The Soliton Equations associated with the Affine Kac–Moody Lie Algebra $G_2^{(1)}$.

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Abstract

We construct in an explict way the soliton equation corresponding to the affine Kac–Moody Lie algebra $G_2^{(1)}$ together with their bihamiltonian structure. Moreover the Riccati equation satisfied by the generating function of the commuting Hamiltonians densities is also deduced. Finally we describe a way to deduce the bihamiltonian equations directly in terms of this latter functions

1 Introduction

One of the most fascinating discoveries of the last decades is surely the deep and fundamental link between the affine Kac–Moody Lie algebras (and their groups as well), and the soliton equations. This relation first studied and described under different points of view in a sequel of seminal papers by Sato [18, 19] Date, Jimbo, Kashiwara and Miwa [11], Hirota [13] Drinfeld and Sokolov [12] and Kac and Wakimoto [16] has inspired almost innumerable further investigations and generalizations (see for example the quite interesting papers of Burroughs, de Groot, Hollowood, Miramontes [1] [2]). Nevertheless, as far as we know, it seems that no explicit description of the hierarchy corresponding in the scheme of Drinfeld and Sokolov to the affine Kac–Moody Lie algebra $G_2^{(1)}$ (even of the first non trivial equations) can be found in the literature, fact probably related to the size of standard realization of G_2 (namely by 7×7 matrices). The aim of this letter is to fill this gap and to show how the bihamiltonian formulation of the Drinfeld-Sokolov reduction [8] [9] [4] makes the computations involved more reasonable. The main ingredient of our construction will be indeed the technique of the transversal submanifold, which can be implemented only in the bihamiltionian reduction theory, and which drops drastically the free variables involved in the computations. The same technique provides also a way to construct a Riccati type equation for the formal Laurent series for the conserved quantities of the corresponding integrable system. Since this equation at least in principle may be iteratively solved in a pure algebraic way, the bihamiltonian technique offers a computational way to construct the whole hierarchy. Moreover what happens in the case of the affine Lie algebras $A_n^{(1)}$ suggests that it could be exists a way to obtain directly the equations of the hierarchy,starting by such conserved quantities, without referring directly to the underlying bihamiltonian structure.

The paper is organized as follows in the first section we perform the bihamiltonian reduction of the Drinfeld-Sokolov hierarchy defined on the affine Kac–Moody Lie algebra $G_2^{(1)}$ obtaining the reduced bihamiltonian structures and the first equations of the hierarchy as well. In the second and last section we explain and perform the so called Frobenius technique for the same algebra obtaining a so called Riccati equation satisfied by the generating function of the conserved densities. Finally we shall show how the knowledge of this function is enough to construct the entire hierarchy of bihamiltonian equation.

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2 The Bihamiltonian Reduction Theory of the Lie algebra $G_2^{(1)}$

The aim of this first section is to obtain the bihamiltonian structure of the soliton equation associated with the Kac–Moody affine Lie algebra $G_2^{(1)}$ in the Drinfeld–Sokolov theory by performing a bihamiltonian reduction process.

For the convenience of the reader, let us start by recalling the main facts of the bihamiltonian reduction theory, referring for more details to the papers [8] [9] [4] where this theory was first developed. A bihamiltonian manifold $\mathcal{M}$ is a manifold equipped with two compatible Poisson structures, i.e., two Poisson tensors P_0 and P_1 such that the pencil $P_\lambda = P_1 - \lambda P_0$ is a Poisson tensor for any $\lambda \in \mathbb{C}$. Let us fix a symplectic submanifold $\mathcal{S}$ of P_0 and consider the distribution $D = P_1\mathrm{Ker}(P_0)$ then the bihamiltonian structure of $\mathcal{M}$, provided that the quotient space $\mathcal{N} = \mathcal{S}/\mathcal{E}$ is a manifold, can be reduced on $\mathcal{N}$ ([8] Prop 1.1).

To construct the reduced Poisson pencil $P_\lambda^{\mathcal{N}}$ from the Poisson pencil P_λ on $\mathcal{M}$ we have to perform the following steps [6]:

1. For any 1–form α on $\mathcal{N}$ we consider the 1–form α^* on $\mathcal{S}$, which obviously belongs to the annihilator E^0 of E in $T^*\mathcal{S}$.

2. We construct a 1–form β on $\mathcal{M}$ which belongs to the annihilator D^0 of D and satisfies
$$i_{\mathcal{S}}^*\beta = \pi^*\alpha \tag{2.1}$$
(i.e., a lifting of α).

3. We compute the vector field $P_\lambda\beta$, which turns out (see [6] Lemma 2.2) to be tangent to $\mathcal{S}$.

4. We project $P_\lambda\beta$ on N:
$$(P_\lambda^{\mathcal{N}})_{\pi(s)}\alpha = \pi_*(P_\lambda)_s\beta.$$

We shall not compute in the next section the reduced bihamiltonian structure related to the affine Kac–Moody Lie algebra $G_2^{(1)}$ using directly the above cited Theorem but rather implementing the technique of the transversal submanifold [9] and [6] in order to avoid most of the computations involved.

A transversal submanifold to the distribution E is a submanifold $\mathcal{Q}$ of $\mathcal{S}$, which intersects every integral leaves of the distribution E in one and only one point. This condition implies the following relations on the tangent space:

$$T_q\mathcal{S} = T_q\mathcal{S} \oplus E_q \qquad \forall q \in \mathcal{Q} \tag{2.2}$$

The importance of the knowledge of a transversal submanifold lies in the following Theorem proved in[8] [4]:

Theorem 2.1 *Let $\mathcal{Q}$ be a transversal submanifold of $\mathcal{S}$ with respect the distribution E. Then $\mathcal{Q}$ is a bihamiltonian manifold isomorphic to the bihamiltonian manifold $\mathcal{N}$ and the corresponding reduced Poisson pair is given by:*

$$\left(P_i^{\mathcal{Q}}\right)_q \alpha = \Pi_*(P_i)_q\tilde{\alpha}) \qquad i = 0, 1 \tag{2.3}$$

where $q \in \mathcal{Q}$, $\alpha \in T_q^*\mathcal{Q}$ $\Pi_* : T_q\mathcal{S} \to T_q\mathcal{Q}$ is the projection with respect the decomposition (2.2) and $\tilde{\alpha} \in T_q^*\mathcal{M}$ satisfies the conditions:

$$\tilde{\alpha}_{|D_q} = 0 \qquad \tilde{\alpha}_{|T_q\mathcal{Q}} = \alpha. \tag{2.4}$$

Actually for our porpoises the hypothesis of this Theorem may be slightly relaxed by considering a submanifold $\mathcal{Q}$ which is only locally transversal (i.e., it satisfies only the weaker condition (2.2)) in this case of course $\mathcal{Q}$ and $\mathcal{N}$ could be only locally isomorphic (see [17] for more details).

The bihamiltionian manifolds which are interesting in, are the bihamiltonian manifold naturally defined on the affine Kac–Moody Lie algebras. An affine non twisted Lie algebra $\widehat{\mathfrak{g}}$ can be realized as a semidirect product of the central extensions of a loop algebra of a simple finite dimensional Lie algebra $\mathfrak{g}$ and a derivation d:

$$\widehat{\mathfrak{g}} = C^\infty(S^1, \mathfrak{g}) \oplus \mathbb{C}d \oplus \mathbb{C}c.$$

Then the Lie bracket of two (typical) elements in $\widehat{\mathfrak{g}}$ of the type

$$X = x_n \otimes x^n + \mu_1 c + \nu_1 d, \qquad Y = x_m \otimes x^m + \mu_2 c + \nu_2 d$$

with $x_n, y_n \in \mathfrak{g}$ and $n, m \in \mathbb{Z}$, $\mu_1, \mu_2, \nu_1, \nu_2 \in \mathbb{C}$ is

$$[X, Y] = [x_n, y_m] \otimes x^{n+m} + (x_n, y_m)c\delta_{n+m,0} - n\nu_2 x_n \otimes x^n + m\nu_1 y_m \otimes x^m \tag{2.5}$$

where $[x_n, y_m]$ is the Lie bracket in $\mathfrak{g}$, and $(\cdot, \cdot)$ is the killing form of $\mathfrak{g}$. (and finally δ is the usual Kronecker delta). In what follows the derivation d will not play any role. Being $\widehat{\mathfrak{g}}$ a affine (infinite dimensional) manifold we may identify it with its tangent space at any point. Moreover using the non degenerated form

$$\langle (V_1, a), (V_2, b) \rangle + \int_S^1 (V_1(x), V_2(x))dx + ab \tag{2.6}$$

we may identify at any point S the tangent space with the corresponding cotangent space $T_S\mathcal{M} = T_S^*\mathcal{M}$. Using these identifications we can write the canonical Lie Poisson tensor of $\widehat{\mathfrak{g}}$ as

$$P_{(S,c)}(V) = c\partial_x V + [S, V]. \tag{2.7}$$

It can be easily shown that this Poisson tensor is compatible with constant Poisson tensor obtained by freezing the tensor (2.7) in any point of $\mathcal{M}$. In particular the hierarchies of Drinfeld and Sokolov turns out to be bihamiltonian with respect to the bihamiltonian pair P_1, P_0 where P_1 is the canonical Poisson tensor (2.7) and P_0 is the constant Poisson tensor

$$(P_0)_{(S,c)}(V) = [A, V]. \tag{2.8}$$

where A is the constant function of $C^\infty(S^1, \mathfrak{g})$ whose value is the element of minimal weight in $\mathfrak{g}$.

3 The reduction process

In this section, following [9], we perform the bihamiltonian reduction of the exceptional Lie algebra $G_2^{(1)}$. It is a rank 2 simple Lie algebra whose Cartan matrix is $\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$. A possible Weyl basis is:

$$
\begin{array}{ll}
H_1 = d_{22} - d_{33} + d_{55} - d_{66} & H_2 = d_{11} - d_{22} + 2d_{33} - 2d_{55} + d_{66} - d_{77} \\
E_1 = d_{23} + d_{56} & E_2 = d_{12} + d_{34} + 2d_{45} + d_{67} \\
F_1 = d_{32} + d_{65} & F_2 = d_{21} + 2d_{43} + d_{54} + d_{76}
\end{array}
\tag{3.1}
$$

where d_{ij} is a 7×7 matrix with 1 in the ij position and zero otherwise.
Thus the elements of the algebra has the form

$$
v = \begin{bmatrix}
h_2 & e_2 & -e_3 & 2\,e_4 & -6\,e_5 & 6\,e_6 & 0 \\
f_2 & h_1 - h_2 & e_1 & e_3 & -2\,e_4 & 0 & 6\,e_6 \\
f_3 & f_1 & -h_1 + 2\,h_2 & e_2 & 0 & -2\,e_4 & 6\,e_5 \\
4\,f_4 & -2\,f_3 & 2\,f_2 & 0 & 2\,e_2 & -2\,e_3 & 4\,e_4 \\
6\,f_5 & -2\,f_4 & 0 & f_2 & h_1 - 2\,h_2 & e_1 & e_3 \\
6\,f_6 & 0 & -2\,f_4 & f_3 & f_1 & -h_1 + h_2 & e_2 \\
0 & 6\,f_6 & -6\,f_5 & 2\,f_4 & -f_3 & f_2 & -h_2
\end{bmatrix}.
$$

As already noted on $G_2^{(1)}$ is defined a bihamiltonian structure given by canonical Lie Poisson tensor and by the tensor (2.8) where in the present case the element of minimal weight is $A = F_6$.
To perform the Marsden-Ratiu reduction process of such bihamiltonian structure we need to compute first a symplectic leaf $\mathcal{S}$ of P_0 and second the distribution $E = P_1(\mathrm{Ker}(P_0)$ on the point of $\mathcal{S}$. As proved in [9] the symplectic leaves of the constant Poisson tensor are affine subspaces modelled on the subspace of $G_2^{(1)}$ orthogonal to the isotropic algebra of the element A. Following Drinfeld and Sokolov let us choose that passing through the point $b = E_1 + E_2$:

$$
S = b + h(t)(2H_1 + H_2) + f_1(t)F_1 + f_3(t)F_3 + f_4(t)F_4 + f_5(t)F_5 + f_6(t)F_6.
$$

Then the (constant) distribution $E = P_1 \ker P_0$ evaluated on the points of $\mathcal{S}$ is

$$\begin{bmatrix} t_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t_1 & 0 & 0 & 0 & 0 & 0 \\ t_2 & t_3 & 0 & 0 & 0 & 0 & 0 \\ 2\,t_4 & -2\,t_2 & 0 & 0 & 0 & 0 & 0 \\ t_5 & -t_4 & 0 & 0 & 0 & 0 & 0 \\ t_6 & 0 & -t_4 & t_2 & t_3 & -t_1 & 0 \\ 0 & t_6 & -t_5 & t_4 & -t_2 & 0 & -t_1 \end{bmatrix}.$$

Luckily enough we may apply Theorem 2.1 since the submanifold $\mathcal{Q}$ of $\mathcal{S}$

$$\mathcal{Q} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & u_0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 6\,u_1 & 0 & 0 & 0 & u_0 & 0 & 1 \\ 0 & 6\,u_1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is transversal to E.

The actually computations of the explicit form of the reduced Poisson pencil as observed in [5] boils down to find (given a 1-form $v = (v_0, v_1) \in T^*(\mathcal{Q})$) a section $V(v)$ in $i^*_{\mathcal{Q}}(T^*\mathcal{M})$ (where $i_{\mathcal{Q}}\mathcal{Q} \hookrightarrow \mathcal{M}$ is the canonical inclusion) such that $P_\lambda V(v) \in TQ$. This implies that the entries of $V(v)$ are polynomials functions of the elements $(8\,e_1, 288\,e_6)$ and that the reduced Poisson pencil

$$\frac{d\,q}{dt_\lambda} = (P^Q_\lambda)_q v = V(v)_x + [V(v) + \lambda\,a, q],$$

becomes

$$
\begin{aligned}
\frac{d\,u_0}{dt_\lambda} \;=\;& \beta\Big(\frac{7}{32}u_0'v_1^{(4)}+\frac{3}{4}u_1v_1'-\frac{1}{8}\left(u_0'\right)^2v_1'+\frac{11}{32}u_0''v_1'''+\frac{5}{32}u_0^{(4)}v_1'+\frac{5}{16}u_0'''v_1''+\frac{1}{16}u_0v_1^{(5)}+\\
&+\frac{5}{8}u_1'v_1-\frac{1}{32}u_0^2v_1'''+\frac{1}{32}u_0^{(5)}v_1+\frac{1}{4}u_0v_0'+\frac{1}{8}u_0'v_0-\frac{7}{4}v_0'''-\frac{1}{32}v_1^{(7)}-\frac{1}{8}u_0u_0''v_1'+\\
&-\frac{5}{32}u_0u_0'v_1''-\frac{3}{32}u_0'u_0''v_1-\frac{1}{32}u_0u_0'''v_1\Big)-\lambda\frac{3}{4}v_1'
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\,u_1}{dt_\lambda} \;=\;& \Big(\frac{1}{96}u_0'v_1^{(8)}-\frac{7}{144}(u_0''')^2v_1'-\frac{1}{1728}u_0^4v_1'''+\frac{1}{1728}u_0^{(9)}v_1-\frac{1}{32}u_0^2v_0'''+\frac{3}{4}u_1v_0'+\\
&+\frac{1}{72}u_1v_1^{(5)}+\frac{47}{576}u_0^{(4)}v_1^{(5)}+\frac{13}{144}u_1''v_1'''+\frac{5}{288}u_0^2u_0'v_1^{(4)}+\frac{7}{864}u_0u_0'u_0^{(4)}v_1+\\
&-\frac{61}{576}(u_0'')^2v_1'''+\frac{1}{432}u_0v_1^{(9)}+\frac{5}{96}u_1^{(4)}v_1'+\frac{1}{8}u_1'v_0-\frac{1}{1728}v_1^{(11)}-\frac{23}{576}u_0u_0^{(5)}v_1''+\\
&+\frac{1}{16}u_0v_0^{(5)}+\frac{29}{288}u_1'''v_1''+\frac{85}{1728}u_0^{(6)}v_1'''+\frac{1}{48}u_0^{(7)}v_1''+\frac{3}{32}u_0'v_0^{(4)}-\frac{1}{24}u_1u_0''v_1'+\\
&-\frac{1}{24}u_0u_1'v_1''-\frac{1}{36}u_1u_0v_1'''-\frac{1}{24}u_1u_0'v_1''+\frac{23}{864}u_0''u_0^2v_1'''+\frac{29}{864}u_0(u_0')^2v_1'''+\\
&-\frac{17}{96}u_0''u_0'''v_1''-\frac{5}{64}u_0''u_0^{(4)}v_1'-\frac{1}{72}u_0''u_0^{(5)}v_1-\frac{7}{1728}u_0^2(u_0')^2v_1'-\frac{67}{432}u_0'u_0'''v_1'''+\\
&-\frac{1}{48}u_0^{(4)}u_0'''v_1-\frac{1}{32}u_0u_0'v_0''+\frac{1}{144}u_0^2u_1'v_1+\frac{1}{72}u_0^2u_1v_1'-\frac{1}{432}u_0^3u_0''v_1'-\frac{1}{1728}u_0^3u_0'''v_1+\\
&-\frac{1}{288}u_0^3u_0'v_1''-\frac{1}{1728}u_0(u_0')^3v_1-\frac{11}{864}u_0u_0(6)v_1'-\frac{1}{576}u_0u_0^{(7)}v_1-\frac{5}{48}u_0'u_0^{(4)}v_1''+\\
&-\frac{11}{288}u_0'u_0^{(5)}v_1'-\frac{5}{864}u_0'u_0^{(6)}v_1-\frac{61}{864}u_0u_0^{(4)}v_1'''+\frac{3}{32}u_0''v_0'''+\frac{35}{576}u_0'''v_1^{(6)}+\frac{1}{64}(u_0')^3v_1''+\\
&-\frac{1}{32}v_0^{(7)}+\frac{35}{864}u_0u_0'u_0'''v_1'-\frac{1}{18}u_0u_0''v_1^{(5)}-\frac{1}{36}u_0'u_1'v_1'+\frac{1}{72}u_0u_0'u_1v_1+\\
&+\frac{1}{32}u_0'''v_0''+\frac{5}{144}u_1'v_1^{(4)}-\frac{1}{288}u_0^2v_1^{(7)}+\frac{1}{72}u_0u_0''u_0'''v_1-\frac{1}{432}u_0^2u_0'u_0''v_1+\\
&+\frac{11}{144}u_0^{(5)}v_1^{(4)}+\frac{11}{144}u_0u_0'u_0''v_1''-\frac{5}{64}u_0u_0'''v_1^{(4)}-\frac{5}{288}u_0u_1''v_1-\frac{5}{288}u_0'u_1'v_1+\\
&-\frac{7}{144}u_0u_1''v_1'-\frac{1}{72}u_1'u_0''v_1+\frac{19}{576}(u_0')^2u_0''v_1'+\frac{13}{576}u_0^2u_0'''v_1''+\frac{17}{1728}u_0^2u_0^{(4)}v_1'+\\
&+\frac{1}{96}u_0'(u_0'')^2v_1+\frac{1}{36}u_0(u_0'')^2v_1'+\frac{13}{1728}u_0'''(u_0')^2v_1+\frac{1}{576}u_0^2u_0^{(5)}v_1-\frac{1}{72}u_1u_0'''v_1+\\
&-\frac{7}{288}u_0u_0'v_1^{(6)}-\frac{5}{36}u_0'u_0''v_1^{(4)}+\frac{1}{96}u_1^{(5)}v_1+\frac{1}{432}u_0^3v_1^{(5)}+\frac{1}{192}u_0^{(8)}v_1'+\\
&+\frac{1}{32}u_0''v_1^{(7)}-\frac{7}{192}(u_0')^2v_1^{(5)}\Big)-\lambda\Big(-\frac{1}{72}u_0'''v_1+\frac{1}{72}u_0^2v_1'+\frac{1}{72}v_1^{(5)}-\frac{1}{24}u_0''v_1'+\\
&+\frac{1}{72}u_0u_0'v_1-\frac{1}{24}u_0'v_1''-\frac{1}{36}u_0v_1'''+\frac{3}{4}v_0'\Big)
\end{aligned}
$$

where the prime indicates the space derivative. Having the reduced bihamiltonian structure we are now able to write explicitly the first non-trivial flows of the hierarchy

[8]. Since the Casimirs of P_0 are given by the functionals

$$H_0 = \int_{S^1} dx\, u_0 \quad \text{and} \quad H_1 = \int_{S^1} dx\, u_0'' u_0 - \frac{u_0^3}{3} + 108 u_1 \tag{3.2}$$

we obtain

$$\begin{aligned} u_{0,t_0} &= \frac{1}{8} u_{0x} &(3.3)\\ u_{1,t_0} &= \frac{1}{8} u_{1x} &(3.4) \end{aligned}$$

and

$$\begin{aligned} u_{0,t_1} &= -\frac{1}{864}(u_0^{(5)} + 5u_0' u_0^2 - 5u_0''' u_0 - 5u_0'' u_0' - 540 u_1) &(3.5)\\ u_{1,t_1} &= -\frac{1}{864}(-9u_1^{(5)} + 15 u_1''' u_0 + 15 u_1'' u_0' + 10 u_1' u_0'' - 5 u_1' u_0^2) &(3.6) \end{aligned}$$

4 The Frobenius Technique

In the first section we have found the bihamiltonian structure of the soliton equation associated to the affine Kac–Moody Lie algebra $G_2^{(1)}$ together with its first not trivial equations. This is of course by far not the same thing as to provide a way to actually compute all the soliton equations of the hierarchy. In the setting of the bihamiltonian theory this second important problem is tackled by looking for Casimirs of the Poisson pencil $P_\lambda = P_1 - \lambda P_0$ i.e., solutions of the equations

$$V_x + [V, S + \lambda A] = 0 \qquad s \in \mathcal{S} \tag{4.1}$$

which are formal Laurent series $V(\lambda) = \sum_{k=-1}^{\infty} V_k \lambda^{-k}$ whose coefficients are one forms defined at least on the points of $\mathcal{S}$ which are exact when restricted on $\mathcal{S}$. Indeed once such a solution is found the vector fields of the hierarchy can be written in the bihamiltonian form $X_k = P_0 V_k = P_1 V_{k-1}$.

This latter task is unfortunately usually a very tough problem, but in the contest of the integrable systems defined on affine Lie algebra it can be solved by using the generalization of the dressing method of Zakharov Shabat proposed by Drinfeld and Sokolov [12]. Unfortunately exactly as happens for the Drinfeld–Sokolov reduction for the Lie algebra $G_2^{(1)}$ the computations involved to derive the explicit expression of the bihamiltonian fields of the hierarchy are very complicated. The aim of this last section is to show how the so called Frobenius technique ([7]) provides somehow a shortcut of the Drinfeld–Sokolov procedure.

More precisely this technique will give a way to compute algebraically by a recursive procedure the conserved densities of the hierarchy and therefore the corresponding (maybe without passing through the Poisson tensors) bihamiltonian vector fields as well. However implementing such technique requires to give up the pure geometrical description of the hierarchy of the first section in order to consider also the minimal true loop module $C^\infty(S^1,\mathbb{R}^7)$ of $G_2^{(1)}$ [10] together with its geometrical dual space and the set of its linear automorphisms as well.

The starting point of the theory is indeed to observe ([5],) that $V \in T^*_S\mathcal{S}$ solves (4.1) at the point $S \in (\mathcal{S}, c = 1)$ if and only if it commutes viewed as linear operator in $\mathrm{End}(C^\infty(S^1,\mathbb{C}^7))$ (up the canonical identification explained in the previous section) with the linear differential operator $-c\partial_x + S + \lambda A$. Although this latter task seems not really easier then the first one, it suggests a way (using the action of the affine Lie group $\widetilde{\mathfrak{G}}$ on the representation space $C^\infty(S^1,\mathbb{R}^7)$) to obtain directly the equations of the hierarchy together with their hamiltonians. Following what suggested by Drinfeld and Sokolov we can find the elements V commuting with $-c\partial_x + S + \lambda A$ using the observation that the element $B + \lambda A$ is a regular element and therefore its isotropic subalgebra $\mathfrak{g}_{B+\lambda A}$ is a Heisenberg subalgebra $\mathfrak{H}$ of $\widetilde{\mathfrak{g}}$ spanned (up to the central charge) in our representation by the matrices $\Lambda^{6n+1} = \left(\frac{\lambda}{24}\right)^n (B + \frac{\lambda}{24}A)$, $\Lambda^{6n-1} = \left(\frac{\lambda}{24}\right)^{n-1} (B + \frac{\lambda}{24}A)^5$ with $n \in \mathbb{Z}$. (For sake of simplicity, from now on, we rescale $\frac{\lambda}{24} \to \lambda$). From this fact Drinfeld and Sokolov proves indeed the

Proposition 4.1 *For any operator of the form $-\partial_x + S + \lambda A$ with $s \in \mathcal{S}$ there exists a element T in $\widetilde{\mathfrak{G}}$ such that:*

$$T(-\partial_x + S + \lambda A)T^{-1} = \partial_x + (B + \lambda A) + H, \quad H \in \mathfrak{H}. \tag{4.2}$$

Therefore the set of the elements in $\widetilde{\mathfrak{g}}$ commuting with $-\partial_x + S + \lambda A$ is given by $T^{-1}\mathfrak{H}T$.

The knowledge of a such a T allows us to compute explicitly for any choice of an element in $\mathfrak{H}$ the corresponding hierarchy of vectors fields together with their Hamiltonians.

Proposition 4.2 *Let $C = \sum_{j=\pm 1 \mathrm{mod}(6)} c_j \Lambda^{-j}$ with $c_j \in \mathbb{C}$ be an element in $\mathfrak{H}$ then:*

1. *the element $V_C = T^{-1}CT$ solves equation (4.1);*
2. *its hamiltonian on $\mathcal{S}$ is the function $H_C = \langle J, C\rangle$ where J is defined by the relation*
$$J = T(S + \lambda A)T^{-1} + T_x T^{-1} \tag{4.3}$$
3. *in particular if C has the form $C = \Lambda^j$, $j = \pm 1 mod(6)$ (say $j = 6n \pm 1$) then V_C and H_C simply denoted respectively V^j has the Laurent expansion*
$$V^j = \lambda^n \sum_{p \geq -2} \frac{1}{\lambda^{p+1}} V_{6p\pm 1} \tag{4.4}$$

Proof .

1. It was already proved above.
2. Using equation (4.3) we can rewrite equation (4.2) in the form $T(-\partial_x + S + \lambda A)T^{-1} = -\partial_x + J$ showing the J commutes with C then:

$$\frac{d}{dt}H_C = \langle \dot{J}, C\rangle = \langle T\dot{S}T^{-1} + \left[\dot{T}T^{-1}, J\right], C\rangle$$

   but since C commutes with J we get

$$\frac{d}{dt}H_C = \langle T\dot{S}T^{-1}, C\rangle = \langle \dot{S}, T^{-1}CT\rangle = \langle \dot{S}, V_C\rangle.$$

3. Equation (4.4) follows for $j = 6n + 1$ from the identity

$$\begin{aligned}\mathrm{res}(\lambda^{p-n}V^j) &= \mathrm{res}(\lambda^{p-n}T\Lambda^j T^{-1}) = \mathrm{res}(\lambda^{p-n}T\lambda^n\Lambda T^{-1}) \\ &= \mathrm{res}(T\lambda^p\Lambda T^{-1}) = \mathrm{res}(T\Lambda^{6p+1}T^{-1}) = \mathrm{res}(V^{6p+1})\end{aligned}$$

   while similar computations show that if $j = 6n - 1$ then $\mathrm{res}(\lambda^{p-n}V^j) = \mathrm{res}(V^{6p-1})$.

□

As already pointed out the actually computation of the element T (which by the way provides also the vector fields of the hierarchy) is in our case quite complicated. To avoid such computations let us first solve the related problem of finding the eigenvalues of the operator $-\partial_x + S + \lambda A$:

$$-\psi_x + (S + \lambda A)\psi = \mu\psi. \tag{4.5}$$

This latter problem can be solved by the observation that the integral leaves E are orbits of a group action, completely characterized by the distribution E at the special point B. It holds indeed:

Proposition 4.3 *The subspace $\mathfrak{g}_{AB} := \{V \in \mathfrak{g}_A | V_x + [V, B] \in \mathfrak{g}_A^\perp\}$ is a subalgebra of $\mathfrak{g}$ contained in the nilpotent subalgebra $\mathfrak{n}_-$ of loops with values in the maximal nilpotent subalgebra spanned by the negative (it depends how G_2 is defined). Therefore the corresponding group $G_{AB} = \exp(\mathfrak{g}_{AB})$ is well defined. The distribution E is spanned by the vector fields $(P_1)_B(V)$ with V belonging to $\mathfrak{g}_{AB}$, and its integral leaves are the orbits of the gauge action of G_{AB} on $\mathcal{S}$ defined by:*

$$S' = TST^{-1} + T_xT^{-1}. \tag{4.6}$$

Explicitly the for the Lie algebra G_2 the group G_{AB} is: Now it easily to see that equation (4.6) implies on the space $\mathrm{End}(C^\infty(S^1, \mathbb{C}^7)$ that the linear differential operators $-\partial_x + S + \lambda A$ and $-\partial_x + S' + \lambda A$ are conjugated by an element $T \in G_{AB}$in formula:

$$(-\partial_x + S' + \lambda A) \circ T = T \circ (-\partial_x + S + \lambda A). \tag{4.7}$$

if S and S' satisfy (4.6) with the same T.

Therefore if we define $v^{(0)} = (1,0,0,0,0,0,0)$ and by recurrence

$$v^{(j+1)}(S) = \partial_x v^{(j)}(S) + (S+\lambda A)v^{(j)}(S) \qquad (v^{(0)}(S) = v^{(0)}) \tag{4.8}$$

then we have the

Proposition 4.4 *The vectors $v^{(j)}(S)$ $j \geq 0$ are covariant i.e., $v^{(j)}(S') = v^{(j)}(S)T$ whenever $S' = TST^{-1} - T_xT^{-1}$ with $T \in G_{AB}$. Moreover the subset $\{v^{(j)}\}_{j=0,\ldots,6}$ is for any S in $\mathcal{S}$ a basis for $\mathbb{C}^7$.*

Developing now the first dependent vector namely $v^{(7)}(S)$ we obtain the relation

$$\begin{aligned} v^{(7)}(S) = \;& 2u_0v^{(5)}(S) + 5u_0'v^{(4)}(S) + (6u_0'' - u_0^2)v^{(3)}(S) + (4u_0''' - 3u_0u_0')v^{(2)}(S) \\ & +(24u_1 - 4\lambda - (u_0')^2 - u_0u_0'' + u_0^{(4)})v^{(1)}(S) + 12u_1'v^{(0)}(S) \end{aligned} \tag{4.9}$$

called the characteristic equation, moreover it is not difficult to show that, by construction, u_0 and u_1 are a complete set of invariants for the action of $\mathfrak{g}_{AB}$ i.e., they can be used to parameterize the quotient space $\mathcal{N}$.

These covariant vectors are the main tool to solve the eigenvector problem stated above. It holds namely

Proposition 4.5 *If ψ is the element of $C^\infty(S^1,\mathbb{C}^7)$ defined by the relations $\langle v^{(0)},\psi\rangle = 1$, $\langle v^{(1)},\psi\rangle = h$ and $\langle v^{(k)},\psi\rangle = h^{(k)}$ $k = 2,\ldots,6$ where the function $h^{(k)}$ are defined by the recurrence: $h^{(1)} = h$, $h^{(k+1)} = h_x^{(k)} + h^{(k)}h$ and h satisfies the "Riccati"-type equation*

$$\begin{aligned} h^{(7)} = \;& 2u_0h^{(5)} + 5u_0'h^{(4)} + (6u_0'' - u_0^2)h^{(3)} + (4u_0''' - 3u_0u_0')h^{(2)} \\ & +(24u_1 - 4\lambda - (u_0')^2 - u_0u_0'' + u_0^{(4)})h + 12u_1' \end{aligned} \tag{4.10}$$

then ψ is an eigenvector of $-\partial_x + S + \lambda A$ with eigenvalue $h(z)$. Moreover equation (4.10) admits a solution of the form $h(z) = cz + \sum_{i<0} h_i z^{-i}$ where $z^6 = \lambda$, and the coefficients h_k are obtained iteratively in an algebraic way.

Remark 4.6 *Up to the transformation $h = \frac{\psi_x}{\psi}$ and the change of coordinates*

$$u_0 = -u \qquad u_1 = \frac{1}{12}(u^{(4)} - u''u - (u')^2 - v)$$

the equation (4.10) coincides with the spectral problem arising from the Lax operator for $G_2^{(1)}$ given in [12].

The Laurent expansion can be effectivley computed using (4.10) its first term are:

$$\begin{aligned} c &= 2^{1/3} \\ h_0 &= 0 \\ h_1 &= \tfrac{2^{1/3}}{6} u_0 \\ h_2 &= -\tfrac{2^{1/3}}{6} u_0' \\ h_3 &= \tfrac{1}{9} u_0'' \\ h_4 &= -\tfrac{2^{2/3}}{36} u_0''' \\ h_5 &= \tfrac{2^{1/3}}{108} \left(u_0'' u_0 + u_0^{(4)} - \tfrac{u_0^3}{3} + 108 u_1 \right) \\ h_6 &= \ldots \end{aligned} \tag{4.11}$$

As expected the coefficients of h corresponding to power of z which are not $\pm 1 \mathrm{mod}(6)$ are total derivatives. Moreover h_1 and h_5 are densities of the functionals (3.2) which are elements of the kernel of P_0. The Laurent series $h(z)$ plays the main role in our construction of the hierarchy related to $G_2^{(1)}$, we are indeed going to show that the knowledge of such function together with the existence of a complete set of Casimirs of (4.1) is enough in order to compute all the solitons equations of the hierarchy. Let us indeed first use the invariance under the Weyl group of G_2 of the eigenvalues of the matrix $B+\lambda A$ (which belongs to a Cartan subalgebra of G_2) (or the very expression of equation 4.10 where only $\lambda = z^6$ explicitly appears) sure that if $\psi(z)$ and $\mu(z)$ are respectively an eigenvector and an eigenvalue of the operator $-\partial_x + S + \lambda A$ then also $\psi_k(z) = \psi(e^{\frac{2\pi i k}{6}} z)$ and $\mu_k(z) = \mu(e^{\frac{2\pi i k}{6}} z)$ are for any $k = 0, \ldots, 5$ another pair of respectively an eigenvector and an eigenvalue. Moreover it is easy to show that for any fixed x in S^1 the elements $\psi(e^{\frac{2\pi i k}{6}} z)$ $k = 0, \ldots 5$ together with the obviously existing "zero"–eigenvector $\chi(z)$ form a basis of $\mathbb{C}^7$.

Further using the expansion of V^j in Proposition 4.2 it easy to show that any flow of the hierarchy may be written as

$$\dot{S}_j = [A, V_j]\,. \tag{4.12}$$

Then since V^j is a solution of (4.1) we have that

$$((V^j)_+)_x + \Big[((V^j)_+, S + \lambda A\Big] = -((V^j)_-)_x - \Big[((V^j)_-, S + \lambda A\Big]$$

where $(\cdot)_+$ and $(\cdot)_-$ are respectively the projection on the regular and singular part of the Laurent series V^j. This latter equation implies that

$$[A, V_j] = ((V^j)_+)_x + \Big[((V^j)_+, S + \lambda A\Big]\,.$$

from which follows as usual, together with (4.12), the bihamiltonian form of the equations of the hierarchy

$$\dot{S}_j = [A, V_j] = ((V^j)_+)_x + \left[(V^j)_+, S + \lambda A\right]. \tag{4.13}$$

It remains only to show how to rewrite these equations directly in terms of the function $h(x)$:

Proposition 4.7 *The Laurent series $h(x)$ evolves as*

$$\partial_{t_j} h = \partial_x H^{(j)}, \tag{4.14}$$

where the Laurent series $H^{(j)}$ called in the literature currents are given by $H^{(j)} = \langle v^{(0)}, (V^j)_+ \psi_0 \rangle$.

Proof. From equations (4.5) with $\mu = h$ and (4.13) follows that

$$(-\partial_x + S + \lambda A)\phi = h\phi - h_{t_j}\psi_0 \tag{4.15}$$

where

$$\phi = (-\partial_{t_j} + V^j)\psi_0. \tag{4.16}$$

Let us now decompose ϕ with respect ot the basis $\chi, \psi_0, \ldots, \psi_5$: $\phi = c_6\chi + \sum_{a=0}^{5} c_a\psi_a$, the equation (4.15) implies

$$-c_{6x}\chi + \sum_{a=0}^{5}(-c_{ax} + c_a h_a)\psi_a = hc_6\chi + \sum_{a=0}^{5} hc_a\psi_a - h_{t_j}\psi_1$$

where $h_0 = h$ and therefore $-c_{6x} = c_6 h$ and $-c_{ax} = c_a h_a = c_a h$ for $a = 1, \ldots, 5$, but being c_a a Laurent series in z and h and $h_a - h$ ($a = 1, \ldots, 5$) series of maximal degree 1 these latter equation give $c_a = 0$ for $a = 1, \ldots, 6$. Hence $\phi = c_0\psi_0$ which taking into account (4.16), the definition of $H^{(j)}$ and the normalization of ψ_0 implies $c_0 = H^{(j)}$. Therefore $\phi = H^{(j)}\psi_0$ which plugged in (4.15) yields $h_{t_j} = \partial_x H^{(j)}$.

□

In the case of the Lie algebras of type A the most important property of the function $H^{(j)}$ is that they can be actually computed without using directly the Casimir V^j giving a really powerful way to write down all the equations of the hierarchy using simply equations (4.14). The matter are however much more complicated for the other affine Lie algebras, impeding a straightforward generalization of the theory. The difficulty arises from the fact that equation (4.10) does not imply any more that $\lambda = z^6$ belongs (in the space $\mathcal{L}$ of all Laurent polynomials in z) to the linear span generated by the Faà di Bruno polynomials $H_+ = \langle h^{(i)} \rangle_{i\in\mathbb{N}}$. The only property which seems still survive in our setting is the observation:

Proposition 4.8 *The Laurent series of $H^{(j)}$ is given by*

$$H^{(j)} = (z)^j + \sum_{l\geq 1} H^j_l z^{-l}. \tag{4.17}$$

Proof. By the definition of $H^{(j)}$ we have $H^{(j)} = \langle v^{(0)}, (V^j)_+\psi_0\rangle = \langle v^{(0)}, V^j\psi_0\rangle - \langle v^{(0)}, (V^j)_-\psi_0\rangle = z^j - \langle v^{(0)}, (V^j)_-\psi_0\rangle$, since $V^1\psi_0 = z\psi_0$. Then since $(V^j)_- = V^j_1\lambda^{-1} + \ldots$ and from the definition of the $h^{(k)}$ and the Riccati equation follows that $\psi_0 = (1, cz + O(1), (cz)^2 + O(z), \ldots)^T$ we have that

$$-\langle v^{(0)}, (V^j)_-\psi_0\rangle = \frac{H^j_1}{z} + \ldots$$

Hence $H^{(j)}$ has the form (4.17).

□

Actually we have some fairly guesses how the theory should be modified in order to take into account at least the affine Lie algebras corresponding to the classical simple Lie algebras. For instance in the case of the affine Lie algebras $B^{(1)}_n$ the equations of the hierarchy has the form (4.14) where h is constrained by the request that $z^{2n}h$ is in the span of the positive Faà di Bruno polynomials and the currents $H^{(k)}$ may be directly computed as the projection on H_+ of z^n with respect to the decomposition $\mathcal{L} = H_+ \oplus H_-$ where $H_- = \langle z^j\rangle_{j<0}$ together with the further constrains that the odd currents are linear combination of strictly positive the Faà di Bruno polynomials. From these latters currents (in the case when $n = 3$) those corresponding to the Lie algebra $G^{(1)}_2$ should be obtained by performing a further reduction.

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