Integral transformation and Darboux transformation of Heun's differential equation

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References

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Heun's differential equation

$$\frac{d^2y}{dz^2}+\left(\frac{\gamma}{z}+\frac{\delta}{z-1}+\frac{\epsilon}{z-t}\right)\frac{dy}{dz}+\frac{\alpha\beta z-q}{z(z-1)(z-t)}y=0,$$

with the condition

$$\gamma+\delta+\epsilon=\alpha+\beta+1.$$

Four singularities $\{0,1,t,\infty\}$.

Three singularities: Hypergeometric equation

$$\frac{d^2y}{dz^2}+\left(\frac{\gamma}{z}+\frac{\alpha+\beta-\gamma+1}{z-1}\right)\frac{dy}{dz}+\frac{\alpha\beta}{z(z-1)}y=0,$$

which has been studied very well.

It is much harder to study Heun's equation.

Known solutions of Heun's equation

- Heun polynomial (Quasi-exact solvability)
- Heun function (Approximation)
- Algebraic solutions (Finite monodromy)
- Finite-gap integration

We now change variables.

Elliptic functions

$\wp(x)$: Weierstrass elliptic function.

$$\wp(x) = \frac{1}{x^2} + \sum_{(m,n)\in\mathbb{Z}\times\mathbb{Z}\setminus\{(0,0)\}} \left(\frac{1}{(x-2m\omega_1-2n\omega_3)^2} - \frac{1}{(2m\omega_1+2n\omega_3)^2}\right).$$

Double-periodicity:
$\wp(x) = \wp(x+2\omega_1) = \wp(x+2\omega_3)$.

Set $\omega_2 = -\omega_1 - \omega_3$, $e_i = \wp(\omega_i)$ $(i = 1, 2, 3)$.

Half periods: $0(=\omega_0)$, ω_1, ω_2, ω_3.

Relations:

$$e_1 + e_2 + e_3 = 0,$$
$$(\wp'(x))^2 = 4(\wp(x)-e_1)(\wp(x)-e_2)(\wp(x)-e_3),$$
$$\frac{\wp''(z)}{\wp'(z)^2} = \frac{1}{2}\sum_{i=1}^{3}\frac{1}{\wp(x)-e_i},$$
$$\wp(x+\omega_1) = e_1 + \frac{(e_1-e_2)(e_1-e_3)}{\wp(x)-e_1},\ \text{etc.}$$

Elliptic representation

Heun's differential equation

$$\frac{d^2y}{dz^2}+\left(\frac{\gamma}{z}+\frac{\delta}{z-1}+\frac{\epsilon}{z-t}\right)\frac{dy}{dz}+\frac{\alpha\beta z-q}{z(z-1)(z-t)}y=0,$$

q : accessory parameter.

By setting

$$z=\frac{\wp(x)-e_1}{e_2-e_1},\ t=\frac{e_3-e_1}{e_2-e_1},$$

$$y\tilde{\Phi}(z)=f(x),\quad \tilde{\Phi}(z)=z^{\frac{-l_0}{2}}(z-1)^{\frac{-l_1}{2}}(z-t)^{\frac{-l_2}{2}},$$

Heun's equation is transformed to

$$\left(-\frac{d^2}{dx^2}+\sum_{i=0}^{3}l_i(l_i+1)\wp(x+\omega_i)\right)f(x)=Ef(x).$$

Correspondence

$0\leftrightarrow\omega_1,\quad 1\leftrightarrow\omega_2,\quad t\leftrightarrow\omega_3,\quad \infty\leftrightarrow\omega_0(=0),$
$l_0=\beta-\alpha-1/2,\ l_1=-\gamma+1/2,\ l_2=-\delta+1/2,$
$E=-4q(e_2-e_1)+(*),\ l_3=-\epsilon+1/2.$

The case $l_1=l_2=l_3=0$ ($\gamma=\delta=\epsilon=1/2$):
Lamé's differential equation

$$H^{(l_0,l_1,l_2,l_3)} = -\frac{d^2}{dx^2} + \sum_{i=0}^{3} l_i(l_i+1)\wp(x+\omega_i).$$

Finite-gap integration is applicable for the case $l_0, l_1, l_2, l_3 \in \mathbb{Z}$, all eigenvalues E.
$\Rightarrow$ Monodromy formulas by hyperelliptic integral, Hermite-Krichever Ansatz.

Set

$$H_1 = -\frac{d^2}{dx^2} + 6\wp(x), \ (l_0 = 2, \ l_1 = l_2 = l_3 = 0)$$
$$H_2 = -\frac{d^2}{dx^2} + 2\wp(x) + 2\wp(x+\omega_1) + 2\wp(x+\omega_2),$$
$$(l_0 = l_1 = l_2 = 1, \ l_3 = 0).$$

It is shown that monodromy formulas for H_1 coincide with the ones for H_2.

We explain this phenomera by Darboux transformation.

Moreover, we establish that eigenfunctions of

$$-\frac{d^2}{dx^2}+2l(2l+1)\wp(x)$$

with eigenvalue E is isomonodromic to the ones of

$$-\frac{d^2}{dx^2}+l(l+1)\wp(x)+l(l+1)\wp(x+\omega_1)+l(l+1)\wp(x+\omega_2)+(l-1)l\wp(x+\omega_3)$$

by generalized Darboux transformation.

Darboux transformation

Set

$$H = -\frac{d^2}{dx^2} + q(x)$$

and assume that $\phi_0(x)$ satisfies

$$H\phi_0(x) = E_0\phi_0(x).$$

Then $q(x) = \phi_0''(x)/\phi_0(x) + E_0$. Set

$$L = \frac{d}{dx} - \frac{\phi_0'(x)}{\phi_0(x)}, \quad L^\dagger = -\frac{d}{dx} - \frac{\phi_0'(x)}{\phi_0(x)}.$$

We have

$$\begin{aligned} L^\dagger L &= \left(-\frac{d}{dx} - \frac{\phi_0'(x)}{\phi_0(x)}\right)\left(\frac{d}{dx} - \frac{\phi_0'(x)}{\phi_0(x)}\right) \\ &= -\frac{d^2}{dx^2} + \left(\frac{\phi_0'(x)}{\phi_0(x)}\right)' + \left(\frac{\phi_0'(x)}{\phi_0(x)}\right)^2 \\ &= H - E_0, \\ LL^\dagger &= -\frac{d^2}{dx^2} - \left(\frac{\phi_0'(x)}{\phi_0(x)}\right)' + \left(\frac{\phi_0'(x)}{\phi_0(x)}\right)^2 \\ &= H - 2\left(\frac{\phi_0'(x)}{\phi_0(x)}\right)' - E_0. \end{aligned}$$

Set

$$\tilde{H} = -\frac{d^2}{dx^2} + q(x) + 2\left(\frac{\phi_0'(x)}{\phi_0(x)}\right)',$$

then

$$H = L^\dagger L + E_0, \quad \tilde{H} = LL^\dagger + E_0,$$

$$LH = LL^\dagger L + E_0 L = \tilde{H}L,$$
$$L^\dagger \tilde{H} = L^\dagger LL^\dagger + E_0 L^\dagger = HL^\dagger.$$

If $f(x)$ is an eigenfunction of H with the eigenvalue E, then $Lf(x)$ is an eigenfunction of $\tilde{H}$ with the eigenvalue E, because

$$\tilde{H}(Lf(x)) = LHf(x) = L(Ef(x)) = E(Lf(x)).$$

Note that the operator $L\left(= \frac{d}{dx} - \frac{\phi_0'(x)}{\phi_0(x)}\right)$ annihilates the 1-dimensional space $\mathbb{C}\phi_0(x)$.

Generalized Darboux transf.

$$H = -\frac{d^2}{dx^2} + q(x).$$

U: n-dimensional space of functions

$$L = \left(\frac{d}{dx}\right)^n + \sum_{i=1}^{n} c_i(x) \left(\frac{d}{dx}\right)^{n-i}$$

is the operator that annihilates any elements in U, i.e., $Lf(x) = 0$ for all $f(x) \in U$. Set

$$\tilde{H} = -\frac{d^2}{dx^2} + q(x) + 2c_1'(x).$$

Proposition 1. $\left(\text{\textit{c.f.}} \quad \begin{array}{l} \text{\textit{Crum 1955}} \\ \text{\textit{Aoyama, Sato, Tanaka 2001}} \end{array} \right)$

If the space U is invariant under the action of H, then we have

$$\tilde{H}L = LH.$$

We call L Crum-Darboux transformation (the generalized Darboux transformation).

The case $n = 1$.

$U = \mathbb{C}\phi_0(x)$, $L = \frac{d}{dx} - \frac{\phi_0'(x)}{\phi_0(x)}$, $2c_1'(x) = 2\left(\frac{\phi_0'(x)}{\phi_0(x)}\right)'$.

We reproduce Darboux transformation.

Quasi-solvability of Heun's equation

$$H^{(l_0,l_1,l_2,l_3)}=-\frac{d^2}{dx^2}+\sum_{i=0}^{3}l_i(l_i+1)\wp(x+\omega_i).$$

Proposition 2. *(Quasi-solvability)*
$\alpha_i=-l_i$ *or* l_i+1 $(i=0,1,2,3)$, $d=-\sum_{i=0}^{3}\alpha_i/2$.
Assume $d\in\mathbb{Z}_{\geq 0}$.
Let $V_{\alpha_0,\alpha_1,\alpha_2,\alpha_3}$ *be the* $d+1$*-dimensional space spanned by*

$$\left\{\widehat{\Phi}(\wp(x))\wp(x)^n\right\}_{n=0,\dots,d} \text{ , where}$$

$\widehat{\Phi}(z)=(z-e_1)^{\alpha_1/2}(z-e_2)^{\alpha_2/2}(z-e_3)^{\alpha_3/2}$.
Then the operator $H^{(l_0,l_1,l_2,l_3)}$ *preserves the space* $V_{\alpha_0,\alpha_1,\alpha_2,\alpha_3}$.

Proposition 3. *Write the minimal annihilation operator* $L_{\alpha_0,\alpha_1,\alpha_2,\alpha_3}$ *of* $V_{\alpha_0,\alpha_1,\alpha_2,\alpha_3}$ *as*

$$L_{\alpha_0,\alpha_1,\alpha_2,\alpha_3}=\left(\frac{d}{dx}\right)^{d+1}+\sum_{i=1}^{d+1}c_i(x)\left(\frac{d}{dx}\right)^{d+1-i}.$$

Then

$$c_1(x)=-\frac{d+1}{4}\left(\sum_{i=1}^{3}\frac{2\alpha_i+d}{\wp(x)-e_i}\right)\wp'(x),$$

and $c_i(x)$ $(i=1,\dots,d+1)$ *are doubly-periodic.*

Crum-Darboux transformation for Heun's equation

Theorem 4.
$\alpha_i = -l_i$ *or* $l_i + 1$ $(i = 0, 1, 2, 3)$,
$d = -\sum_{i=0}^{3} \alpha_i/2$. *Assume* $d \in \mathbb{Z}_{\geq 0}$.
Let $L_{\alpha_0,\alpha_1,\alpha_2,\alpha_3}$ *be the operator defined in Proposition 3. Then we have*

$$H^{(\alpha_0+d,\alpha_1+d,\alpha_2+d,\alpha_3+d)} L_{\alpha_0,\alpha_1,\alpha_2,\alpha_3} = L_{\alpha_0,\alpha_1,\alpha_2,\alpha_3} H^{(l_0,l_1,l_2,l_3)}.$$

Proof. It follows from Proposition 1 that

$$(H^{(l_0,l_1,l_2,l_3)} + 2c_1'(x)) L_{\alpha_0,\alpha_1,\alpha_2,\alpha_3} = L_{\alpha_0,\alpha_1,\alpha_2,\alpha_3} H^{(l_0,l_1,l_2,l_3)}.$$

It is shown that

$$H^{(l_0,l_1,l_2,l_3)} + 2c_1'(x) = H^{(\alpha_0+d,\alpha_1+d,\alpha_2+d,\alpha_3+d)}.$$

□

If $d = 0$ (the case of Darboux transformation), then the theorem was essentially obtained by Khare and Sukhatme (2005).

Monodromy

$f_1(x,E)$, $f_2(x,E)$: a basis of solutions to $(H^{(l_0,l_1,l_2,l_3)}-E)f(x)=0$.
$f_1(x+2\omega_k,E)$, $f_2(x+2\omega_k,E)$ $(k=1,3)$ are also solutions to the differential equation, and

$$\begin{aligned}&(f_1(x+2\omega_k,E)\ f_2(x+2\omega_k,E))\\ &=(f_1(x,E)\ f_2(x,E))\begin{pmatrix} a_{11}^{(k)} & a_{12}^{(k)} \\ a_{21}^{(k)} & a_{22}^{(k)} \end{pmatrix}.\end{aligned}$$

Set $\tilde{f}_i(x,E)=L_{\alpha_0,\alpha_1,\alpha_2,\alpha_3}f_i(x,E)$ $(i=1,2)$. Then
$H^{(\alpha_0+d,\alpha_1+d,\alpha_2+d,\alpha_3+d)}\tilde{f}_i(x,E)=E\tilde{f}_i(x,E)$.
Since $L_{\alpha_0,\alpha_1,\alpha_2,\alpha_3}$ is doubly-periodic, we have

$$\begin{aligned}&(\tilde{f}_1(x+2\omega_k,E)\ \tilde{f}_2(x+2\omega_k,E))\\ &=(\tilde{f}_1(x,E)\ \tilde{f}_2(x,E))\begin{pmatrix} a_{11}^{(k)} & a_{12}^{(k)} \\ a_{21}^{(k)} & a_{22}^{(k)} \end{pmatrix}.\end{aligned}$$

Proposition 5. *The monodromy structure of* $H^{(l_0,l_1,l_2,l_3)}$ *coincides with the one of* $H^{(\alpha_0+d,\alpha_1+d,\alpha_2+d,\alpha_3+d)}$.

Namely, the operator $L_{\alpha_0,\alpha_1,\alpha_2,\alpha_3}$ *defines an isomonodromic transformation from* $H^{(l_0,l_1,l_2,l_3)}$ *to* $H^{(\alpha_0+d,\alpha_1+d,\alpha_2+d,\alpha_3+d)}$.

Example

The case $l_0 = 2l$ $(l \in \mathbb{Z}_{\geq 1})$, $l_1 = l_2 = l_3 = 0$.

Set $\alpha_0 = -2l$, $\alpha_1 = \alpha_2 = 1$, $\alpha_3 = 0$.
Then $d = -(\alpha_0 + \cdots + \alpha_3)/2 = l - 1$.

$$H^{(-l-1,l,l,l-1)} L_{-2l,1,1,0} = L_{-2l,1,1,0} H^{(2l,0,0,0)},$$
$$H^{(-l-1,l,l,l-1)} = H^{(l,l,l,l-1)}.$$

$H^{(2l,0,0,0)}$ is isomonodromic to $H^{(l,l,l,l-1)}$.

If $l = 1$, then $d = 0$,

$$H^{(1,1,1,0)} L_{-2,1,1,0} = L_{-2,1,1,0} H^{(2,0,0,0)},$$

and the operator $L_{-2,1,1,0}$ is written as

$$L_{-2,1,1,0} = \frac{d}{dx} - \frac{\wp'(x)}{2(\wp(x) - e_1)} - \frac{\wp'(x)}{2(\wp(x) - e_2)}.$$

Hence $H_1 = -\frac{d^2}{dx^2} + 6\wp(x)$ is isomonodromic to $H_2 = -\frac{d^2}{dx^2} + 2\wp(x) + 2\wp(x+\omega_1) + 2\wp(x+\omega_2)$.

Application to finite-gap integration

A feature of finite-gap integration:
Existence of an differential operator $\tilde{A}$ s.t. $[\tilde{A}, H] = 0$ ($H = -d^2/dx^2 + v(x)$) and $\deg(\tilde{A})$ is odd.
($\Leftrightarrow$ the potential $v(x)$ satisfies stationary higher order KdV equation)

Theorem 6.
If $l_0, l_1, l_2, l_3 \in \mathbb{Z}$, then we can construct an odd-order differential operator $\tilde{A}$ such that $[\tilde{A}, H^{(l_0,l_1,l_2,l_3)}] = 0$ by composing four Crum-Darboux transformations.

If $l_0 = 2$, $l_1 = l_2 = l_3 = 0$, then
$\tilde{A} = L_{2,-1,-1,0} L_{1,-2,1,0} L_{0,2,-1,-1} L_{-2,0,0,0}$.

Integral transformation of Heun's equation

Middle convolution for 2×2 Fuchsian system with four singularities $\{0, 1, t, \infty\}$

$$\frac{dY}{dz} = \left(\frac{A_0}{z} + \frac{A_1}{z-1} + \frac{A_t}{z-t}\right)Y, \;\; Y = \begin{pmatrix} y_1(z) \\ y_2(z) \end{pmatrix}.$$

⇒ Integral transformation of 2×2 Fuchsian system with four singularities $\{0, 1, t, \infty\}$ (T. JMAA 2008).
⇒ Integral transformation of Heun's equation (T. SIGMA 2009).

But it was already established by Kazakov and Slavyanov (1996) by another method.

Theorem 7. *([KS1996]) Set*

$$\begin{aligned}
&\{\mu - (2-\alpha)\}\{\mu - (2-\beta)\} = 0, \; \gamma' = \gamma + \mu - 1, \\
&\delta' = \delta + \mu - 1, \; \epsilon' = \epsilon + \mu - 1, \\
&\alpha' = \mu, \; \beta' = 2\mu + \alpha + \beta - 3, \\
&q' = q + (1-\mu)(\epsilon + \delta t + (\gamma - \mu)(t+1)).
\end{aligned}$$

Let $y(w)$ be a solution to

$$\frac{d^2y}{dw^2} + \left(\frac{\gamma}{w} + \frac{\delta}{w-1} + \frac{\epsilon}{w-t}\right)\frac{dy}{dw} + \frac{\alpha\beta w - q}{w(w-1)(w-t)}y = 0.$$

Then the functions $(i \in \{0, 1, t, \infty\})$

$$\tilde{y}(z) = \int_{[\alpha_z, \alpha_i]} y(w)(z-w)^{-\mu} dw$$

are solutions to

$$\frac{d^2\tilde{y}}{dz^2} + \left(\frac{\gamma'}{z} + \frac{\delta'}{z-1} + \frac{\epsilon'}{z-t}\right)\frac{d\tilde{y}}{dz} + \frac{\alpha'\beta' z - q'}{z(z-1)(z-t)}\tilde{y} = 0.$$

Elliptic representation of integral transformation

$\sigma(x)$: Weierstrass sigma function,.
$\sigma_i(x)$ $(i = 1, 2, 3)$: Weierstrass co-sigma function which has a zero at $x = \omega_i$.
Let $\alpha_0 \in \{-l_0, l_0 + 1\}$ and set

$$\eta = \frac{\alpha_0 + l_1 + l_2 + l_3 + 3}{2}, \; l_0' = \frac{-\alpha_0 + l_1 + l_2 + l_3 + 1}{2},$$
$$l_1' = \frac{-\alpha_0 + l_1 - l_2 - l_3 - 1}{2}, \; l_2' = \frac{-\alpha_0 - l_1 + l_2 - l_3 - 1}{2},$$
$$l_3' = \frac{-\alpha_0 - l_1 - l_2 + l_3 - 1}{2}.$$

Proposition 8. *If $f(x)$ is a solution of*

$$\left(-\frac{d^2}{dx^2} + \sum_{i=0}^{3} l_i(l_i + 1)\wp(x + \omega_i) - E\right) f(x) = 0,$$

then the function

$$\tilde{f}(x) = \sigma(x)^{-l_0'}\sigma_1(x)^{-l_1'}\sigma_2(x)^{-l_2'}\sigma_3(x)^{-l_3'} \int_{I_l} f(y)\sigma(y)^{-\alpha_0+1} \cdot$$
$$\sigma_1(y)^{l_1+1}\sigma_2(y)^{l_2+1}\sigma_3(y)^{l_3+1}(\sigma(x+y)\sigma(x-y))^{-\eta}dy$$

($l \in \{0, 1, 2, 3\}$) is a solution of

$$\left(-\frac{d^2}{dx^2} + \sum_{i=0}^{3} l_i'(l_i' + 1)\wp(x + \omega_i) - E\right) \tilde{f}(x) = 0.$$

I_l $(l = 0, 1, 2, 3)$: suitable cycle on $\mathbb{C}$ with variable y.

If $\eta \in \mathbb{Z}_{\geq 1}$, then we essentially recover generalized Darboux transformation.

Application to monodromy

$f_1(x,E), f_2(x,E)$: independent solutions of

$$\left(-\frac{d^2}{dx^2}+\sum_{i=0}^{3} l_i(l_i+1)\wp(x+\omega_i)-E\right)f(x)=0, \quad (1)$$

$M_{2\omega_k}(E)$ $(k\in\{1,3\})$: 2×2 matrix;
Monodromy matrix of Eq.(1) w.r.t. $x\to x+2\omega_k$

$$(f_1(x+2\omega_k,E)\ f_2(x+2\omega_k,E))=(f_1(x,E)\ f_2(x,E))M_{2\omega_k}(E)$$

$$\left(-\frac{d^2}{dx^2}+\sum_{i=0}^{3} l'_i(l'_i+1)\wp(x+\omega_i)-E\right)\tilde{f}(x)=0, \quad (2)$$

$M'_{2\omega_k}(E)$: Monod. matrix of Eq.(2) w.r.t. $x\to x+2\omega_k$.

Theorem 9.

$$\mathrm{tr}M'_{2\omega_k}(E)=\mathrm{tr}M_{2\omega_k}(E), \quad (k=1,3)$$

Corollary 10. *Let* $k\in\{1,3\}$.
$\exists f(x,E)$*: solution of Eq.(1)*
s.t. $f(x+2\omega_k,E)=C_k(E)f(x,E)$
$\Rightarrow \exists\tilde{f}(x,E)$*: solution of Eq.(2)*
s.t. $\tilde{f}(x+2\omega_k,E)=C_k(E)\tilde{f}(x,E)$.
In other word, periodicity is preserved by the integral transformation.

Summary

Isomonodromic transformations for Heun's equation by Crum-Darboux transformations.

Relationship to finite-gap integration.
Consruction of a commuting operator.

Integral transformations for Heun's equation:
A genaralization of Crum-Darboux transformation.

Invariance of monodromy