On the Nonautonomous Nonlinear Schrödinger Equations and Soliton Management

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(Joint work with Hong-Gang Luo and et al.)

- This talk is based on the recently published papers

  - Xu-Gang He, Dun Zhao, Lin Li and Hong-Gang Luo, Engineering integrable nonautonomous nonlinear Schrodinger equations, Phys. Rev. E 79, 056610 (2009).

  - Hong-Gang Luo, Dun Zhao and Xu-Gang He, Exactly controllable transmission of nonautonomous optical solitons, Phys. Rev. A 79, 063802 (2009).

  - Dun Zhao, Xu-Gang He and Hong-Gang Luo, Transformation from the nonautonomous to standard NLS equations, Eur. Phys. J. D, 53, 213-216(2009).

  - Dun Zhao, Hong-Gang Luo and Hua-Yue Chai, Integrability of the Gross-Pitaevskii equation with Feshbach resonance management, Physics Letters A 372, 5644-5650 (2008).

Outline

1 Background

- Soliton management, such as dispersion and nonlinearity management in optics, Feshbach-resonance management and dispersion management in atomic and condensed-matter physics, is a new concept in nonlinear dynamical systems described by nonautonomous nonlinear Schrödinger equations (NNLSEs) with varying dispersion, nonlinearity and gain or loss. Due to their great importance, the study on NNLSEs has attracted extensive attention in recent years.

- Reference

  - B. A, Malomed, Soliton Management in Periodic Systems, Springer, (2006).

Our study is motivated by the soliton management in Bose-Einstein condensation and in optical soliton transmission.

- When a gas of bosonic particles is cooled below a critical temperature, it condenses into a Bose-Einstein condensate. The condensate consists of a macroscopic number of particles, which are all in the same ground state of the system.

- Bose-Einstein condensation was predicted by Einstein in 1925 and was produced in the laboratory for the first time in 1995. The realization of Bose-Einstein condensates of dilute gases allows for the direct observation of wave packet dynamics in real space on a macroscopic scale, usually called matter-wave solitons.

Experiments observations

- N. H . Anderson and J. R. Ensher, Matthews M R, et al., Observation of Bose-Einstein condensation in a dilute atomic vapor. Science, 269: 198-201(1995).
- J. Denschlag, J.E. Simsarian, D.L. Feder, et al. Generating Solitons by Phase Engineering of a Bose-Einstein Condensate, Science 287, 97-101(2000).
- Kevin E. Strecker, Guthrie B. Partridge, Andrew G. Truscott, and Randall G. Hulet, Formation and propagation of matter-wave soliton trains, Nature 417, 150-153 (2002).
- L. Khaykovich, F. Schreck, G. Ferrari, et al., Formation of a Matter-Wave Bright Soliton, Science 296, 1290-1293 (2002).

- Hasegawa and Tappert found that a soliton can also be formed in fiber systems, i.e., the optical soliton. Due to the particlelike property of the soliton, they proposed that the optical soliton could be an ideal subject to transmit optical signals. The realization of the optical soliton transmission was first reported by Mollenauer et al.. Since then the fundamental properties of optical solitons and their applications in optical communication have been extensively investigated.

- Reference

  - A. Hasegawa and F. Tappert, Appl. Phys. Lett. 23, 142 (1973).
  - A. Hasegawa, Optical Solitons in Fibers (Springer-Verlag, Berlin, 1990).
  - A. Hasegawa and Y. Kodama, Solitons in Optical Communications (Oxford, New York, 1995).
  - L. F. Mollenauer, R. H. Stolen, and J. P. Gordon, Phys. Rev. Lett. 45, 1095 (1980).

Soliton management

- A soliton can be well defined by its four basic parameters namely (i) amplitude (or width); (ii) frequency (or velocity); (iii) phase and (iv) time position. So it is possible to control the soliton dynamics by defining the parameter functions. This is the meaning of soliton management.

- Dispersion and nonlinearity (Feshbach-resonance) management have been found to be a robust method for creating solitonlike wave packet, in another word, solitary pulses can exist as robust solutions to nonautonomous NLS equations.

- **Some references on soliton management**
  - P.G. Kevrekidis, G. Theocharis, D. J. Frantzeskakis, and Boris A. Malomed, Feshbach Resonance Management for Bose-Einstein Condensates, Phys. Rev. Lett. 90, 230401 (2003).
  - B. Eiermann, P. Treutlein, Th. Anker, M. Albiez, M. Taglieber, K.-P. Marzlin, and M. K. Oberthaler, Dispersion management for atomic matter waves, Phys. Rev. Lett. 91, 060402 (2003).
  - K. Porsezian et al., Dispersion and nonlinear management for femtosecond optical solitons, Phys. Lett. A, 361, 504 – 508 (2007).
  - M. Centurion, M. A. Porter, P. G. Kevrekidis, and D. Psaltis, Nonlinearity management in optics: experiment, theory, and simulation, Phys. Rev. Lett. 97, 033903 (2006).
  - B. A, Malomed, Soliton Management in Periodic Systems, Springer, (2006).

Model equations

- Manipulation of matter waves and matter wave solitons

$$i\frac{\partial u(x,t)}{\partial t}+f(t)\frac{\partial^2 u(x,t)}{\partial x^2}+g(t)|u(x,t)|^2u(x,t)+V_{ext}(x,t)u(x,t)+i\gamma(t)=0.$$

  - Usually, $V_{ext}(x,t) = V_0x^2$ or $V(x) = V_0 sin^2 x$. When $f(t) = constant$ and $g(t) = constant$, it is the classical Gross-Pitaevskii equation.
  - f, g, V_{ext} and $\gamma(t)$ denote the dispersion management, Feshbach-resonance, the external potential and loss/gain respectively.

- Optical soliton transmission

$$i\frac{\partial u(z,t)}{\partial z}+f(z)\frac{\partial^2 u(z,t)}{\partial t^2}+g(z)|u(z,t)|^2u(z,t)+i\gamma(z)=0,$$

  - f and g denote the dispersion management and nonlinearity management respectively, and $\gamma(z) = \gamma_{loss} + \gamma_R$, where γ_{loss} means the fiber loss ($\gamma_{loss} > 0$) and γ_R the Raman gain ($\gamma_R < 0$).

Two questions:

- What happens with a classical soliton *beyond the autonomy* when the external fields are functions of time?
- How to manipulate the above-mentioned managements more efficiently?

Nonautonomous soliton

- Zabusky and Kruskal introduced for the first time the soliton concept to characterize nonlinear solitary waves that do not disperse and preserve their identity during propagation and after a collision. The classical soliton concept was developed for nonlinear and dispersive systems that have been autonomous, modeling uniform media.
- Reference

  - N. J. Zabusky and M. D. Kruskal, Phys. Rev. Lett. 15, 240 (1965).

- Chen and Liu give the first extension of the classical soliton concept in 1976, they found that the soliton can be accelerated in a linearly inhomogeneous plasma, the model NLS equation possesses the time-varying eigenvalue, and the Inverse Scattering Transformation method was generalized. At the same time the analytical solitonlike solutions for the Korteweg-de Vries equation with varying nonlinearity and dispersion were also found by Calogero and Degasperis. It was shown that the basic property of classical solitons, such as interact elastically, was preserved.
- Reference

  - H.-H. Chen and C.-S. Liu, Phys. Rev. Lett. 37, 693 (1976).
  - F. Calogero and A. Degasperis, Lett. Nuovo Cimento Soc. Ital. Fis. 16, 425 (1976); 16, 434 (1976).

- More recently, the solitonlike interaction among nonautonomous systems have been studied systematically by Serkin et al..
- Reference

  - V. N. Serkin and A. Hasegawa, Phys. Rev. Lett. 85, 4502 (2000); JETP Lett. 72, 89 (2000).
  - V. N. Serkin and T. L. Belyaeva, ibid. 74, 573 (2001).
  - V. N. Serkin and A. Hasegawa, IEEE J. Sel. Top. Quantum Electron. 8, 418 (2002).
  - V. N. Serkin, A. Hasegawa, and T. L. Belyaeva, Phys. Rev. Lett. 92, 199401 (2004).
  - V. N. Serkin, A. Hasegawa, and T. L. Belyaeva, Phys. Rev. Lett. 98, 074102 (2007).

- The concept of nonautonomous soliton was introduced by Serkin et al. in 2007. Simply speaking, a nonautonomous soliton is a *solitonlike* solution of a nonautonomous system generally moves with varying amplitudes and speeds.
- The nonautonomous soliton concept deals with nonlinear and dispersive systems that have been nonautonomous.
- Reference

  – Serkin, Hasegawa, and Belyaevain, Phys. Rev. Lett. 98, 074102 (2007).

2 On the Nonautonomous NLSEs

We study the following (1+1)-dimensional nonautonomous nonlinear Schrödinger equation (NLSE), which is also called the nonautonomous Gross-Pitaevskii (GP) equation:

$$i\frac{\partial u(x,t)}{\partial t} + f(x,t)\frac{\partial^2 u(x,t)}{\partial x^2} + g(x,t)|u(x,t)|^2 u(x,t) + V(x,t)u(x,t) + i\gamma(x,t)u(x,t) = 0. \tag{1}$$

where $f(x,t)$ and $g(x,t)$ are the dispersion and nonlinearity management parameters, respectively. $V(x,t)$ denotes the external potential and $\gamma(x,t)$ the dissipation (loss)($\gamma > 0$) or gain ($\gamma < 0$). These coefficients are usually assumed to be real.

Background
On the . . .
Main results
Discussions
Summary
Home Page
Title Page
Go Back
Full Screen
Close
Quit

Some known results

- Vladimir N. Serkin and Akira Hasegawa, Novel Soliton Solutions of the Nonlinear Schrödinger Equation Model, Phys. Rev. Lett., 85, 4502-4505(2000).
  - Model equation
$$i\frac{\partial\Psi}{\partial Z} \pm \frac{1}{2}D(Z)\frac{\partial^2\Psi}{\partial^2 T} + R(Z)|\Psi|^2\Psi = i\Gamma(Z)\Psi.$$
  - Found solitary wave solutions with the form
$$\Psi = \sqrt{\frac{D}{R}}PQ(S)exp(i\frac{T^2}{2} + i\int_0^Z K(\zeta)d\zeta).$$
    where the real function $Q(S)$ describes a canonical form of bright or dark solitons.

- V. I. Kruglov, A.C. Peacock, and J. D. Harvey, Exact Self-Similar Solutions of the Generalized Nonlinear Schrödinger Equation with Distributed Coefficients, Phys. Rev. Lett., 90, 113902(2003).
  - Model equation

$$i\psi_z = \frac{\beta(z)}{2}\psi_{\tau\tau} - \gamma(z)|\psi|^2\psi + i\frac{g(z)}{2}\psi.$$

  - Found solitary wave solutions with the form

$$\psi(z,\tau) = U(z,\tau)\exp i(a(z) + c(z)(\tau - \tau_c)^2).$$

where $U(z,\tau) = \frac{\Lambda\sqrt{|\rho(z)|}}{1-c_0 D(z)}$, $D(z) = 2\int_0^z \beta(s)ds$.

- H. Sakaguchi and B. A. Malomed, Resonant nonlinearity management for nonlinear Schrödinger solitons, Phys. Rev. E, 70, 066613 (2004).
  - Model equation

$$i\phi_t = -\frac{1}{2}\phi_{xx} + U(x)\phi + (g_0 + g_1 \sin(\omega t)|\phi|^2\phi.$$

  - Found solitary wave solutions with the form

$$\psi(x,t) = Asech[A(x-x_0)]exp(i\frac{A^2 t}{2}).$$

  - Investigated the effects of the periodic modulation of the nonlinearity coefficient on fundamental and higher-order solitons, found resonant splitting of higher-order solitons.

- Z. X. Liang, Z. D. Zhang, and W. M. Liu, Dynamics of a Bright Soliton in Bose-Einstein Condensates with Time-Dependent Atomic Scattering Length in an Expulsive Parabolic Potential, Phys. Rev. Lett. 94, 050402 (2005).
  - Model equation
$$i\phi_t + \phi_{xx} + 2g_0 e^{\lambda t}|\phi|^2\phi + \frac{1}{4}\lambda^2 x^2 \phi = 0.$$
  - Found a family of exact solutions by Bäcklund transformation, described the phenomenon of soliton compression in BEC and the dynamic stability of the number of atoms in the bright soliton.

- Serkin, Hasegawa, and Belyaevain, Nonautonomous solitons in external potentials, Phys. Rev. Lett. 98, 074102 (2007).
  - Model equation

$$i\frac{\partial Q}{\partial t} + \frac{D(t)}{2}\frac{\partial^2 Q}{\partial^2 x} + \sigma R(t)|Q|^2 Q - 2\alpha(t) x Q - \frac{\Omega(t)^2}{2} x^2 Q.$$

  - Found integrability condition and Lax pairs

$$-\Omega(t)^2 D(t) = \frac{d^2}{dt^2} lnD(t) + R(t)\frac{d^2}{dt^2}\frac{1}{R(t)} - \frac{d}{dt} lnD(t)\frac{d}{dt} lnR(t).$$

  - Introduced the conception of nonautonomous soliton.

3 Main Results

Based on the well-known Painlevé test for partial differential equations, i.e., the well-known WTC test (J. Weiss, M. Tabor, and G. Carnevale, J. Math. Phys. 24 (1983)), we get

Theorem 1 The GP equation (1) can pass the Painlevé test for PDE if and only if $f(x,t) = f(t)$, $g(x,t) = g(t)$, $\gamma(x,t) = \gamma(t)$, $V(x,t) = V_0(t) + V_1(t)x + V_2(t)x^2$, where $V_0(t)$ and $V_1(t)$ are arbitrary, and $f(t)$, $g(t)$, $\gamma(t)$, $V_2(t)$ satisfy the relation

$$(4f^2gg_t - 2ff_tg^2)\gamma - 4f^2g^2\gamma^2 - 2f^2g^2\gamma_t - g^2ff_{tt}$$
$$+f^2gg_{tt} - 2f^2g_t^2 + f_t^2g^2 + f_tgfg_t + 4V_2f^3g^2 = 0. \qquad (2)$$

- Theorem 1 suggests that Eq. (1) which is Painlevé integrable should be of the form

$$i\frac{\partial u(x,t)}{\partial t} + f(t)\frac{\partial^2 u(x,t)}{\partial x^2} + g(t)|u(x,t)|^2 u(x,t) + \left(V_0(t) + V_1(t)x + V_2(t)x^2\right)u(x,t) + i\gamma(t)u(x,t) = 0, \tag{3}$$

where $f(t), g(t), \gamma(t)$ and $V_2(t)$ satisfy relation (2).

- Setting

$$\frac{g_t(t)}{g(t)} - \frac{f_t(t)}{f(t)} - 2\gamma(t) = \theta(t),$$

Eq. (2) can be rewritten as the following Riccati equation of $\theta(t)$

$$\theta_t - \theta^2 - \frac{f_t}{f}\theta + 4fV_2 = 0,$$

thus all functions satisfying Eq. (2) can be presented as

$$g(t) = f(t)e^{\int(\theta(t)+2\gamma(t))dt}, \quad V_2(t) = -\frac{f\theta_t - f\theta^2 - f_t\theta}{4f^2}.$$

with $\theta(t)$ and $f(t)$ given arbitrarily.

Setting $\Gamma(t) = \int_0^t 2\gamma(t)dt$ and $z(t) = C_1 + \int_0^t \frac{f\,V_1(t)}{g} e^{\Gamma(t)} dt$, we have

Theorem 2 Under condition (2), Eq.(3) can be converted into the standard NLS equation

$$i\frac{\partial}{\partial T}Q(X,T) + \varepsilon \frac{\partial^2}{\partial X^2}Q(X,T) + \delta \left|Q(X,T)\right|^2 Q(X,T) = 0 \tag{4}$$

by the transformation $u(x,t) = Q(X(x,t),T(t))e^{ia(x,t)+c(t)}$, where

$$\begin{aligned}
a(x,t) &= \frac{1}{4f(t)}\left(\left(\ln\frac{f(t)}{g(t)}\right)_t - 2\gamma(t)\right)x^2 + \frac{g(t)}{f(t)}e^{-\Gamma(t)}z(t)x \\
&\quad - \int_0^t \left(\frac{g(t)^2}{f(t)}e^{-2\Gamma(t)}z(t)^2 - V_0(t)\right)d\,t + C_2, \\
X(x,t) &= \frac{\varepsilon g(t)}{\delta f(t)}e^{-\Gamma(t)}x - \frac{2\,\varepsilon}{\delta}\int_0^t \frac{g(t)^2}{f(t)}z(t)d\,t. \\
T(t) &= \frac{\varepsilon}{\delta^2}\int_0^t \frac{g^2(t)}{f(t)}e^{-2\Gamma(t)}z(t)d\,t + C_3, \qquad c(t) = \frac{1}{2}\ln\frac{\varepsilon g(t)}{\delta f(t)} - \Gamma(t).
\end{aligned}$$

Obviously, only when

$$sign(\varepsilon\delta) = sign(f(t)g(t)),$$

the transformation in Theorem 2 is well-defined. This means that the type of solitonlike solution is determined by the sign of $f(t)g(t)$.

By Theorem 2, we have

Corollary 1 Under condition (2), each solution $Q(X,T)$ of Eq.(4) presents a solution of Eq.(3) by

$$u(x,t) = Q(X(x,t), T(t))e^{ia(x,t)+c(t)}. \tag{5}$$

Remark

Under condition (2), the Lax pairs of Eq.(1) can also be constructed.

Some special cases of Eq.(1) have been studied by other authors in other situations, for examples

- Joshi, N., Painlevé property of general variable-coefficient versions of the Korteweg-De Vries and non-linear Schrödinger equations, Phys. Le tt. A, 125(9) (1987), 456-460.
- M. J. Ablowitz and P. A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering* (Cambridge University, Press,1991).

4 Discussions

We have known that under condition (2), a canonical soliton $Q(x,t)$ of Eq.(4) can give a solitonlike solution $u(x,t)$ of Eq.(1). It is easy to see from Eq.(5) that

$$|u(x,t)| = |Q(X(x,t),T(t))|e^{c(t)} = |Q(X(x,t),T(t))|e^{-\Gamma(t)}\sqrt{\frac{\varepsilon g(t)}{\delta f(t)}}. \qquad (6)$$

Equation (6) provides an explicit way to control the amplitude of the nonautonomous soliton on the basis of the canonical soliton.

- $c(t) = 0$ means that the amplitude of the corresponding nonautonomous soliton remains unchanged.
- $c(t) > 0$ (or < 0) means that the amplitude of the corresponding nonautonomous soliton increases (or decreases) exponentially.
- When $c(t)$ changes periodically its sign during the propagation, as a result, the the amplitude of the corresponding nonautonomous soliton oscillates with the same period.

An immediate result is

Corollary 2 Let $R(t) = \mathrm{e}^{-\int_0^t \frac{-\frac{d}{dt}\gamma(t)+2(\gamma(t))^2}{\gamma(t)}dt}$. If

$$f(t) = -\frac{R(t)}{2\int_0^t \frac{R(t)V_2(t)}{\gamma(t)}dt - C},$$
$$g(t) = -\frac{\delta}{\varepsilon}\frac{R(t) + e^{2\Gamma(t)}}{2\int_0^t \frac{R(t)V_2(t)}{\gamma(t)} - C}, \tag{7}$$

then we have

$$|u(x,t)| = |Q(X(x,t), T(t))|.$$

Corollary 2 tells us how to rectify the decrease of the amplitude caused by the dissipation through tuning the real functions for dispersion and nonlinearity. Some further discussions and examples can be found in our papers listed previously.

5 Summary

We obtain a condition to ensure a class of NNLSEs with loss/gain to be Painlevé integrable, and we also provide a correspondence from the solutions of the standard NLS equation to the solutions of the Painlevé integrable NLS equation with time-varying dispersion, nonlinearity, loss/gain and time-dependent confining parabolic external potential, and thus give a further understanding to the relation between classical solitons and nonautonomous solitons, this would be helpful for understanding the dynamical behavior of the BECs by using the Feshbach resonance management and dispersion management, also for the transmission of optical solitons by using the dispersion management and nonlinearity management.

Thanks for your attention!