Baroclinic Equivalent and Nonequivalent Barotropic Modes for Rotating Stratified Flows

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OUTLINE

- Introduction
- Baroclinic equivalent barotropic (EB) modes
- Baroclinic non-EB elliptic and hyperbolic modes
- Summary and discussions

Why rotating stratified flow?

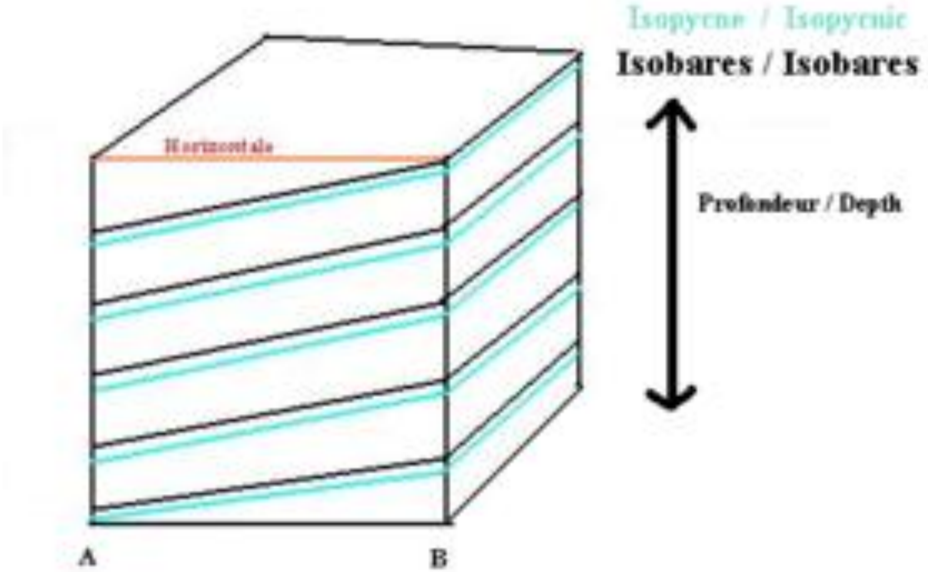
- All the natural phenomena on the earth should be treated under rotating coordinate.
- Atmosphere is separated into thermal layers due to temperature variations.
- Stratification (water) can occur due to gradients in salinity or temperature.

Baroclinic & Barotropic mode

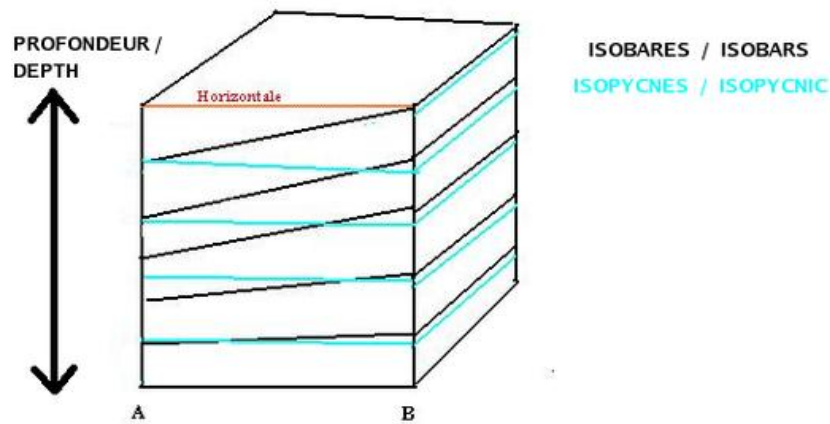
- Baroclinic mode (斜压模式): a baroclinic atmosphere is one for which the density depends on both the temperature and the pressure.

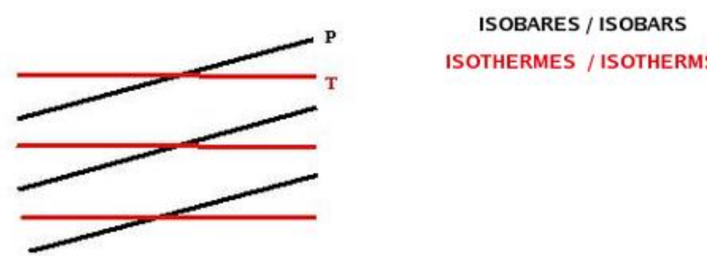
- Barotropic mode (正压模式): barotropic atmosphere, for which the density depends only on the pressure, that is, isobaric surfaces and isopycnal surfaces coincide.

Fluide barotrope / Barotropic fluid



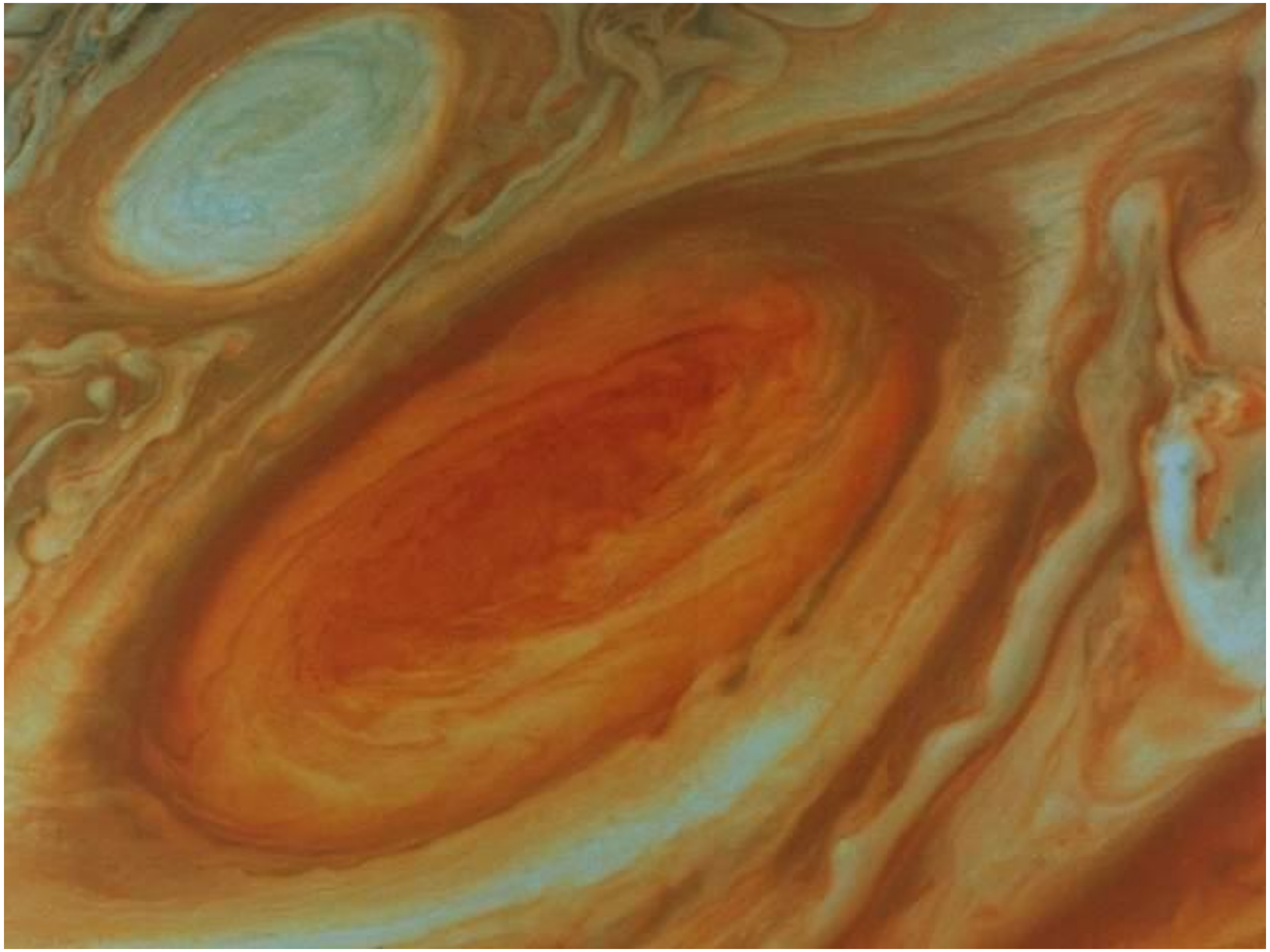
Fluide barocline / Baroclinic fluid





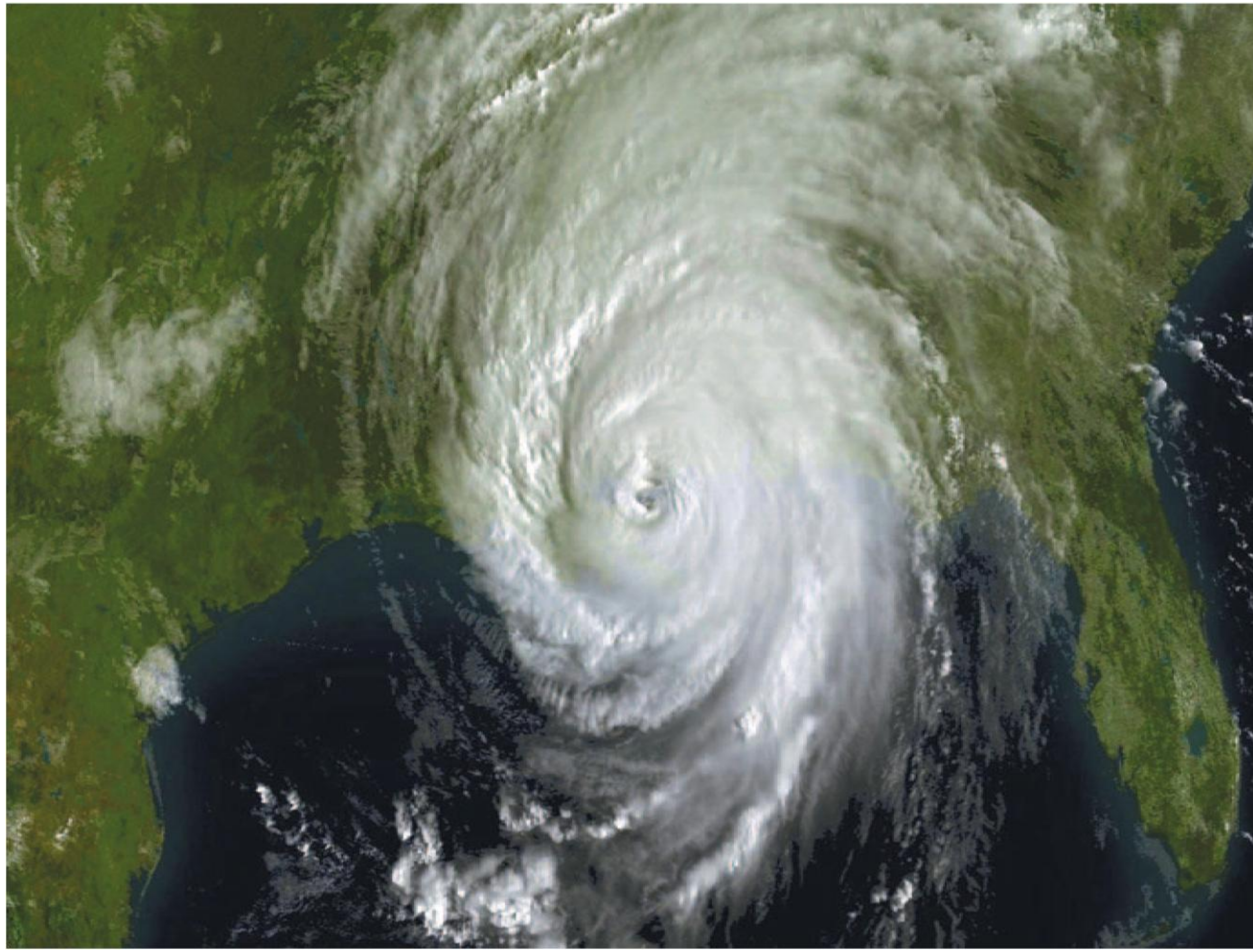
In fluid dynamics, the **baroclinity** is a measure of the stratification in a fluid.

1). Baroclinic vorticity: the Great Jupiter's Red Spot



J. C. McWilliams, J. B.Weiss and I. Yavneh, Science, 264,410 (1994)
E. J. Hoppinger, F. K. Browand, Nature, 295, 393 (1982).

2). Hurricanes and tropical cyclone



Hurricane Katrina 2005, S. Y. Lou, M. Jia, X. Y. Tang and F. Huang, Phys. Rev. E, 75 056318 (2007)

A steady baroclinic laminar model:

$$uu_x + vu_y - fv = -p_x,\ uv_x + vv_y + fu = -p_y,$$
$$p_z = -\rho, \quad u_x + v_y = 0, \quad u\rho_x + v\rho_y = 0, \qquad (1)$$

- f: Coriolis parameter
- p: pressure perturbations divided by a mean density ρ_0
- ρ: density perturbation scaled by ρ_0/g
- u, v: horizontal velocities

Remark:

- To derive the model, the author has hypothesized that the formation mechanism for coherent structures in rotating stratified flows is fundamentally baroclinic.
- Vertical velocity w has been dropped out because of its weakness.
- It is the late-time equilibrium state in the free decay of rotating stratified.

C. Sun, J. Atmos. Sci., 65, 2740 (2008)

- one type of special barotropic tilting vortex solution and four special types of baroclinic equivalent-barotropic (EB) vortices
- either barotropic or EB, a conjecture is proposed: ***Baroclinic solutions to the model are always EB***.

Questions:

- How to find possible baroclinic modes of the model?
- Is the conjecture correct?

2. Baroclinic EB modes

incompressible condition $u_x + v_y = 0$

stream function $u = -\psi_y, \ v = \psi_x$

$$J(\psi, K_z) - (\zeta + f)J(\psi, \psi_z) = 0,$$
$$J(\psi, \zeta) = 0.$$

$$K \equiv \frac{1}{2}\psi_x^2 + \frac{1}{2}\psi_y^2 \qquad \zeta \equiv \psi_{xx} + \psi_{yy}$$

$$J(a, b) \equiv a_x b_y - a_y b_x$$

Definition:

A baroclinic flow is EB if the stream lines on each plane align vertically or, equivalently, if the horizontal velocity vector does not change direction vertically.

The fluid (1) is called baroclinic EB iff $\psi_x = F\psi_y$ for arbitrary F=F(x, y).

$$\psi_x = F\psi_y$$

$$J(\psi, K_z) - (\zeta + f)J(\psi, \psi_z) = 0,$$
$$J(\psi, \zeta) = 0.$$

The only two possible cases of baroclinic EB

- A. $\psi_{yz} = 0$
- B. $F_y + FF_x = 0$

A. $\psi_{yz} = 0$ Baroclinic EB with an arbitrary nonlinear Poison flow

The stream function is $\psi = \phi(x,y) + \psi_0(z)$

where

$$\phi_{xx} + \phi_{yy} = g(\phi)$$

and the solutions to (1) are

$$u = -\phi_y, \ v = \phi_x,$$
$$p = \frac{1}{2}\phi_y^2 + \int \phi_x \phi_{yy} \mathrm{dx} + f\phi + \phi_0(y) + p_0(z),$$

If we take

$$g(\phi) = -(b^2 + c^2)(1 + a^2)\sin(\phi)$$

the stream function is

$$\psi = \phi = 4\arctan\left(a\mathrm{sn}(bx, m)\mathrm{sn}(cy, n)\right)$$

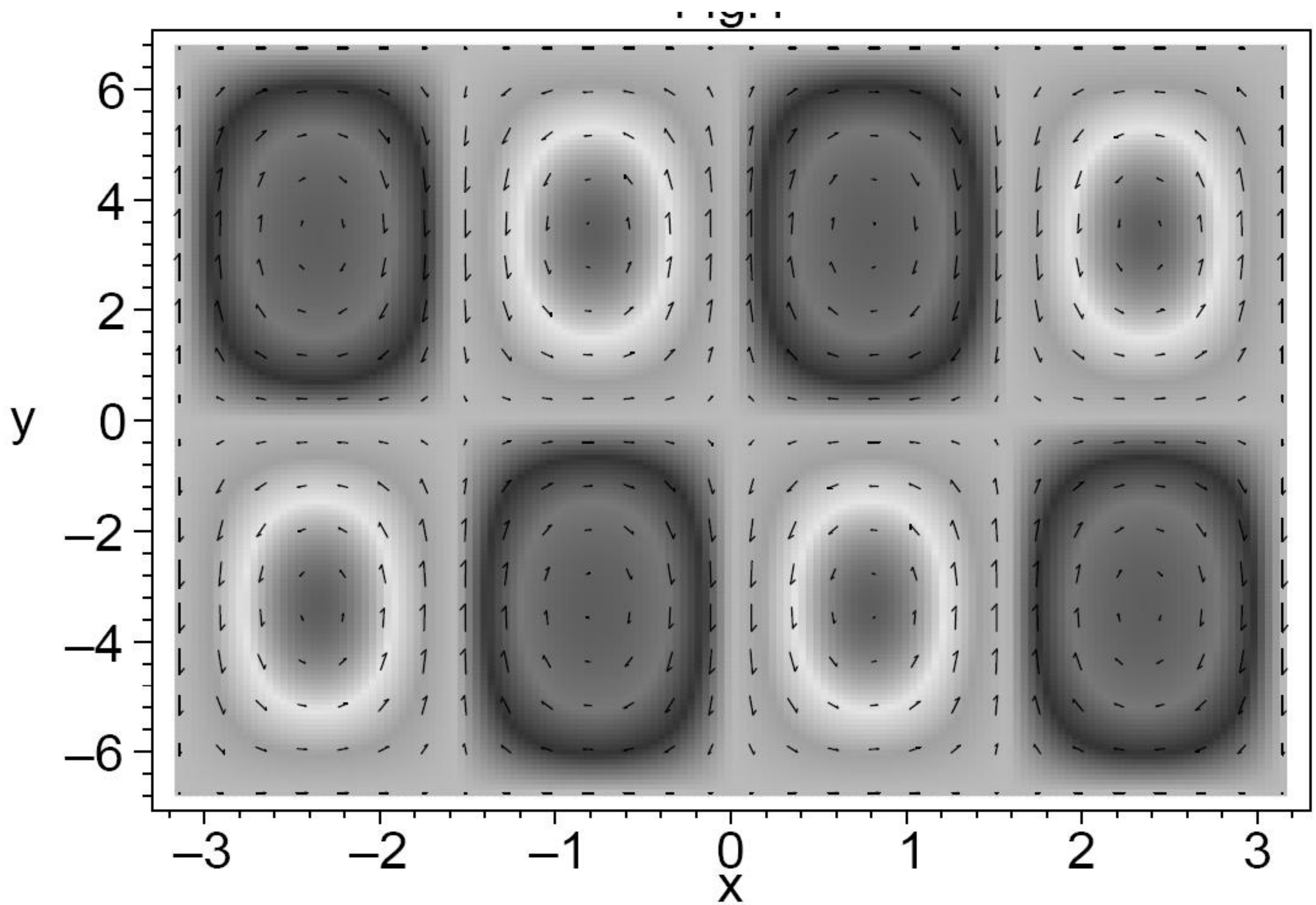
and other physcial quantities are

$$u = -\frac{4ac\mathrm{sn}(bx, m)\mathrm{cn}(cy, n)\mathrm{dn}(cy, n)}{1 + a^2\mathrm{sn}^2(bx, m)\mathrm{sn}^2(cy, n)},$$

$$v = \frac{4ab\mathrm{cn}(bx, m)\mathrm{dn}(bx, m)\mathrm{sn}(cy, n)}{1 + a^2\mathrm{sn}^2(bx, m)\mathrm{sn}^2(cy, n)},$$

$$p = f\phi - 8b^2\frac{c^2\mathrm{sn}^2(bx, m) + b^2\mathrm{sn}^2(cy, n)}{1 + a^2\mathrm{sn}^2(bx, m)\mathrm{sn}^2(cy, n)} + g$$

density plot and the corresponding velocity field for the vortex street solution with $a = \frac{1}{8},\ b = 2,\ c = \frac{1}{2}$



B. $F_y + FF_x = 0$ Baroclinic EB symmetric circulations

$$\psi = \psi_0(r,\ z), \quad r \equiv c_1(x^2 + y^2) + c_2 x + c_3 y,$$
$$u = -\psi_{0r}(2c_1 y + c_3),\ v = \psi_{0r}(2c_1 x + c_2),$$
$$p = 2c_1 \int \psi_{0r}^2 \mathrm{dr} + f\psi_0 + p_0(z),$$

$$r \equiv x^2 + y^2 - 2$$

the stream function is

$$\psi = \operatorname{sech}(1-z)\arctan\{\sinh[(1-z)r]\}$$

and the velocity are

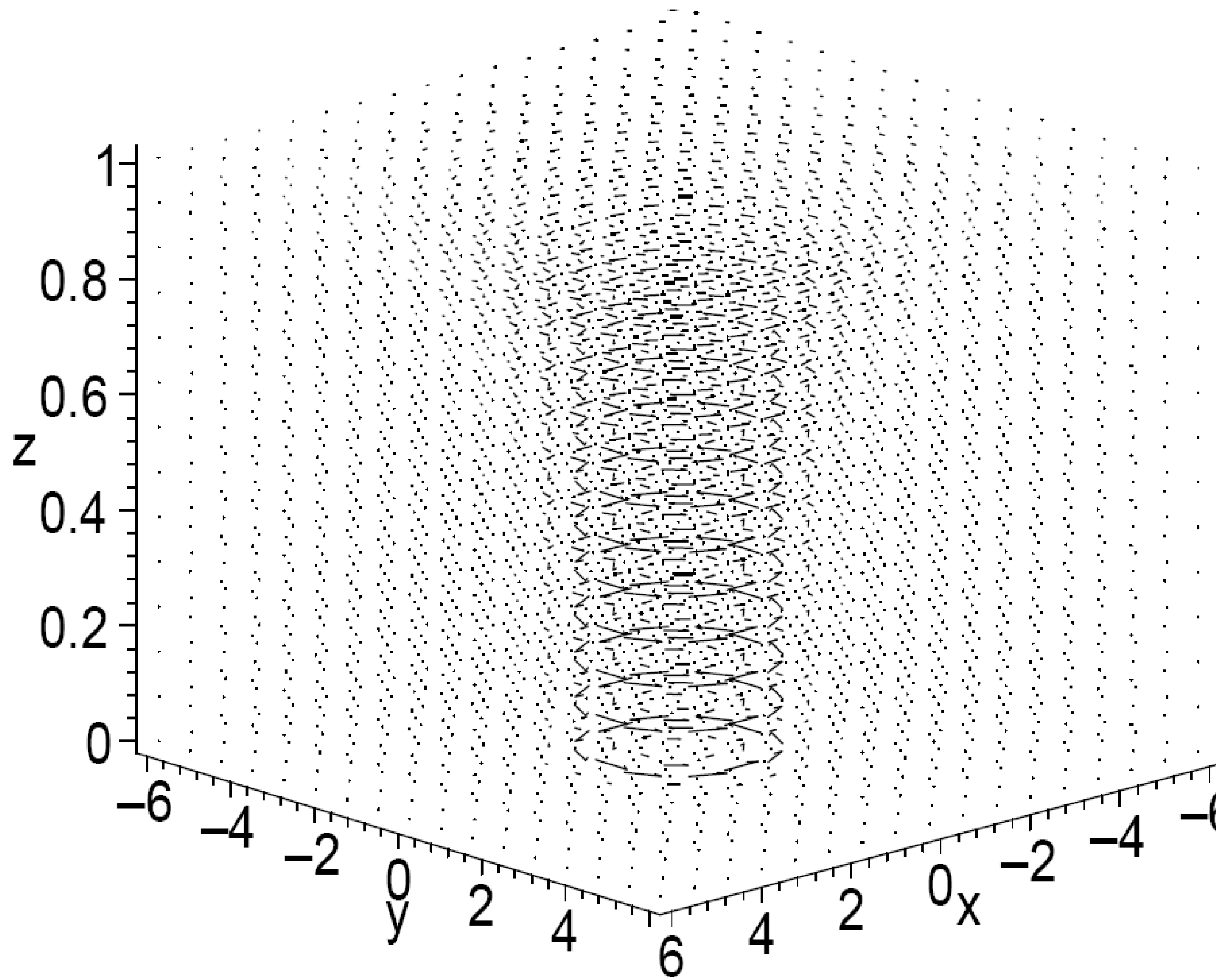
$$u = 2(z-1)y\operatorname{sech}[(1-z)r]\operatorname{sech}(1-z)$$

$$v = 2(1-z)x\operatorname{sech}[(1-z)r]\operatorname{sech}(1-z)$$

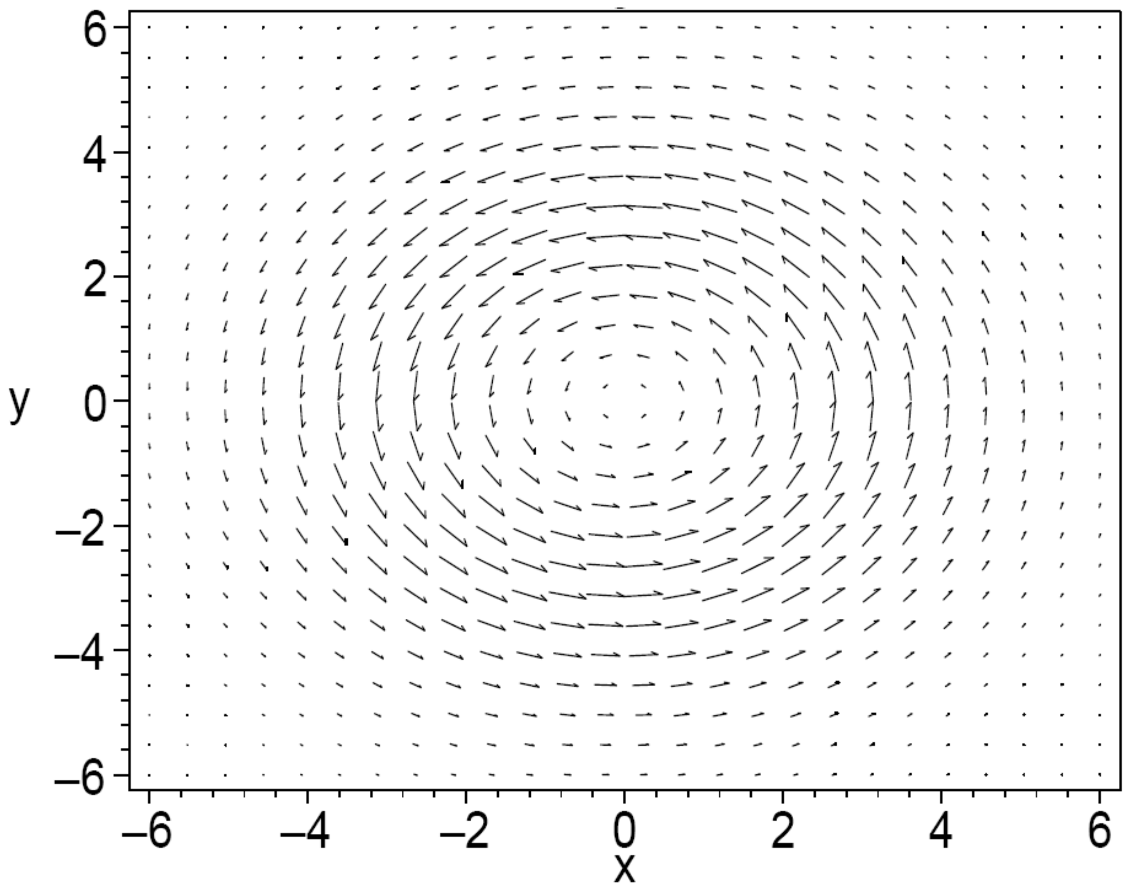
the pressure is

$$p = 2(1-z)\tanh[(1-z)r]\operatorname{sech}^2[(z-1)r] - 2f\operatorname{sech}(1-z)\arctan\{\exp[(z-1)r]\} + p_0$$

The 3-dimensional vortex solution described by the vector velocity field.



The hurricane like structure which is the bird's eye view.



3. Baroclinic non-EB elliptic and hyperbolic modes

An elliptic or hyperbolic mode is defined as its stream lines are elliptic and/or hyperbolic curves.

The stream function has the form

$$\psi = \psi(a_1(z)(x - x_0(z))^2 + a_2(z)(y - y_0(z))^2, z),$$

- Baroclinic elliptic or hyperbolic non-EB modes with rotational shape as the height z changes
- Baroclinic elliptic or hyperbolic non-EB mode with skew center

$$\psi_\pm = \pm\frac{1}{2h} g_\pm \eta_\pm^2 + \psi_0$$

$$u_\pm = \frac{1}{h} g_\pm (y - y_0), \; v = g_\pm h (x - x_0),$$

$$p_\pm = p_0 + \frac{1}{2} g_\pm \left(g_\pm \eta^2 \pm f \frac{\eta_\pm^2}{h} \right),$$

$$\eta_\pm^2 \equiv (y - y_0)^2 \pm h^2 (x - x_0)^2$$

$$\eta^2 \equiv (x - x_0)^2 + (y - \bar{y_0})^2$$

$$g_+ \equiv c_1 - \operatorname{arctanh}(h)$$

$$g_- \equiv c_1 - \arctan(h)$$

The “+” sign is related to the baroclinic elliptic circulation while the “-” sign corresponds to the baroclinic hyperbolic wave case.

$$\psi_\pm = \pm\frac{1}{2h} g_\pm \eta_\pm^2 + \psi_0$$

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$$\eta^2 \equiv (x - x_0)^2 + (y - y_0)^2$$

$$g_+ \equiv c_1 - \operatorname{arctanh}(h)$$

$$g_- \equiv c_1 - \arctan(h)$$

The circulation center is independent of the height z.

$$\psi_{\pm} = \pm \frac{1}{2h} g_{\pm} \eta_{\pm}^{2} + \psi_{0}$$

$$u_{\pm} = \frac{1}{h} g_{\pm} (y - y_{0}), \;\; v = g_{\pm} h (x$$

$$p_{\pm} = p_{0} + \frac{1}{2} g_{\pm} \left(g_{\pm} \eta^{2} \pm f \frac{\eta_{\pm}^{2}}{h} \right)$$

The length of the elliptic axes are changeable as z and then the circulation shape is rotated as z changes.

$$\eta_{\pm}^{2} \equiv (y - y_{0})^{2} \pm h^{2} (x - x_{0})^{2}$$

$$\eta^{2} \equiv (x - x_{0})^{2} + (y - y_{0})^{2}$$

$$g_{+} \equiv c_{1} - \mathrm{arctanh}(h)$$

$$g_{-} \equiv c_{1} - \arctan(h)$$

$$\psi_\pm = \pm \frac{1}{2h} g_\pm \eta_\pm^2 + \psi_0$$

$$u_\pm = \frac{1}{h} g_\pm (y - y_0), \; v = g_\pm h (x - x_0),$$

$$p_\pm = p_0 + \frac{1}{2} g_\pm \left(g_\pm \eta^2 \pm f \frac{\eta_\pm^2}{h} \right),$$

$$\eta_\pm^2 \equiv (y - y_0)^2 \pm$$

$$\eta^2 \equiv (x - x_0)^2 +$$

$$g_+ \equiv c_1 - \mathrm{arct}$$

$$g_- \equiv c_1 - \arctan(h)$$

All the quantities, the stream function, the velocity field and the pressure and density, possess elliptic distributions.

$$\psi_\pm = \pm\frac{1}{2h} g_\pm \eta_\pm^2 + \psi_0$$

$$u_\pm = \frac{1}{h} g_\pm (y - y_0), \; v = g_\pm h(x - x_0),$$

$$p_\pm = p_0 + \frac{1}{2} g_\pm \left(g_\pm \eta^2 \pm f \frac{\eta_\pm^2}{h} \right),$$

$$\eta_\pm^2 \equiv (y - y_0)^2 \pm h^2 (x - x_0)^2$$

$$\eta^2 \equiv (x - x_0)^2 + (y - y_0)^2$$

$$g_+ \equiv c_1 - \text{arctanh}(h)$$

$$g_- \equiv c_1 - \arctan(h)$$

The rotation direction of the vortex may be changeable if $g_+ = c_1 - \text{arctanh}(h) = 0$ has a solution.

- Baroclinic elliptic or hyperbolic non-EB modes with rotational shape as the height z changes
- Baroclinic elliptic or hyperbolic non-EB mode with skew center

Baroclinic elliptic or hyperbolic non-EB mode with skew center

$$\psi = c(x - x_0(z))^2 - \frac{f}{2}(y - y_0)^2 + \psi_0(z),$$
$$u = f(y - y_0), \ v = 2c(x - x_0(z)),$$
$$p = p_0(z) + \frac{1}{2}f(2c + f)(y - y_0)^2,$$

c is arbitrary constant and x_0, y_0 and ψ_0 are arbitrary functions of z.

Baroclini… on-EB mode w…

c<0, the solution is related to the baroclinic elliptic non-EB circulation; c>0 is the baroclinic hyperbolic non-EB mode.

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The center is changeable as the height changes while the length of axes of the circulation is independent of z.

c is arbitrary constant a...
arbitrary functions of z...

Baroclini[c] ... [n]on-EB

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$$p = p_0(z) + \frac{1}{2}f(2c + f)(y - y_0)^2,$$

The pressure and the density distributions have no circulation structure though the stream function and the velocity field do.

...tant a... of z

The center is changeable as the height changes while the length of axes of the circulation is independent of z.

Conjecture: *Baroclinic solutions to the model are always EB.*

- Baroclinic elliptic or hyperbolic non-EB modes with rotational shape as the height z changes
- Baroclinic elliptic or hyperbolic non-EB mode with skew center

The conjecture is disproved!

4. Summary and Discussion

- All the possible baroclinic EB models are obtained.

  1. Baroclinic EB with an arbitrary nonlinear Poison flow

  2. Baroclinic EB symmetric circulations

- All the possible (two types of) elliptic circulations and/or hyperbolic modes are found.

  Disproves the conjecture!

Thank you!