N-soliton solutions of two-dimensional soliton cellular automata

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Motivation

Soliton solutions of continuous and discrete soliton equations

Grammian $\Leftrightarrow$ Wronskian

$\Downarrow$

Hirota form

(perturbation form)

$\Downarrow$ Ultradiscretization

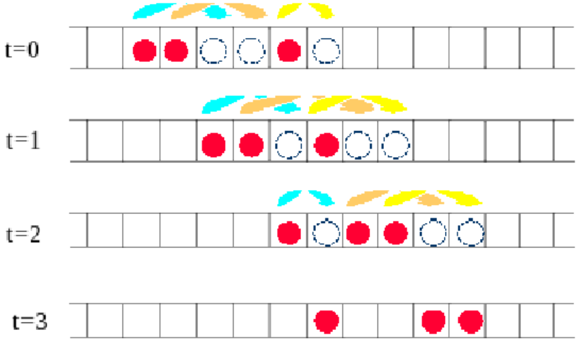
?????

Q1: Ultradiscrete (tropical) analogue of Wronskian and Grammian including all types of line soliton solutions?

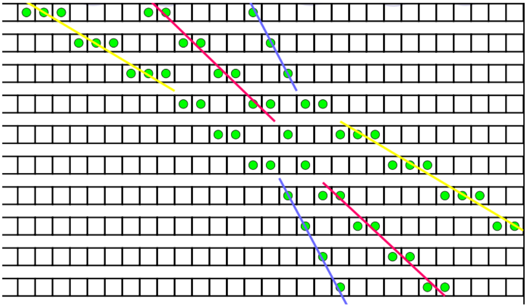
Q2: Grammian form for general line soliton solutions for 2-dimensional soliton systems??

Soliton Cellular Automata (SCA)= Box and Ball System

D. Takahashi & J. Satsuma (1989) J. Phys. Soc. Japan



3 soliton interaction



$$T_j^{t+1} = -\max(T_j^t - 1, \sum_{i=-\infty}^{j-1} (T_i^{t+1} - T_i^t))$$

Ultradiscretization

T.Tokihiro, D.Takahashi, J.Matsukidaira, J.Satsuma: Phys. Rev. Lett. 76 (1996)

The relationship between Soliton equations and Soliton Cellular Automata

Key formula

$$\lim_{\epsilon \to 0+} \epsilon \ln(e^{A/\epsilon} + e^{B/\epsilon}) = \max(A, B)$$

KdV eq. ⇔ semi-discrete KdV eq. ⇔ discrete KdV eq. ⇒ SCA

space discretization — time discretization — ultradiscretization

KdV equation $V_t + 6VV_x + V_{xxx} = 0$

$\Downarrow$ Space discretization

Semi-discrete KdV (Lotka-Volterra) equation $\frac{dv_n}{dt} = v_n(v_{n-1} - v_{n+1})$

$\Downarrow$ Time discretization

Discrete KdV (discrete Lotka-Volterra) equation

$\frac{u_n^{t+1} - u_n^t}{\delta} = u_n^t u_{n-1}^t - u_n^{t+1} u_{n+1}^{t+1}$

$\Downarrow$ Ultradiscretization

Ultradiscrete KdV equation

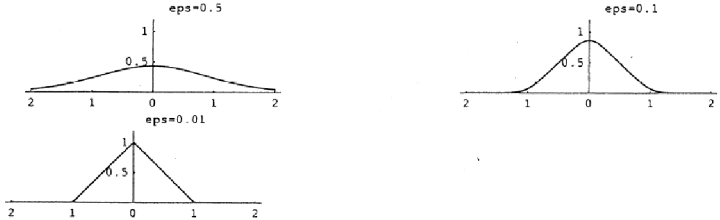
$U_n^{t+1} - U_n^t = \max(0, U_{n-1}^t - 1) - \max(0, U_{n+1}^{t+1} - 1)$

$\Downarrow$ $U_n^t = \sum_{j=-\infty}^{n+1} T_j^t - \sum_{j=-\infty}^{n+1} T_j^{t+1}$

Box-Ball system $T_j^{t+1} = -\max(T_j^t - 1, \sum_{i=-\infty}^{j-1}(T_i^{t+1} - T_i^t))$

1 soliton solution of the Toda lattice

$$\lim_{\varepsilon \to +0} \varepsilon \log(e^{A/\varepsilon} + e^{B/\varepsilon} + \cdots) = \max(A, B, \cdots)$$



$$a + b \Rightarrow \max(A, B), \quad ab \Rightarrow A + B$$

Question

How to obtain exact solutions?

$\downarrow$

Take the ultradiscrete limit of exact solutions of difference equations.

However, we can take ultradiscrete limit of exact solutions only in which all terms are non-negative! It is not easy to find which type of exact solutions exists in ultradiscrete limit.

Many works use the Hirota form (which is equivalent to Gram type) of N-soliton solutions. $\longrightarrow$ Some soliton solutions were missed in previous studies!

Question

R. Hirota & D. Takahashi(J.Phys.Soc.Japan, 2007):

They proposed a new type of solutions of ultradiscrete soliton equations.

It is similar to 'Permanent'.

Q1. Show the root of permanent type solutions in ultradiscrete soliton systems. Derive permanent type solutions systematically from the Determinant solutions.

Q2. Systematic method to construct simple forms of soliton solutions of 2D and 1D ultradiscrete soliton equations.

Determinant and Permanent

Determinant

$$\det(A) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} a_{i\,\sigma(i)}$$

Permanent

$$\mathrm{perm}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\,\sigma(i)}$$

where

$$A = (a_{ij})_{1 \le i,j \le N}$$

is a matrix.

e.g. 2×2-matrix

$\det(A) = a_{11}a_{22} - a_{12}a_{21}, \quad \mathrm{perm}(A) = a_{11}a_{22} + a_{12}a_{21}.$

Permanent-type solution of Ultradiscrete Systems

Ultradiscrete Lotka-Volterra (ultradiscrete KdV) equation

$$V_n^{m+1} - V_n^m = \max(0, V_{n-1}^m - 1) - \max(0, V_{n+1}^{m+1} - 1)$$

$$V_n^m = F_{n-1}^m + F_{n+2}^{m+1} - F_n^m - F_{n+1}^{m+1}$$

$$F_n^m = \frac{1}{2}\max[\,|s_i(m + 2(j-1), n)|\,]_{1\le i,j\le N}$$

$$\max[a_{ij}] = \max_{\sigma\in S_n}\left(\sum_{i=1}^{n} a_{i,\sigma(i)}\right)$$

Note

$$\lim_{\epsilon\to 0+} \epsilon \ln(\mathrm{perm}[e^{a_{ij}/\epsilon}]) = \max[a_{ij}]$$

Does this solution exist only in ultradiscrete systems???

Hirota Bilinear form, τ-function

KP (Kadomtsev-Petviashvili) equation

$$\frac{\partial}{\partial x}\left(-4\frac{\partial u}{\partial t}+\frac{\partial^3 u}{\partial x^3}+6u\frac{\partial u}{\partial x}\right)+3\frac{\partial^2 u}{\partial y^2}=0\,.$$

Transformation

$$u(x,y,t)=2\frac{\partial^2}{\partial x^2}\log\tau(x,y,t)\,,$$

Hirota bilinear form

$$[-4D_xD_t+D_x^4+3D_y^2]\tau\cdot\tau=0\,,$$

$$D_x^m f\cdot g=(\partial_x-\partial_{x'})^m f(x,y,t)g(x',y,t)|_{x=x'}\,.$$

Hirota Bilinear form, τ-function

$$\tau = \begin{vmatrix} f_1^{(0)} & \cdots & f_N^{(0)} \\ \vdots & \ddots & \vdots \\ f_1^{(N-1)} & \cdots & f_N^{(N-1)} \end{vmatrix}, \quad f_i^{(n)} := \frac{\partial^n}{\partial x^n} f_i \,.$$

where $f_i(x,y,t)$ is a set of M linearly independent solutions of the linear equations

$$\frac{\partial f_i}{\partial y} = \frac{\partial^2 f_i}{\partial x^2}, \qquad \frac{\partial f_i}{\partial t} = \frac{\partial^3 f_i}{\partial x^3},$$

for $1 \le i \le N$. A finite dimensional solutions:

$$f_i(x,y,t) = \sum_{j=1}^{M} a_{ij} E_j(x,y,t)\,, \quad i = 1\,, \cdots, N < M\,,$$

$$E_j(x,y,t) := e^{\theta_j} = \exp(k_j x + k_j^2 y + k_j^3 t + \theta_j^0)\,.$$

KP τ-function A-matrix determines the type of soliton & non-negativity

$$\tau(x,y,t) = \det(A\Theta K)$$
$$= \sum_{1\le m_1<\cdots<m_N\le M} V_{m_1,\ldots,m_N} A_{m_1,\ldots,m_N} \exp\left(\theta_{m_1,\cdots,m_N}\right),$$

$A = (a_{n,m})$ is the $N \times M$ coefficient matrix,
$\Theta = \mathrm{diag}(e^{\theta_1}, \cdots, e^{\theta_M})$,
the $M \times N$ matrix K is given by $K = (k_m^{n-1})$,
$\theta_{m_1,\cdots,m_N}(x,y,t) = \theta_{m_1} + \cdots + \theta_{m_N}$, $k_1 < k_2 < \cdots < k_M$
$V_{m_1,\ldots,m_N}$ is the vandermonde determinant
$V_{m_1,\ldots,m_N} = \prod_{1\le j<l\le N}(k_{m_j} - k_{m_l})$, and $A_{m_1,\ldots,m_N}$ is determined by columns the $N \times N$-minor of A.

A-matrix for non-singular soliton solution

τ-function is totally non-negative if A-matrix is the following:

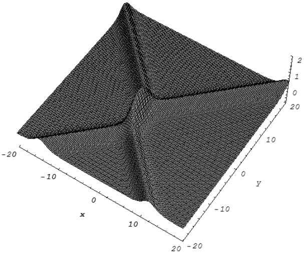
$$A_{\rm O} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad A_{\rm P} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix},$$

$$A_{\rm T} = \begin{pmatrix} 1 & 0 & - & - \\ 0 & 1 & + & + \end{pmatrix},$$

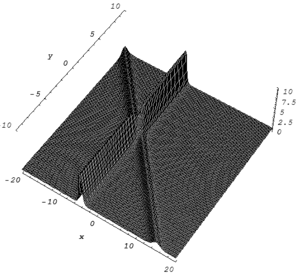
$$A_{\rm I} = \begin{pmatrix} 1 & 1 & 0 & -a \\ 0 & 0 & 1 & 1 \end{pmatrix}, \qquad A_{\rm II} = \begin{pmatrix} 1 & 0 & -a & -a \\ 0 & 1 & 1 & 1 \end{pmatrix},$$

$$A_{\rm III} = \begin{pmatrix} 1 & 0 & 0 & -a \\ 0 & 1 & 1 & 1 \end{pmatrix}, \qquad A_{\rm IV} = \begin{pmatrix} 1 & 0 & -a & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix},$$

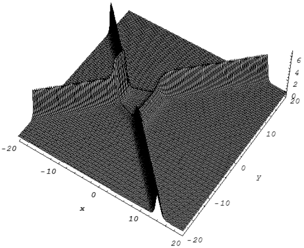
Y. Kodama (2004), G. Biondini & S. Chakravarty (2006), S. Chakravarty & Y. Kodama (2008)



O-type (2143)



P-type (4321)



T-type (3412)

The 2-dimensional Toda lattice

$$\frac{\partial^2}{\partial x \partial t} Q_n(x,t) = e^{Q_{n+1}(x,t)} - 2e^{Q_n(x,t)} + e^{Q_{n-1}(x,t)}$$

$$\frac{\partial^2 \tau_n}{\partial x \partial t}\, \tau_n - \frac{\partial \tau_n}{\partial t}\, \frac{\partial \tau_n}{\partial x} = \tau_{n+1}\, \tau_{n-1} - \tau_n^2$$

$$Q_n(x,t) = \log\left(1 + \frac{\partial^2}{\partial x \partial t} \log \tau_n(x,t)\right) .$$

Discretization of 2D Toda lattice

$$\Delta_l^+ \Delta_m^- Q_{l,m,n}$$
$$= V_{l,m-1,n+1} - V_{l+1,m-1,n} - V_{l,m,n} + V_{l+1,m,n-1} \,,$$
$$V_{l,m,n} = (\delta\kappa)^{-1} \log[1 + \delta\kappa \,(\exp Q_{l,m,n} - 1)] \,,$$

$$\Delta_l^+ f_{l,m,n} = \frac{f_{l+1,m,n} - f_{l,m,n}}{\delta}, \quad \Delta_m^- f_{l,m,n} = \frac{f_{l,m,n} - f_{l,m-1,n}}{\kappa}.$$

$$(\Delta_l^+ \Delta_m^- \tau_{l,m,n})\, \tau_{l,m,n} - (\Delta_l^+ \tau_{l,m,n})\, \Delta_m^- \tau_{l,m,n}$$
$$= \tau_{l,m-1,n+1} \tau_{l+1,m,n-1} - \tau_{l+1,m-1,n} \tau_{l,m,n},$$

$$V_{l,m,n} = \Delta_l^+ \Delta_m^- \log \tau_{l,m,n} \,, \quad Q_{l,m,n} = \log \frac{\tau_{l+1,m+1,n-1} \tau_{l,m,n+1}}{\tau_{l+1,m,n} \tau_{l,m+1,n}} \,.$$

Ultradiscretization of 2D Toda lattice

Using $\lim_{\epsilon \to 0+} \epsilon \ln(e^{A/\epsilon} + e^{B/\epsilon}) = \max(A, B)$, we can create the ultradiscrete 2D Toda lattice

$$\Delta_l^+ \Delta_m^+ v_{l,m,n} = \Delta' \max(0, v_{l,m,n} - \mathrm{r} - \mathrm{s}),$$

$$\Delta' f_{l,m,n} \equiv f_{l+1,m+1,n-1} + f_{l,m,n+1} - f_{l+1,m,n} - f_{l,m+1,n} .$$

Construction of soliton solutions of ultradiscrete 2D Toda lattice

The past research : Some special cases. Take ultradiscrete limit of some special soliton solutions. The solution is not in determinant or permanent.

Idea

Determinant solution of fully discrete 2D Toda

$\Downarrow$ ultradiscretization

Determinant-tye solution of ultradiscrete 2D Toda?

KM & G. Biondini (2004): resonant type determinant-type (actually, permanent-type) solution

Difficulty: Ultradiscretization is available only in which all terms in τ-function are non-negative.

(Resonant-type) determinant solution has total non-negativity.

Theorem

The tau-function of N line soliton solutions of the ultradiscrete 2D Toda lattice, i.e. the tropical tau-function, is

$$\rho(l,m,n) = \lim_{\epsilon\longrightarrow 0+} \epsilon \log \tau^{\epsilon}_{l,m,n} = \lim_{\epsilon\longrightarrow 0+} \epsilon \log |\mathbf{A}\mathbf{\Phi}_{\epsilon}\mathbf{K}_{\epsilon}|$$

where $\mathbf{A} = (a_{ij})_{1\le i\le N, 1\le j\le M}$, $\mathbf{\Phi}_{\epsilon} = \mathrm{diag}(\phi_1, \phi_2, \dots, \phi_M)$,
$\phi_j = k_j^n (1+\delta k_j)^l (1+\kappa k_j^{-1})^{-m} \phi_{j,0}$, $k_j = e^{K_j/\epsilon}$,
$\delta = \mathrm{e}^{-r/\epsilon}$, $\kappa = \mathrm{e}^{-s/\epsilon}$, $\phi_{j,0} = \mathrm{e}^{\psi_{j,0}/\epsilon}$ *and* $\mathbf{K}_{\epsilon} = (k_i^{j-1})_{1\le i,j\le N}$.

Moreover, the tropical tau-function $\rho(l,m,n)$ is a tropical determinant

$$\rho(l,m,n) = |\,\mathbf{A}\Phi_{trop}\mathbf{K}_{trop}\,|_{trop}$$

where Φ_{trop}, $\mathbf{K}_{\mathbf{trop}}$ are tropical matrices and $|\quad|_{trop}$ is a tropical determinant, these are obtained by taking ultradiscrete limit of matrices Φ_ϵ, $\mathbf{K}_\epsilon$ and a determinant. Here a matrix $\mathbf{A}$ must be chosen for satsfying that each term of tau-function $\tau^{\epsilon}_{l,m,n}$ is non-negative. In other words, the tropical τ-function can be considered as ultradiscrete limit of permanent if an appropriate A is chosen because all terms are non-negative.

Soliton solution of ultradiscrete 2D Toda

Theorem

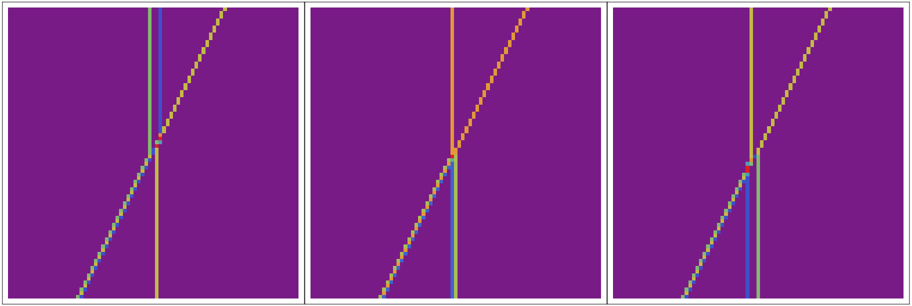
By using Binet-Cauchy formula, a tropical tau-function is expressed in the form of

$$\rho(l,m,n) = \max \left(\phi(h_1, \cdots, h_N) + \sum_{j=1}^{N} (j-1) K_{h_j} \right)$$

where the phase combination is defined by
$\phi(h_1, \cdots, h_N) = \sum_{j=1}^{N} \phi_{h_j}^{trop}$*, the phase is*
$\phi_j^{trop} = nK_j + l \max(0, K_j - r) - m \max(0, -K_j - s) + \psi_{j,0}$.
Note that each term in the tropical tau-function ρ *corresponds to non-zero minor* $A(h_1, h_2, \cdots, h_N)$ *which denotes the* $N \times N$ *minor of a matrix* A *obtained by selecting columns* $h_1, h_2, \cdots, h_N$.

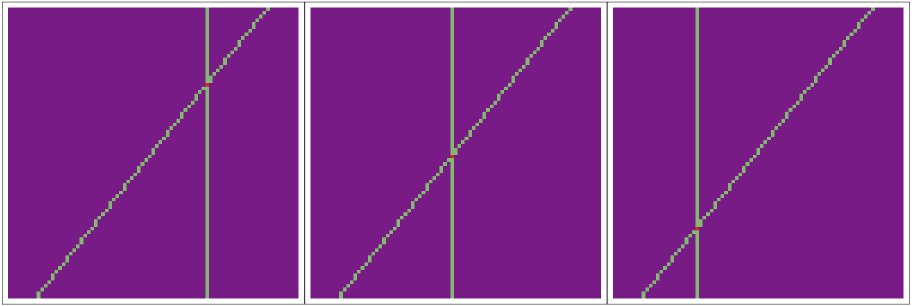
2-soliton (T-type, (3412))

$$A = \begin{pmatrix} 1 & 0 & - & - \\ 0 & 1 & + & + \end{pmatrix},$$



2-soliton (O-type, (2143))

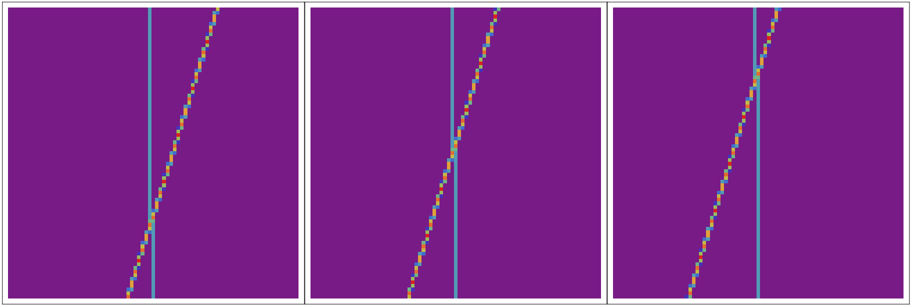
$$A = \begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} \end{pmatrix},$$



Soliton interaction of ultradiscrete 2D Toda

2-soliton (P-type, (4321))

$$A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix},$$



Grammian

Question: Are there general line soliton solutions in Grammian form?

Grammian solution for the KP equation

$$\tau = \det(I + BF\hat{B}^T) = \det(I + CF)\,,$$

where B is an $N \times r$ matrix and $\hat{B}$ is an $N \times N$ matrix, $C = \hat{B}^T B$ is an $N \times r$ matrix of constant coefficients and F is an $r \times N$ matrix whose entries are given by

$$F_{mn} = \frac{e^{\phi(p_m) - \phi(q_n)}}{p_m - q_n}\,, \quad (m = 1, 2, ..., r, \quad n = 1, 2, ..., N).$$

The O-type N-soliton solution is obtained by setting $r = N$ and $C = I$. In this case the resulting τ-function $\tau = \det(I + F)$ is positive for all x, y, t if the parameters are ordered as

$p_N < p_{N-1} < ... < p_1 < q_1 < q_2 < .... < q_N$.

Grammian form of general line soliton solutions

Wronskian form $\tau = \det(AEK)$, $E = \mathrm{diag}(e^{\theta_i})_{i=1}^{M}$, K is vandermonde matrix, A is $N \times M$ coefficient matrix.

Grammian form of general line soliton solutions

$$\hat{\tau} = \det(I + J\hat{E}_2\chi\hat{E}_1{}^{-1}),$$

with $\hat{E}_1 = E_1 D_1 = \mathrm{diag}(e^{\hat{\phi}(q_i)})_{i=1}^{N}$ and $\hat{E}_2 = E_2 D_2 = \mathrm{diag}(e^{\hat{\phi}(p_i)})_{i=1}^{M-N}$, $(\chi)_{ij} = \frac{1}{p_i - q_j}$ for $i = 1, \ldots, M-N$, $j = 1, \ldots, N$. $D_1 = \mathrm{diag}(\prod_{j=1, j\neq i}(q_i - q_j))_{i=1}^{N}$, $D_2 = \mathrm{diag}(\prod_{j=1}(p_i - q_j))_{i=1}^{M-N}$, $E_1 = \mathrm{diag}(e^{\phi(q_i)})_{i=1}^{N}$, $E_2 = \mathrm{diag}(e^{\phi(p_i)})_{i=1}^{M-N}$, $A = (I, J)P$, I is $N \times N$ submatrix of the pivot columns of A, J is $N \times (M-N)$ submatrix of the non-pivot columns of A, P is $M \times M$ permutation matrix.

$$\begin{aligned}\tau &= |AEK| = |(I,J)PEK| = |(I,J)(PEP^{-1})PK| \\ &= \left|(I,J)\begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}\begin{pmatrix} K_1 \\ K_2 \end{pmatrix}\right| = |IE_1K_1 + JE_2K_2| \\ &= |E_1K_1(I + JE_2K_2K_1^{-1}E_1^{-1})| = |E_1K_1|\hat{\tau}\end{aligned}$$

$\hat{\tau} = |I + JE_2K_2K_1^{-1}E_1^{-1})|$, E_1, E_2 are respectively $N \times N$ and $(M-N) \times (M-N)$ block diagonal matrices whose elements are permutations of the set $\{e^{\theta_1}, e^{\theta_2}, \cdots, e^{\theta_M}\}$. K_1, K_2 are respectively $N \times N$ and $(M-N) \times (M-N)$ matrices obtained by permuting the rows of the vandermonde matrix K by P.

P induces a permutation π of the ordered set $\{k_1, ..., k_M\}$:

$$\pi(\{k_1, k_2, ..., k_M\}) = \{q_1, q_2, ..., q_N, p_1, p_2, ..., p_{M-N}\},$$

Using a formula

$$K_2 K_1^{-1} = D_2 \chi D_1^{-1} \,,$$

we obtain

$$\begin{aligned}\hat{\tau} &= |I + J E_2 K_2 K_1^{-1} E_1^{-1}| \\ &= |I + J E_2 D_2 \chi D_1^{-1} E_1^{-1}| = |I + J \hat{E}_2 \chi \hat{E}_1{}^{-1}| \,.\end{aligned}$$

where $(\chi)_{ij} = \frac{1}{p_i - q_j}$ for $i = 1, ..., M - N$, $j = 1, ..., N$,
$D_1 = \mathrm{diag}(\prod_{j=1, j\neq i}(q_i - q_j))_{i=1}^{N}$,
$D_2 = \mathrm{diag}(\prod_{j=1}(p_i - q_j))_{i=1}^{M-N}$, $K_1 = (q_i^{j-1})$, $K_2 = (p_i^{j-1})$,
$E_1 = \mathrm{diag}(e^{\phi(q_i)})_{i=1}^{N}$, $E_2 = \mathrm{diag}(e^{\phi(p_i)})_{i=1}^{M-N}$.

- We can also take ultradiscrete limit of Grammian solution. This will give another form of soliton solutions of ultradiscrete systems.
- Soliton solutions of 1D soliton cellular automata (CA) are obtained by using the reduction technique from line soliton solutions of ultradiscrete 2D-Toda or KP.
- We can recover known solutions for 1D soliton CA. It is easy to find possible soliton-type solutions of ultradiscrete soliton equations if we start from our formula.

Conclusions

- We proposed a systematic method to obtain soliton solutions of ultra-discrete soliton systems.
- Determinant solutions of discrete integrable systems lead to tropical determinant solutions (permanent-type solution) in ultradiscrete limit.
- Why do soliton solutions look like permanent in ultradiscrete?⟶ Because of loss of vandermonde determinant in ultradiscrete limit.
- We can also do same computation in the ultradiscrete KP equation. N-soliton solutions of 1D soliton cellular automata can be obtained from solutions of ultradiscrete 2D Toda and KP equations.
- We found the corresspondence between Wronskian form and Grammian form of general line soliton solutions. This correspondence also survive in the ultradiscrete systems.
- B-type, C-type, D-type 2D Cellular Automata?