The Integrals of Motion for the Elliptic Deformed W-algebra

An elliptic quantization of the KdV theory

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Summary

We construct infinitely many commutative operators I_m, G_m $(m=1,2,3,\cdots)$ associated with Elliptic Quantum Group $U_{q,p}$.

$$[I_m, I_n] = [I_m, G_n] = [G_m, G_n] = 0.$$

They can be regarded as Elliptic Quantization of KdV theory.

KdV Theory	I_m	Conservation Laws
	G_m	Monodromy

KdV equation

- $$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} + 6u \cdot \frac{\partial u}{\partial x}$$

Integrals of Motion

$$\frac{d I_m}{dt} = 0$$ Conservation Laws

$$I_1 = \int u\,dx \;,\quad I_3 = \int u^2\,dx \;,\quad I_5 = \int \left(u^3 - \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2\right)dx, \cdots$$

Poisson Bracket $\{\,,\,\}_{PB}$

- $$\frac{\partial u}{\partial t} = \{u, I_3\}_{PB} = \frac{\partial^3 u}{\partial x^3} + 6u \cdot \frac{\partial u}{\partial x}$$
- $$\{I_m, I_n\}_{PB} = 0 \qquad (m, n = 1, 2, 3, \cdots)$$

KdV theory

$$l_n = \oint x^{n+1} u(x) \frac{dx}{2\pi i}$$

- $\{l_m, l_n\}_{PB} = (m-n) l_{m+n} + \frac{1}{2}(m^3 - m)\delta_{m+n,0}$

Virasoro algebra

$$L_m \to \frac{c}{6} l_m, \quad \underset{c\to\infty}{[A,B] = \frac{6}{c}\{A,B\}_{PB}}$$

- $[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}$

- Quantum KdV Theory is Conformal Field Theory (CFT) described by Virasoro algebra.

$\boxed{\text{CFT}}$

[Sasaki, Yamanaka (1988)]
[Zamolodchikov (1989)]
[Bazhanov, Lukyanov, Zamolodchikov (1996)]
[Feigin, Frenkel (1996)]

- $[I_m, I_n] = 0 \quad (m, n = 1, 2, 3, \cdots)$

$\boxed{\text{Generating function}}$ $\quad [\mathbb{T}(z), \mathbb{T}(w)] = 0$

- $\mathbb{T}(z) = \sum_{m \in \mathbb{N}} G_m z^m \leftsquigarrow$ Quantum Group $U_q(\mathfrak{g})$

- $\log \mathbb{T}(z) \sim \sum_{m \in \mathbb{N}} I_{2m-1} z^{1-2m} \quad (z \to \infty)$

$$[I_m, I_n] = [I_m, G_n] = [G_m, G_n] = 0$$

KdV Theory — Lie algebra $\mathfrak{g}$

$\{I_m, I_n\}_{PB} = 0$

$\Downarrow$ Quantum Field Theory

CFT — Virasoro algebra, $U_q(\mathfrak{g})$

$[I_m, I_n] = 0$

$\Downarrow$ Elliptic Version

Today's Talk — $U_{qp}(\mathfrak{g})$

$\boxed{\text{Outline}}$

§1 Elliptic Algebra $U_{q,p}(\widehat{gl_N})$

§2 Level $k=1$

- Frenkel – Jing Realization
- Deformed W-algebra $W_{q,t}(\widehat{gl_N})$
- Commutative Operators I_m, G_m

§3 Level k $(\neq 0, -N)$ Noncritical !

- Wakimoto Realization
- Commutative Operators G_m

$$\boxed{\text{Lie algebra} \quad \mathfrak{sl}(2,\mathbb{C})}$$

- $\mathfrak{sl}(2,\mathbb{C}) = \{ X \in \mathrm{Mat}(2,\mathbb{C}) \mid \mathrm{tr}\, X = 0 \}$

  $[X, Y] = X \cdot Y - Y \cdot X$

- $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

  $\mathfrak{sl}(2,\mathbb{C}) = \mathbb{C}E \oplus \mathbb{C}H \oplus \mathbb{C}F$

$$\boxed{[H,E] = 2E,\ [H,F] = -2F,\ [E,F] = H}$$

Quantum Group $U_q(sl(2,\mathbb{C}))$

$q \neq 0$
$(q^2 \neq 1)$

Generator q^h, e, f

Relation

$$q^h e q^{-h} = q^2 e$$

$$q^h f q^{-h} = q^{-2} f$$

$$[e, f] = \frac{q^h - q^{-h}}{q - q^{-1}}$$

$\boxed{\text{Relation between } \mathfrak{g} \text{ and } U_q(\mathfrak{g})}$ $\quad (q = \exp(\varepsilon))$

- $[e, f] = \dfrac{q^h - q^{-h}}{q - q^{-1}} \longrightarrow h \quad (q \to 1)$

$$\left(\begin{array}{l} q^h e q^{-h} = e + \varepsilon [h, e] + O(\varepsilon^2) \\ q^2 e = e + \varepsilon \cdot 2e + O(\varepsilon^2) \end{array} \right. \Longrightarrow [h, e] = 2e$$

$$q^h f q^{-h} = q^{-2} f \quad \Longrightarrow \quad [h, f] = -2f$$

- $U_q(\mathfrak{g})$ is one parameter deformation of $\mathfrak{g}$.

$\boxed{\text{Quantum Group} \quad U_q(\widehat{sl}(2,\mathbb{C}))}$

Jimbo

$\boxed{\text{Generator}}$

- $e_1, e_2, h_1, h_2, f_1, f_2$

$\boxed{\text{Relation}}$

- $q^{h_i} e_j q^{-h_i} = q^{A_{ij}} e_j$
- $q^{h_i} f_j q^{-h_i} = q^{-A_{ij}} f_j \qquad (1 \leq i, j \leq 2)$
- $[e_i, f_j] = \delta_{ij} \dfrac{q^{h_i} - q^{-h_i}}{q - q^{-1}}$
- $e_i^3 e_j - [3]_q e_i^2 e_j e_i + [3]_q e_i e_j e_i^2 - e_j e_i^3 = 0$
- $f_i^3 f_j - [3]_q f_i^2 f_j f_i + [3]_q f_i f_j f_i^2 - f_j f_i^3 = 0 \quad (i \neq j)$

$$(A_{ij})_{1 \leq i,j \leq 2} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}, \quad [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$$

Drinfeld Realization of $U_q(\widehat{sl}(2,\mathbb{C}))$

Relation

- $[a_m, X^{\pm}(z)] = \pm \frac{[2m]_q}{m} q^{\mp k\frac{|m|}{2}} z^m X^{\pm}(z)$
- $(z_1 - q^{\pm 2} z_2) X^{\pm}(z_1) X^{\pm}(z_2) = (q^{\pm 2} z_1 - z_2) X^{\pm}(z_2) X^{\pm}(z_1)$
- $[X^+(z_1), X^-(z_2)]$

$$= \frac{1}{(q-q^{-1})z_1 z_2}\Big(\delta\Big(q^{-k}\frac{z_1}{z_2}\Big) q^{h_1} \exp\Big((q-q^{-1})\sum_{m=1}^{\infty} a_m q^{\frac{km}{2}} z_1^{-m}\Big)$$
$$- \delta\Big(q^{k}\frac{z_1}{z_2}\Big) q^{-h_1} \exp\Big(-(q-q^{-1})\sum_{m=1}^{\infty} a_{-m} q^{\frac{km}{2}} z_1^{m}\Big)\Big)$$

- $X^{\pm}(z) = \sum_{m\in\mathbb{Z}} x_m^{\pm} z^{-m-1}$, a_m, $\delta(z) = \sum_{m\in\mathbb{Z}} z^m$

Two Realization of $U_q(\widehat{sl}(2,\mathbb{C}))$

- $x_0^+ = e_1$, $x_0^- = f_1$, $x_1^- = e_0 q^{h_1}$, $x_{-1}^+ = q^{-h_1} f_0$

Elliptic Deformation

$$(z_1 - q^2 z_2) X^+(z_1) X^+(z_2) = (q^2 z_1 - z_2) X^+(z_2) X^+(z_1)$$

$$\Downarrow$$

$$[u_1 - u_2 + 1]_r F(z_1) F(z_2) = [u_1 - u_2 - 1]_r F(z_2) F(z_1)$$

- $[u]_r$ Elliptic Theta function , $z = x^{2u}$

Elliptic Theta function

$\left(z = x^{2u},\ \tau = \pi\sqrt{-1}/\log x\right)$

- $$[u]_r = x^{\frac{u^2}{r} - u} \frac{\Theta_{x^{2r}}(x^{2u})}{(x^{2r}; x^{2r})_\infty^3}$$
- $$\Theta_q(z) = (z; q)_\infty (q/z; q)_\infty (q; q)_\infty$$
- $$(z; q)_\infty = \prod_{m=0}^{\infty} (1 - q^m z)$$

Quasi-Periodicity

- $$[u+r]_r = -[u]_r$$
- $$[u+r\tau]_r = -e^{2\pi i (u + \frac{\tau}{2})/r} [u]_r$$

$\boxed{\text{Elliptic Algebra } U_{q,p}(\widehat{gl_N})}$

[Jimbo et al. (1999)]
[Kojima, Konno (2003)]

$$\left(0<x<1,\ r,\ s>0,\ r^* = r-k>0 \right)$$

- $E_{\bar{j}}(z),\ F_{\bar{j}}(z),\ H_{\bar{j}}(z) \quad (\bar{j}=1,2,\cdots,N-1)$

Relations

- $[u_1-u_2-\frac{s}{N}]_r F_{\bar{j}}(z_1) F_{\bar{j}+1}(z_2) = [u_2-u_1+\frac{s}{N}-1]_r F_{\bar{j}+1}(z_2) F_{\bar{j}}(z_1)$
- $[u_1-u_2+1]_r F_{\bar{j}}(z_1) F_{\bar{j}}(z_2) = [u_1-u_2-1]_r F_{\bar{j}}(z_2) F_{\bar{j}}(z_1)$
- $F_{\bar{j}}(z_1) F_{\bar{i}}(z_2) = F_{\bar{i}}(z_2) F_{\bar{j}}(z_1)$ otherwise

Relations $\quad (r^* = r - k)$

- $[u_1 - u_2 + 1 - \frac{s}{N}]_{r^*} E_{\bar{j}}(z_1) E_{\overline{j+1}}(z_2) = [u_2 - u_1 + \frac{s}{N}]_{r^*} E_{\overline{j+1}}(z_2) E_{\bar{j}}(z_1)$
- $[u_1 - u_2 - 1]_{r^*} E_{\bar{j}}(z_1) E_{\bar{j}}(z_2) = [u_1 - u_2 + 1]_{r^*} E_{\bar{j}}(z_2) E_{\bar{j}}(z_1)$
- $E_{\bar{j}}(z_1) E_{\bar{i}}(z_2) = E_{\bar{i}}(z_2) E_{\bar{j}}(z_1)$ otherwise
- $[E_{\bar{i}}(z_1), F_{\bar{j}}(z_2)] = \frac{\delta_{\bar{i}\bar{j}}}{x - x^{-1}} \times$

$$\times \left(\delta\left(x^k \frac{z_2}{z_1}\right) H_{\bar{i}}(x^r z_2) - \delta\left(x^k \frac{z_1}{z_2}\right) H_{\bar{i}}(x^{-r} z_2) \right)$$

etc.

δ-function $\quad \delta(z) = \sum_{m \in \mathbb{Z}} z^m$

$\boxed{\text{Frenkel - Jing Realization}}$ $U_{q,p}(\widehat{gl_N})$, Level $k=1$

Boson $\quad B_m^i \quad (i=1,2,\cdots,N) \quad (m\in\mathbb{Z}_{\neq 0})$

- $$[B_m^i, B_n^j] = m\frac{[(r-1)m]}{[rm]} \times \begin{cases} \dfrac{[(s-1)m]}{[sm]}, & (i=j) \\ (-1)\dfrac{[m]}{[sm]}x^{s\cdot m\cdot \mathrm{sgn}(i-j)} & (i\neq j) \end{cases}$$

- q-integer
$$[a] = \frac{x^a - x^{-a}}{x - x^{-1}}$$
[Asai et.al. (1996)]

Zero Mode $\quad P_\lambda, Q_\lambda \quad (\lambda \in \bigoplus_{j=1}^{N} \mathbb{Z}\bar{\epsilon}_j)$

- $$[iP_\lambda, Q_\mu] = (\lambda|\mu)$$

- $\{\epsilon_j\}_{j=1}^{N}$ Orthonormal Basis $\quad (\epsilon_i|\epsilon_j) = \delta_{ij}$

$$\bar{\epsilon}_j = \epsilon_j - \frac{1}{N}\sum_{j=1}^{N}\epsilon_j$$

Fock space $\quad \mathcal{F}_{\ell,k} \quad (\ell, k \in \bigoplus_{j=1}^{N} \mathbb{Z}\bar{\epsilon}_j)$

- $\beta^{i}_{-m}|\ell,k\rangle = 0 \quad , \quad (m>0\ ;\ j=1,2,\cdots,N)$
- $P_\alpha|\ell,k\rangle = \left(\alpha \middle| \ell\sqrt{\frac{r}{r-1}} - k\sqrt{\frac{r-1}{r}}\right)|\ell,k\rangle$

Free Field Realization

$N=3,4,5 \quad U_{q,p}(\widehat{gl_N})$, $\quad \alpha_j = \bar{\epsilon}_j - \bar{\epsilon}_{j+1}$

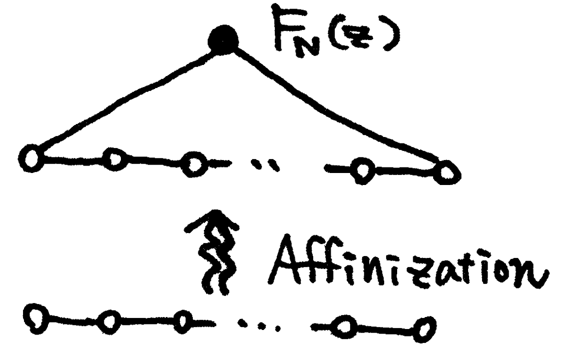
- $$F_j(z) = e^{i\sqrt{\frac{r-1}{r}}Q_{\alpha_j}} \left(x^{(\frac{2s}{N}-1)j} z\right)^{\sqrt{\frac{r-1}{r}}P_{\alpha_j} + \frac{r-1}{r}}$$
$$\times :\exp\left(\sum_{m\neq 0} \frac{1}{m}(\beta_m^j - \beta_m^{j+1})(x^{\frac{2s}{N}} z)^{-m}\right):$$

- $$E_j(z) = e^{-i\sqrt{\frac{r}{r-1}}Q_{\alpha_j}} \left(x^{(\frac{2s}{N}-1)j} z\right)^{-\sqrt{\frac{r}{r-1}}P_{\alpha_j} + \frac{r}{r-1}}$$
$$\times :\exp\left(-\sum_{m\neq 0} \frac{1}{m}\frac{[rm]}{[(r-1)m]}(\beta_m^j - \beta_m^{j+1})(x^{\frac{2s}{N}} z)^{-m}\right):$$

$(j=1,2,\cdots,N-1)$

[Asai et.al. (1996)]

Affinization of Current



Additional Operator $F_N(z)$, $E_N(z)$

- $[u_1-u_2-\frac{s}{N}]_h F_N(z_1)F_1(z_2) = [u_2-u_1+\frac{s}{N}-1]_h F_1(z_2)F_N(z_1)$
- $[u_1-u_2-\frac{s}{N}]_h F_{N-1}(z_1)F_N(z_2) = [u_2-u_1+\frac{s}{N}-1]_h F_N(z_2)F_{N-1}(z_1)$
- $[u_1-u_2+1]_h F_N(z_1)F_N(z_2) = [u_1-u_2-1]_h F_N(z_2)F_N(z_1)$

$$F_N(z_1)F_j(z_2) = F_j(z_2)F_N(z_1) \quad (j=2,3,\cdots,N-2)$$

Similar for $E_N(z)$ [Kojima, Shiraishi (2008)]

Free Field Realization   Level $k=1$

- $$F_N(z)=e^{i\sqrt{\frac{r-1}{r}}Q_{\alpha_N}}\left(x^{2s-N}z\right)^{\sqrt{\frac{r-1}{r}}P_{\bar{\epsilon}_1}+\frac{r-1}{2r}}z^{-\sqrt{\frac{r-1}{r}}P_{\bar{\epsilon}_1}+\frac{r-1}{2r}}$$
$$\times :\exp\left(\sum_{m\neq 0}\frac{1}{m}\left(x^{-2sm}\beta_m^N-\beta_m^1\right)z^{-m}\right):$$

- $$E_N(z)=e^{-i\sqrt{\frac{r}{r-1}}Q_{\alpha_N}}\left(x^{2s-N}z\right)^{-\sqrt{\frac{r}{r-1}}P_{\bar{\epsilon}_N}+\frac{r}{2(r-1)}}z^{\sqrt{\frac{r}{r-1}}P_{\bar{\epsilon}_1}+\frac{r}{2(r-1)}}$$
$$\times :\exp\left(-\sum_{m\neq 0}\frac{1}{m}\frac{[rm]}{[(r-1)m]}\left(x^{-2sm}\beta_m^N-\beta_m^1\right)z^{-m}\right):$$

Def $\boxed{\text{Nonlocal integrals of motion}}$ $N=3,4,5,\cdots$

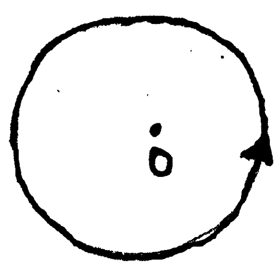
$$G_m = \int\cdots\int_C \prod_{t=1}^{N}\prod_{j=1}^{m} \frac{dz_j^{(t)}}{z_j^{(t)}} F_1(z_1^{(1)})F_1(z_2^{(1)})\cdots F_1(z_m^{(1)})$$

$$\times F_2(z_1^{(2)})F_2(z_2^{(2)})\cdots F_2(z_m^{(2)})\cdots F_N(z_1^{(N)})F_N(z_2^{(N)})\cdots F_N(z_m^{(N)})$$

$$\times \frac{\prod_{t=1}^{N}\prod_{1\le j<k\le m}[u_j^{(t)}-u_k^{(t)}]_r[u_k^{(t)}-u_j^{(t)}-1]_r}{\prod_{t=1}^{N-1}\prod_{j,k=1}^{m}[u_j^{(t)}-u_k^{(t+1)}+1-\frac{s}{N}]_r\prod_{j,k=1}^{m}[u_j^{(1)}-u_k^{(N)}+\frac{s}{N}]_r}$$

$$\times \prod_{t=1}^{N}\Big[\sum_{j=1}^{m}(u_j^{(t)}-u_{j+1}^{(t+1)})-\sqrt{r(r-1)}P_{\epsilon_{t+1}}\Big]_r$$

C : $\boxed{z_j^{(t)}}$ $|z_j^{(t)}|=1$



$\left(z_j^{(t)}=x^{2u_j^{(t)}}\right)$

[Kojima, Shiraishi (2008)]

- G_m^* is defined similarly by using $E_{\bar{j}}(z)$

$$\boxed{\begin{array}{l}\text{Theorem} \\ [G_m, G_n] = [G_m, G_n^*] = [G_m^*, G_n^*] = 0 \\ \qquad (m, n = 1, 2, 3, \cdots)\end{array}}$$

Deformed W-algebra $W_{qt}(\widehat{gl_N})$

$(q = x^{2h},\ t = x^{2-2h})$

Relations $\quad T_i(z) \quad (1 \le i \le N)$

$$\cdot\ f_{ij}\left(\frac{z_2}{z_1}\right) T_i(z_1) T_j(z_2) - f_{ji}\left(\frac{z_1}{z_2}\right) T_j(z_2) T_i(z_1)$$

$$= c \sum_{\ell=1}^{i} \prod_{m=1}^{\ell-1} \Delta\left(x^{2m-1}\right) \times$$

$$\times \Big(\delta\left(x^{j-i+2m} \frac{z_2}{z_1}\right) f_{i-\ell, j+\ell}\left(x^{-j+i}\right) T_{i-\ell}\left(x^{-\ell} z_1\right) T_{j+\ell}\left(x^{\ell} z_2\right)$$

$$- \delta\left(x^{-j+i-2m} \frac{z_2}{z_1}\right) f_{i-\ell, j+\ell}\left(x^{j-i}\right) T_{i-\ell}\left(x^{\ell} z_1\right) T_{j+\ell}\left(x^{-\ell} z_2\right) \Big)$$

[Odake (2001)]
[Shiraishi et.al. (1996)]

Structure functions

$s, r > 0$

- $f_{ij}(z)$

$$= \exp\left(\sum_{m=1}^{\infty} \frac{1}{m} \frac{(1-x^{2rm})(1-x^{-2(r-1)m})(1-x^{2m\,\mathrm{Min}(i,j)})(1-x^{2m(s-\mathrm{Max}(i,j))})}{(1-x^{2m})(1-x^{2sm})} \times x^{|i-j|} z^{m} \right)$$

- $\Delta(z) = \dfrac{(1-x^{2r-1}z)(1-x^{1-2r}z)}{(1-xz)(1-x^{-1}z)}, \quad C = -\dfrac{(1-x^{2r})(1-x^{-2r+2})}{(1-x^{2})}$

- $\delta(z) = \sum_{m \in \mathbb{Z}} z^{m}$

- $T_i(z) = \sum_{m \in \mathbb{Z}} T_m^{(i)} z^{-m} \quad (i = 1, 2, \cdots, N)$

$N = S = 2$

$$W_{q,t}(\widehat{\mathfrak{gl}_2})\Big|_{S=2} = \mathrm{Vir}_{q,t}$$

Deformed Virasoro

- $T_1(z) = \sum_{m\in\mathbb{Z}} T_m z^{-m}$
- $\sum_{\ell=0}^{\infty} f_\ell (T_{n-\ell} \cdot T_{m+\ell} - T_{m-\ell} \cdot T_{n+\ell}) = c(x^{2n} - x^{-2n})\delta_{m+n,0}$
- $\sum_{\ell=0}^{\infty} f_\ell z^\ell = \exp\left(\sum_{m=1}^{\infty} \frac{1}{m} \frac{(1-x^{2rm})(1-x^{(-2r+2)m})}{(1+x^{2m})} z^m \right)$

In CFT limit $(x\to 1)$ we get Virasoro algebra.

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12} C_{CFT}\, m(m^2-1)\,\delta_{m+n,0}$$

$$C_{CFT} = 1 - \frac{6}{r(r-1)}$$

Free Field Realization

Level $k=1$

- $$T_{\bar{j}}(z) = \sum_{1\le s_1 < s_2 < \cdots < s_{\bar{j}} \le N} : \Lambda_{s_1}(x^{-\bar{j}+1}z)\Lambda_{s_2}(x^{-\bar{j}+3}z)\cdots\Lambda_{s_{\bar{j}}}(x^{\bar{j}-1}z):$$

- $$\Lambda_{\bar{j}}(z) = x^{-\sqrt{r(r-1)}P_{\bar{\epsilon}_{\bar{j}}}} : \exp\left(\sum_{m\neq 0}\frac{x^{rm}-x^{-rm}}{m}\beta_m^{j} z^{-m}\right):$$

Correspondence

Level $k=1$

- $[\, W_{qt}(\widehat{gl_N}) \,,\, \overline{X}_{\alpha_j} \,] = 0$

- $[\, T_j(z) \,,\, \overline{X}_{\alpha_i} \,] = 0$ $\quad (i, j = 1, 2, \cdots, N-1)$

$$\overline{X}_{\alpha_i} = \oint \frac{dz}{2\pi i z} F_j(z) \frac{[u + \frac{1}{2} - \sqrt{r(r-1)} P_{\alpha_j}]_r}{[u - \frac{1}{2}]_r}$$

- $W_{qt}(\widehat{gl_N}) \Longleftrightarrow U_{qp}(\widehat{gl_N})$

Deformed W Algebra $\qquad$ Elliptic Quantum Group

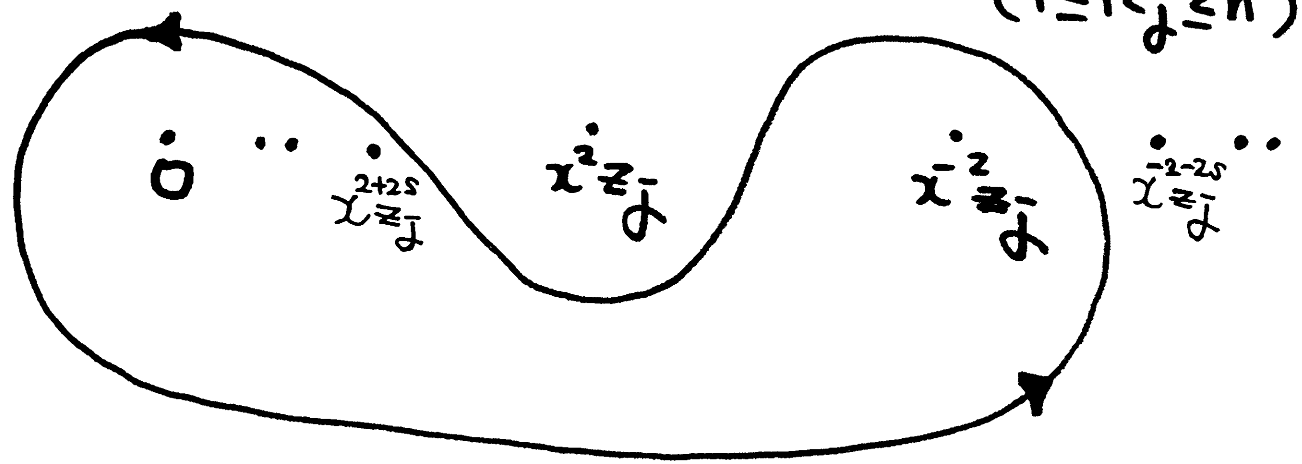
[Feigin et.al. (1998)]

Def

Local integrals of motion

- $$I_n = \int \cdots \int_C \prod_{j=1}^{n} \frac{dz_j}{z_j} \prod_{1 \le i < j \le n} \frac{[u_j - u_i]_s [u_j - u_i + r]_s}{[u_j - u_i + 1]_s [u_j - u_i + r - 1]_s} \times T_1(z_1) T_1(z_2) \cdots T_1(z_n)$$

$C : z_i$ $\quad (1 \le i < j \le n)$



- I_n^* is defined by similar way.

[Kojima, Shiraishi (2008)]

Th
$$\boxed{[I_m, I_n] = 0}$$

We constructed infinitly many commutative operators I_m, G_m $(m=1,2,\cdots)$ associated with the elliptic deformed W-alg. $W_{qt}(\widehat{gl_N})$.

$$[I_m, I_n] = [I_m, G_n] = [G_m, G_n] = 0.$$

$\boxed{\text{Wakimoto Realization}}$ Level $k \neq 0, -N$

$U_q(\widehat{sl_N})$

$\boxed{\text{Boson}}$ $a_m^i \quad (1 \leq i \leq N-1)$

$b_m^{ij}, c_m^{ij} \quad (1 \leq i < j \leq N)$

$$\boxed{\begin{aligned}
\cdot\ [a_m^i, a_n^j] &= \frac{1}{m}[(k+N)m][A_{ij}m]\,\delta_{ij}\,\delta_{m+n,0} \\
\cdot\ [b_m^{ij}, b_n^{i'j'}] &= -\frac{1}{m}[m]^2\,\delta_{ii'}\delta_{jj'}\,\delta_{m+n,0} \\
\cdot\ [c_m^{ij}, c_n^{i'j'}] &= \frac{1}{m}[m]^2\,\delta_{ii'}\delta_{jj'}\,\delta_{m+n,0}
\end{aligned}}$$

$$(A_{ij}) = \begin{pmatrix} 2 & -1 & & 0 \\ -1 & 2 & -1 & \\ & -1 & & \\ 0 & & 2 & -1 \\ & & -1 & 2 \end{pmatrix}$$

Cartan matrix

[Wakimoto 1986]
[Feigin, Frenkel 1990]
[Awata et.al. 1994]

$\boxed{\text{Zero Mode}}$ $\quad P_a^{\bar{i}} \quad (1 \leq \bar{i} \leq N)$

$$P_b^{\bar{i}\bar{j}}, \; P_c^{\bar{i}\bar{j}} \quad (1 \leq \bar{i} < \bar{j} \leq N)$$

$$\boxed{\begin{aligned} &\cdot \; [\, P_a^{\bar{i}}, P_a^{\bar{j}} \,] = (k+N) A_{\bar{i}\bar{j}} \\ &\cdot \; [\, P_b^{\bar{i}\bar{j}}, P_b^{\bar{i}'\bar{j}'} \,] = -\delta_{\bar{i}\bar{i}'} \delta_{\bar{j}\bar{j}'} \\ &\cdot \; [\, P_c^{\bar{i}\bar{j}}, P_c^{\bar{i}'\bar{j}'} \,] = \delta_{\bar{i}\bar{i}'} \delta_{\bar{j}\bar{j}'} \end{aligned}}$$

In what follows we consider operators acting on Fock space.

Auxiliary Field

- $a^{i}_{\pm}(z) = \pm(q-q^{-1})\sum_{n>0} a^{i}_{\pm n} z^{\mp n} \pm P^{i}_{a} \log q$
- $b^{i\bar{\alpha}}_{\pm}(z) = \pm(q-q^{-1})\sum_{n>0} b^{i\bar{\alpha}}_{\pm n} z^{\mp n} \pm P^{i\bar{\alpha}}_{b} \log q$
- $b^{\bar{i}\bar{\alpha}}(z) = -\sum_{n\neq 0} \frac{1}{[n]} b^{\bar{i}\bar{\alpha}}_{n} z^{-n} + q^{\bar{i}\bar{\alpha}}_{b} + P^{\bar{i}\bar{\alpha}}_{b} \log z$
- $C^{\bar{i}\bar{\alpha}}(z) = -\sum_{n\neq 0} \frac{1}{[n]} C^{\bar{i}\bar{\alpha}}_{n} z^{-n} + q^{\bar{i}\bar{\alpha}}_{c} + P^{\bar{i}\bar{\alpha}}_{c} \log z$

Free Field Realization

$U_q(\widehat{sl_N})$, Level $k \neq 0, -N$

$(i = 1, 2, \cdots, N-1)$

$$\cdot\; x_i^+(z) = \frac{-1}{(q-q^{-1})z} \sum_{j=1}^{i} :\exp\left((b+c)^{ji}(q^{j-1}z)\right)$$
$$\times \Big(\exp\left(b_+^{j\,i+1}(q^{j-1}z) - (b+c)^{j\,i+1}(q^{j}z)\right)$$
$$- \exp\left(b^{j\,i+1}(q^{j-1}z) - (b+c)^{j\,i+1}(q^{j}z)\right)\Big)$$
$$\times \exp\Big(\sum_{l=1}^{j-1}\left(b_+^{l\,i+1}(q^{l-1}z) - b_+^{l\,i}(q^{l}z)\right)\Big):$$

$$\cdot\; x_i^-(z) = \frac{-1}{(q-q^{-1})z} \Big(\sum_{j=1}^{i-1} :\exp\left((b+c)^{j\,i+1}(q^{-(k+j)}z)\right)$$
$$\times \Big(\exp\left(-b_-^{j,i}(q^{-(k+j)}z) - (b+c)^{ji}(q^{-(k+j-1)}z)\right)$$
$$- \exp\left(-b_+^{ji}(q^{-(k+j)}z) - (b+c)^{ji}(q^{-(k+j-1)}z)\right)\Big) \times$$

$$
\begin{aligned}
&\times \exp\Big(\sum_{l=j+1}^{i} \Big(b_-^{l\, i+1}(q^{-k-l+1} z) - b_-^{l i}(q^{-k-l} z) \Big) \\
&\qquad + a_-^{i}(q^{-\frac{k+N}{2}} z) + \sum_{l=i+1}^{N} \Big(b_-^{il}(q^{-k-l} z) - b_-^{i+1\, l}(q^{-k-l+1} z) \Big) \Big) : \\
&+ :\exp\Big((b+c)^{i\, i+1}(q^{-k-i} z) \Big) \\
&\qquad \times \exp\Big(a_-^{i}(q^{-\frac{k+N}{2}} z) + \sum_{l=i+1}^{N} \Big(b_-^{il}(q^{-k-l} z) - b_-^{i+1\, l}(q^{-k-l+1} z) \Big) \Big) : \\
&- :\exp\Big((b+c)^{i\, i+1}(q^{k+i} z) \Big) \\
&\qquad \times \exp\Big(a_+^{i}(q^{\frac{k+N}{2}} z) + \sum_{l=i+1}^{N} \Big(b_+^{il}(q^{k+l} z) - b_+^{i+1\, l}(q^{k+l-1} z) \Big) \Big) : \\
&- \sum_{j=i+2}^{N} :\exp\Big((b+c)^{ij}(q^{k+j-1} z) \Big) \\
&\qquad \times \Big(\exp\Big(b_+^{i+1\, j}(q^{k+j-1} z) - (b+c)^{i+1\, j}(q^{k+j} z) \Big) \\
&\qquad\quad - \exp\Big(b_-^{i+1\, j}(q^{k+j-1} z) - (b+c)^{i+1\, j}(q^{k+j-2} z) \Big) \Big) \\
&\qquad \times \exp\Big(a_+^{i}(q^{\frac{k+N}{2}} z) + \sum_{l=i}^{N} \Big(b_+^{il}(q^{k+l} z) - b_+^{i+1\, l}(q^{k+l-1} z) \Big) \Big) : \Big)
\end{aligned}
$$

Free Field Realization

$U_{q,p}(\widehat{sl_N})$ Level $k \neq 0, -N$

$(s=N,\ p=q^{2h},\ q=x)$

[Kojima, 2008, 2009]
[Chang, Ding, 2008, 2009]

Zero Mode

$P_i, Q_i \quad (1 \leq i \leq N-1)$

- $$[P_i, Q_j] = \frac{A_{ij}}{2}$$

- $$h_i = \sum_{j=1}^{i} (P_b^{j,i+1} - P_b^{j,i}) + P_a^{i} + \sum_{j=i+1}^{N} (P_b^{i,j} - P_b^{i+1,j})$$

Free Field Realization

$U_{q,p}(\widehat{sl_N})$, Level $k \neq 0, -N$

- $E_i(z) = U^{*i}(z)\, \chi_i^{+}(z)\, e^{2Q_i} z^{-\frac{P_i - 1}{r^*}}$
- $F_i(z) = \chi_i^{-}(z)\, U^{i}(z)\, z^{\frac{h_i + P_i - 1}{r}}$ $\quad (1 \leq i \leq N-1)$

- $$U^{*i}(z) = \left(\prod_{j=1}^{i-1} B_+^{*j\,i+1}(q^{2-j}z)\, B_-^{*j\,i}(q^{1-j}z)\right) \times B_+^{*i\,i+1}(q^{2-i}z)\, B_+^{*i\,i+1}(q^{-i}z) \times \left(\prod_{j=i+2}^{N} B_+^{*i\,j}(q^{-j+1}z)\, B_-^{*i+1\,j}(q^{-j+2}z)\right) \times A^{*i}(q^{\frac{k-N}{2}}z)$$

- $$U^{i}(z) = \left(\prod_{j=1}^{i-1} B_-^{j\,i+1}(q^{-2+j}z)\, B_+^{j\,i}(q^{-1+j}z)\right) \times B_-^{i\,i+1}(q^{-2+i}z)\, B^{i\,i+1}(q^{i}z) \times \left(\prod_{j=i+2}^{N} B_-^{i\,j}(q^{j-1}z)\, B_+^{i+1\,j}(q^{j-2}z)\right) \times A^{i}(q^{-\frac{k-N}{2}}z)$$

Auxiliary Field

$(1 \leq i < j \leq N)$

- $$\mathcal{B}^{*\,ij}_{\pm}(z) = \exp\left(\pm \sum_{n>0} \frac{1}{[(r-k)n]} b^{ij}_{-n} \left(q^{r-k-1} z\right)^{n} \right)$$

- $$\mathcal{B}^{ij}_{\pm}(z) = \exp\left(\pm \sum_{n>0} \frac{1}{[rn]} b^{ij}_{n} \left(q^{-r+k+1} z\right)^{-n} \right)$$

- $$A^{*\,i}(z) = \exp\left(\sum_{n>0} \frac{1}{[(r-k)n]} a^{i}_{-n} \left(q^{r-k} z\right)^{n} \right)$$

- $$A^{i}(z) = \exp\left(- \sum_{n>0} \frac{1}{[rn]} a^{i}_{n} \left(q^{-r+k} z\right)^{-n} \right)$$

Affinization of Current

$U_{q,p}(\widehat{sl_2})$

$N=2$, Level k

- $$E_2(z) = B_-^{*12}(qz)\,B_-^{*12}(q^{-1}z)\,A^{*1}\!\left(q^{\frac{k-2}{2}}z\right) \times \frac{-1}{(q-q^{-1})z}$$
$$\times \Big(:\exp\left(-b_+^{12}(z) + (b+c)^{12}(qz)\right): - :\exp\left(-b_-^{12}(z) + (b+c)^{12}(q^{-1}z)\right): \Big)$$

- $$F_2(z) = \frac{-1}{(q-q^{-1})z} \times \Big(:\exp\left(-b_-^{12}(z) - (b+c)^{12}(qz) + a_-^{1}\left(q^{-\frac{k+2}{2}}z\right)\right):$$
$$- :\exp\left(-b_+^{12}(z) - (b+c)^{12}(q^{-1}z) + a_+^{1}\left(q^{\frac{k+3}{2}}z\right)\right): \Big)$$
$$\times\ B_+^{12}(q^{-1}z)\,B_+^{12}(qz)\,A^{1}\!\left(q^{\frac{k-2}{2}}z\right)$$

Def **Nonlocal integrals of motion** Level k $U_{q,p}(\widehat{sl_2})$

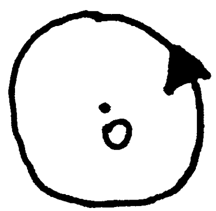
$$G_m^* = \int \cdots \int_C \prod_{j=1}^{m} \frac{dz_j^{(1)}}{z_j^{(1)}} \prod_{j=1}^{m} \frac{dz_j^{(2)}}{z_j^{(2)}}$$

$$\times E_1(z_1^{(1)}) E_1(z_2^{(1)}) \cdots E_1(z_m^{(1)}) E_2(z_1^{(2)}) E_2(z_2^{(2)}) \cdots E_2(z_m^{(2)})$$

$$\times \frac{\prod_{t=1,2} \prod_{1 \le i < j \le m} [u_i^{(t)} - u_j^{(t)}]_{r-k} [u_j^{(t)} - u_i^{(t)} + 1]_{r-k}}{\prod_{1 \le i, j \le m} [u_i^{(1)} - u_j^{(2)} - 1]_{r-k} [u_j^{(2)} - u_i^{(1)}]_{r-k}}$$

$$\times \left[\sum_{j=1}^{m} (u_j^{(1)} - u_{j+1}^{(2)}) - (P_b^{12} + P_c^{12}) \right]_{r-k} \left[\sum_{j=1}^{m} (u_j^{(1)} - u_{j+1}^{(2)}) + (P_b^{12} + P_c^{12}) \right]_{r-k}$$

$C:$ $z_j^{(t)}$ $|z_j^{(t)}| = 1$



Theorem

$$[G_m^*, G_n^*] = 0 \qquad (m, n = 1, 2, \cdots)$$
$$[G_m, G_n] = 0$$

[Kojima 2009]

Level k Generalization of $W_{qt}(\widehat{sl_N})_k$

$$[\, W_{qt}(\widehat{sl_N})_k \,,\, \overline{X}_{\alpha_i\, k} \,] = 0 \qquad (i = 1, 2, \cdots, N-1)$$

Open Problem   Our Homework

- Free Field Realization of $W_{qt}(\widehat{sl_N})_k$
- Local Integrals of Motion I_m for Level k

- Elliptic deformed Baxter's T-Q operator

  $\rightsquigarrow$ Elliptic Hirota-Miwa equation

Summary

- We constructed infinitely many commutative operators I_m, G_m for deformed W algebra $W_{qt}(\widehat{sl_N})$.

- We constructed Level k realization of Elliptic Algebra $U_{qp}(\widehat{sl_N})$ which we call Wakimoto realization.

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