepsilon = \pm$

$\varepsilon = +$: focusing

$\varepsilon = -$: defocusing

$$iu_t = u_{xx} + 2\varepsilon|u|^2 u + i(u_{xxx} + \cdots)$$

Purpose

Construct solutions for plane wave boundary condition $|u| \to 1$ as $x \to \pm\infty$ for $\varepsilon = \pm$.

$$u = \frac{g}{f} \underbrace{e^{i(\lambda x - \lambda^3 t)}}_{\text{carrier wave}} \qquad (f : \text{real})$$

$$\begin{cases} \dfrac{g}{f} \to 1 & \text{as } x \to -\infty \\ \dfrac{g}{f} \to \alpha & \text{as } x \to +\infty \quad (|\alpha| = 1) \end{cases}$$

$$\begin{cases} D_x^2 f \cdot f = 4\varepsilon(|g|^2 - f^2) \\ (D_x^3 - D_t + 3i\lambda D_x^2 + 3(2\varepsilon - \lambda^2)D_x + 6i\varepsilon\lambda)\, g \cdot f = 6i\varepsilon\lambda r g \\ (D_x + 2i\lambda)\, g \cdot g^* = 2irf \end{cases}$$

r : auxiliary variable (real) $\qquad \boxed{\lambda \neq 0}$

- CKP hierarchy (cf. bright soliton)
- 1-component (for non-vanishing boundary condition)
- 2-different discrete shift for g and g^*

$$f = \tau_{00} \qquad g = \tau_{10} \qquad g^* = \tau_{01} \qquad r = \tau_{11}$$

(cf. $f = \tau_0$, $g = \tau_1$, $g^* = \tau_{-1}$ for NLS)

$$\text{CKP} \Rightarrow \tau_{10} = \tau_{0,-1}, \quad \tau_{01} = \tau_{-1,0}$$

- complicated reduction condition

$$D_x^2 f\cdot f = 4\varepsilon(|g|^2 - f^2)$$

$$\Uparrow$$

$$\begin{cases} D_x D_u f\cdot f = 2(g g^* - f^2) \\ D_x D_v f\cdot f = 2(g^* g - f^2) \\ D_u + D_v = \varepsilon D_x \qquad \text{(reduction)} \end{cases}$$

$$\Uparrow \quad f = \tau_{00},\ g = \tau_{10} = \tau_{0,-1},\ g^* = \tau_{01} = \tau_{-1,0}$$

$$\begin{cases} D_x D_u \tau_{00}\cdot\tau_{00} = 2(\tau_{10}\tau_{-1,0} - \tau_{00}\tau_{00}) \\ D_x D_v \tau_{00}\cdot\tau_{00} = 2(\tau_{01}\tau_{0,-1} - \tau_{00}\tau_{00}) \\ D_u + D_v = \varepsilon D_x \end{cases}$$

Solution $\tau_{kl} = \det\left(m_{ij}^{kl}\right)_{1\le i,j\le 2N}$

$$m_{ij}^{kl} = \delta_{j,2N+1-i} + \frac{1}{p_i+p_j}\left(\frac{i\lambda-p_i}{i\lambda+p_j}\right)^k\left(\frac{i\lambda+p_i}{i\lambda-p_j}\right)^l e^{\xi_i+\xi_j}$$

$$\xi_i = p_i x + p_i^3 t + \frac{1}{i\lambda-p_i}u - \frac{1}{i\lambda+p_i}v + \xi_i^{(0)}$$

$(u=v=0)$

$$\tau_{kl} \doteq \det\left(\delta_{j,2N+1-i}\, e^{-\xi_i-\xi_{2N+1-i}} + \frac{1}{p_i+p_j}\left(\frac{i\lambda-p_i}{i\lambda+p_j}\right)^k\left(\frac{i\lambda+p_i}{i\lambda-p_j}\right)^l\right)$$

reduction condition $\left(D_u+D_v=\varepsilon D_x \quad \text{i.e. } (\partial_u+\partial_v)\tau_{kl} = \varepsilon\,\partial_x\tau_{kl}\right)$

$$\frac{1}{i\lambda-p_i}+\frac{1}{i\lambda-p_{2N+1-i}}-\frac{1}{i\lambda+p_i}-\frac{1}{i\lambda+p_{2N+1-i}} = \varepsilon\left(p_i+p_{2N+1-i}\right)$$

$$p_i = q_i + i\sqrt{\lambda^2 - \varepsilon - q_i^2 + \sqrt{1 - 4\lambda^2(\varepsilon + q_i^2)}}$$

$$p_{2N+1-i} = q_i - i\sqrt{\qquad ''\qquad} \qquad 1 \leq i \leq N$$

(1) $1 - 4\lambda^2(\varepsilon + q_i^2) > 0$

$\lambda^2 - \varepsilon - q_i^2 + \sqrt{1 - 4\lambda^2(\varepsilon + q_i^2)} > 0 \quad \Longrightarrow \quad p_{2N+1-i} = p_i^*$

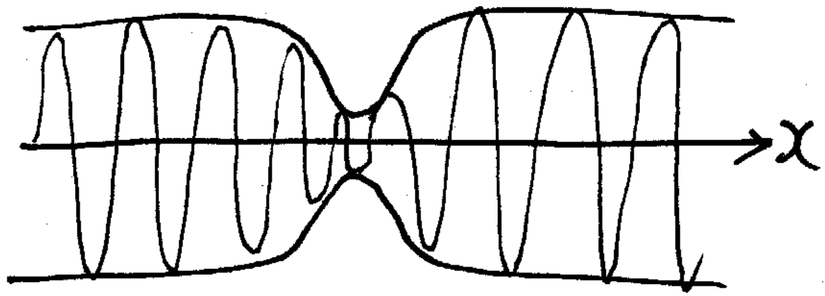
(2) $1 - 4\lambda^2(\varepsilon + q_i^2) > 0$

$\lambda^2 - \varepsilon - q_i^2 - \sqrt{1 - 4\lambda^2(\varepsilon + q_i^2)} < 0 \quad \Longrightarrow \quad p_i, \, p_{2N+1-i}$: real

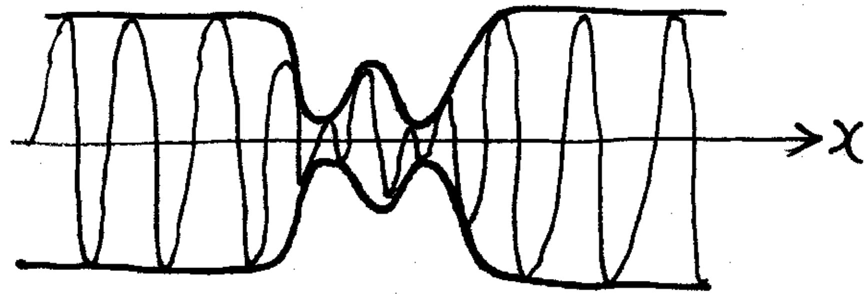
possible to get regular solutions for both $\varepsilon = +1$ and -1.

$\varepsilon = -1$ defocusing

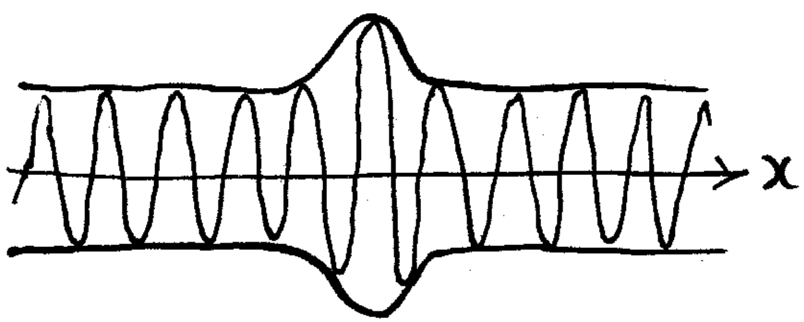
hole soliton



double hole soliton without oscillation



$\varepsilon = +1$ focusing



- 1-component CKP + reduction

$\Rightarrow$ Sasa-Satsuma eq. and its solution for non-vanishing boundary condition for both defocusing ($\varepsilon = -1$) and focusing ($\varepsilon = +1$) cases.

- Soliton of double hole type (for $\varepsilon = -1$)
- No oscillation of double hole

No internal freedom (Soliton is characterized by its wave number and phase parameter only.)

- Breather type solution for $\varepsilon = +1$ and -1.

(homoclinic orbit solution)

Ablowitz and Herbst, SIAM J. Appl. Math. <u>50</u> ('90) 339.