Supersymmetric Fifth Order Equations

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a joint work with Kai TIAN

Outline

- SUSY: basics
- 5th order SUSY integrable systems
- A SUSY Sawada-Kotera equation
- A Novel SUSY 5th KdV

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Direct way to SUSY systems:

Independent variables:	Dependent variables
$x \Rightarrow (x, \theta)$	fermionic or bosonic functions

Example (Kupershmidt: Phys. Letts. A (1984))

$$\begin{aligned} u_t &= 6uu_x - u_{xxx} + 3\xi\xi_{xx}, \\ \xi_t &= -4\xi_{xxx} + 3u_x\xi + 6u\xi_x \end{aligned}$$

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Example (Manin and Radul: Commun. Math. Phys. (1985))

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$$D = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial x}, \ \Phi = \xi + \theta u$$

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Soliton Equations and their SUSY Counterparts

Equation	Authors	Journals (Years)
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Fifth order evolution equations

We consider

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Question: Is it integrable ?
Yes, but only for three cases

- 5th KdV:

$$u_t + u_{xxxxx} + 10uu_{xxx} + 25u_x u_{xx} + 20u^2 u_x = 0$$

- Sawada-Kotera or Caudrey-Dodd-Gibbon

$$u_t + u_{xxxxx} + 5uu_{xxx} + 5u_x u_{xx} + 5u^2 u_x = 0$$

- Kaup-Kupershimdt:

$$u_t + u_{xxxxx} + 10u_x u_{xx} + 30u^2 u_x = 0$$

Fujimoto & Watanabe: Math. Japonica **28** (1983)
Harada & Oishi: J Phys. Soc. Japan **54** (1985)

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$$\phi_t+\phi_{xxxxx}+\left[\alpha\phi_{xx}(\mathcal{D}\phi)+\alpha\phi_x\mathcal{D}\phi_x+\alpha\phi(\mathcal{D}\phi)_{xx}+\frac{2}{5}\alpha^2\phi(\mathcal{D}\phi)^2\right]_x=0$$

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$$\phi_t+\phi_{xxxxx}+\left[10(\mathcal{D}\phi)\phi_{xx}+5(\mathcal{D}\phi)_{xx}\phi+15(\mathcal{D}\phi)^2\phi\right]_x=0$$

$\phi=\phi(x,t,\theta)$, a fermioic variable, $\mathcal{D}=\partial_\theta+\theta\partial_x$

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$$S_x(D_t+D_x^5)\tau\cdot\tau=0$$

with

$$\phi=2\mathcal{D}(\ln\tau)_x$$

where S_x is the super Hirota derivative McArthur & Yung (1993)

$$S_xf\cdot g=(Df)g-(-1)^{|f|}f(Dg)$$

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Symmetry Approach

We form the following equation

$$\begin{aligned}\Phi_t &= \Phi_{xxxxx} + \alpha_1 \Phi_{xxx}(D\Phi) + \alpha_2 \Phi_{xx}(D\Phi_x) + \alpha_3 \Phi_x (D\Phi_{xx}) \\ &\quad + \alpha_4 \Phi_x (D\Phi)^2 + \alpha_5 \Phi(D\Phi_{xxx}) + \alpha_6 \Phi(D\Phi_x)(D\Phi)\end{aligned}$$

which is the most general equation with the degree

$$\deg = \frac{13}{2}$$

where

$$\Phi(x, t, \theta) = \xi(x, t) + \theta u(x, t)$$

is fermionic field

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Question: integrability ?

1.
$$\begin{aligned}\Phi_t &= \Phi_{xxxxx} + \alpha\Phi_{xxx}(D\Phi) + 2\alpha\Phi_{xx}(D\Phi_x) + 2\alpha\Phi_x(D\Phi_{xx}) \\ &\quad + \tfrac{2}{5}\alpha^2\Phi_x(D\Phi)^2 + \alpha\Phi(D\Phi_{xxx}) + \tfrac{4}{5}\alpha^2\Phi(D\Phi_x)(D\Phi)\end{aligned}$$

2.
$$\begin{aligned}\Phi_t &= \Phi_{xxxxx} + \alpha\Phi_{xxx}(D\Phi) + \alpha\Phi_{xx}(D\Phi_x) + \alpha\Phi_x(D\Phi_{xx}) \\ &\quad + \tfrac{3}{10}\alpha^2\Phi_x(D\Phi)^2\end{aligned}$$

3.
$$\Phi_t = \Phi_{xxxxx} + \alpha\Phi_{xxx}(D\Phi) + \alpha\Phi_x(D\Phi_{xx}) + \tfrac{1}{5}\alpha^2\Phi_x(D\Phi)^2$$

4.
$$\begin{aligned}\Phi_t &= \Phi_{xxxxx} + \alpha\Phi_{xxx}(D\Phi) + \tfrac{3}{2}\alpha\Phi_{xx}(D\Phi_x) + \alpha\Phi_x(D\Phi_{xx}) \\ &\quad + \tfrac{1}{5}\alpha^2\Phi_x(D\Phi)^2\end{aligned}$$

5.
$$\Phi_t = \Phi_{xxxxx} + \alpha\Phi_{xxx}(D\Phi) + \alpha\Phi_{xx}(D\Phi_x) + \tfrac{1}{5}\alpha^2\Phi_x(D\Phi)^2$$

6.
$$\begin{aligned}\Phi_t &= \Phi_{xxxxx} + 2\alpha\Phi_{xxx}(D\Phi) + 3\alpha\Phi_{xx}(D\Phi_x) + \alpha\Phi_x(D\Phi_{xx}) \\ &\quad + \tfrac{3}{5}\alpha^2\Phi_x(D\Phi)^2 + \tfrac{3}{5}\alpha^2\Phi(D\Phi_x)(D\Phi)\end{aligned}$$

equations in components

2nd equation:

$$\begin{cases} u_t = u_{xxxxx} + \alpha u_{xxx} + 2\alpha u_x u_{xx} + \frac{3\alpha^2}{10} u^2 u_x \\ \xi_t = \xi_{xxxxx} + \alpha u \xi_{xxx} + \alpha u_x \xi_{xx} + \alpha u_{xx} \xi_x + \frac{3\alpha^2}{10} u^2 \xi_x \end{cases}$$

3rd one:

$$\begin{cases} u_t = u_{xxxxx} + \alpha u u_{xxx} + \alpha u_x u_{xx} + \frac{\alpha^2}{5} u^2 u_x \\ \\ \xi_t = \xi_{xxxxx} + \alpha u \xi_{xxx} + \alpha u_{xx} \xi_x + \frac{\alpha^2}{5} u^2 \xi_x \end{cases}$$

4th one:

$$\begin{cases} u_t = u_{xxxxx} + \alpha u u_{xxx} + \frac{5\alpha}{2} u_x u_{xx} + \frac{\alpha^2}{5} u^2 u_x \\ \xi_t = \xi_{xxxxx} + \alpha u \xi_{xxx} + \frac{3\alpha}{2} u_x \xi_{xx} + \alpha u_{xx} \xi_x + \frac{\alpha^2}{5} u^2 \xi_x \end{cases}$$

For last two, we have

$$\begin{cases} u_t = u_{xxxxx} + \alpha u u_{xxx} + \alpha u_x u_{xx} + \frac{\alpha^2}{5} u^2 u_x - \alpha \xi_{xxx} \xi_x \\ \xi_t = \xi_{xxxxx} + \alpha u \xi_{xxx} + \alpha u_x \xi_{xx} + \frac{\alpha^2}{5} u^2 \xi_x \end{cases}$$

susy SK

$$\begin{cases} u_t = u_{xxxxx} + 2\alpha u u_{xxx} + 4\alpha u_x u_{xx} + \frac{6\alpha^2}{5} u^2 u_x - \alpha \xi_{xxx} \xi_x \\ \qquad + \frac{3\alpha^2}{5} u \xi_{xx} \xi + \frac{3\alpha^2}{5} u_x \xi_x \xi \\ \xi_t = \xi_{xxxxx} + 2\alpha u \xi_{xxx} + 3\alpha u_x \xi_{xx} + \alpha u_{xx} \xi_x + \frac{3\alpha^2}{5} u^2 \xi_x + \frac{3\alpha^2}{5} u u_x \xi \end{cases}$$

a susy 5th KdV

SUSY SK

$$L = \partial_x^3 + \Psi\partial_x\mathcal{D} + U\partial_x + \Phi\mathcal{D} + V.$$

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Following Gel'fand and Dickey, we have

$$\frac{\partial L}{\partial t_n} = [(L^{\frac{n}{3}})_+, L]$$

where $[A, B] = AB - (-1)^{|A||B|}BA$.
Not surprising: a special case of Manin & Radul (Commun. Math. Phys. (1985)).
We consider: $n = 5$ and $t_5 = t$.

$$L + L^* = 0$$

Then we find

$$\Psi = 0, \quad V = \frac{1}{2}(U_x - (\mathcal{D}\Phi))$$

that is

$$L = \partial_x^3 + U\partial_x + \Phi\mathcal{D} + \frac{1}{2}(U_x - (\mathcal{D}\Phi))$$

In this case, we take $B = 9(L^{\frac{5}{3}})_+$, namely

$$\begin{aligned} B &= 9\partial_x^5 + 15U\partial_x^3 + 15\Phi\mathcal{D}\partial_x^2 + 15(U_x + V)\partial_x^2 \\ &\quad +15\Phi_x\mathcal{D}\partial_x + (10U_{xx} + 15V_x + 5U^2)\partial_x \\ &\quad +10(\Phi_{xx} + \Phi U)\mathcal{D} + 10V_{xx} + 10UV + 5\Phi(\mathcal{D}U) \end{aligned}$$

for convenience.

Then, the flow of equations, resulted from

$$\frac{\partial L}{\partial t} = [B, L]$$

reads as

$$U_t + U_{xxxxx} + 5\left(UU_{xx} + \frac{3}{4}U_x^2 + \frac{1}{3}U^3 + \Phi_x(\mathcal{D}U) + \frac{1}{2}\Phi(\mathcal{D}U_x) + \frac{1}{2}\Phi\Phi_x - \frac{3}{4}(\mathcal{D}\Phi)^2\right)_x = 0$$

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- setting $\Phi = 0$, we will have the standard Kaup-Kupershimdt
- Hamiltonian system:

$$\begin{pmatrix} U_t \\ \Phi_t \end{pmatrix} = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix} \delta\mathscr{H}$$

where the Hamiltonian is given by

$$\mathscr{H} = \int \Big[\frac{5}{4}\Phi(\mathcal{D}\phi)^2 - (\mathcal{D}U_x)(\mathcal{D}\Phi_{xx}) - \frac{5}{3}\Phi U^3 - \frac{5}{4}(\mathcal{D}U_x)(\mathcal{D}U)\Phi + \frac{5}{4}(\mathcal{D}U)U_x(\mathcal{D}\Phi) + 5(\mathcal{D}U)U(\mathcal{D}\Phi_x) + \frac{5}{2}(\mathcal{D}U)\Phi_x\Phi\Big] \mathrm{d}x\mathrm{d}\theta.$$

$$\begin{aligned} L &= \partial_x^3 + U\partial_x + \Phi\mathcal{D} + \frac{1}{2}(U_x - (\mathcal{D}\Phi)) \\ &= (\mathcal{D}^3 + W\mathcal{D} + \Upsilon)(\mathcal{D}^3 - \mathcal{D}W + \Upsilon), \end{aligned}$$

gives us a Miura-type transformation

$$\begin{aligned} U &= -2W_x - W^2 + (\mathcal{D}\Upsilon), \\ \Phi &= -\Upsilon_x - 2\Upsilon W, \end{aligned}$$

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and the modified system is given by

$$
\begin{aligned}
& W_t + W_{xxxxx} + 5W_{xxx}(\mathcal{D}\Upsilon) - 5W_{xxx}W_x - 5W_{xxx}W^2 - 5W_{xx}^2 + 10W_{xx}(\mathcal{D}\Upsilon_x) \\
& \quad - 20W_{xx}W_xW - 5W_{xx}W(\mathcal{D}\Upsilon) - 5W_x^3 - 5W_x^2(\mathcal{D}\Upsilon) + 5W_x(\mathcal{D}\Upsilon_{xx}) + 5W_x(\mathcal{D}\Upsilon)^2 \\
& \quad + 5W_xW^4 - 5W_xW(\mathcal{D}\Upsilon_x) + 10W(\mathcal{D}\Upsilon_x)(\mathcal{D}\Upsilon) - 10\Upsilon_x\Upsilon W_x + 5(\mathcal{D}W_{xx})\Upsilon_x \\
& \quad + 5(\mathcal{D}W_x)\Upsilon_{xx} + 5(\mathcal{D}W_x)\Upsilon W_x + 10(\mathcal{D}W_x)\Upsilon W^2 - 5(\mathcal{D}W)\Upsilon_x W_x + 10(\mathcal{D}W)\Upsilon_x W^2 \\
& \quad + 10(\mathcal{D}W)\Upsilon(\mathcal{D}\Upsilon_x) - 5(\mathcal{D}W)\Upsilon W_{xx} + 30(\mathcal{D}W)\Upsilon W_x W = 0,
\end{aligned}
$$

$$
\begin{aligned}
& \Upsilon_t + \Upsilon_{xxxxx} + 5\Upsilon_{xxx}(\mathcal{D}\Upsilon) - 5\Upsilon_{xxx}W^2 + 5\Upsilon_{xx}(\mathcal{D}\Upsilon_x) + 5\Upsilon_{xx}W_{xx} - 25\Upsilon_{xx}W_xW \\
& \quad + 5\Upsilon_{xx}W(\mathcal{D}\Upsilon) + 5\Upsilon_x(\mathcal{D}\Upsilon)^2 + 5\Upsilon_xW_{xxx} - 25\Upsilon_xW_{xx}W - 25\Upsilon_xW_x^2 + 5\Upsilon_xW_x(\mathcal{D}\Upsilon) \\
& \quad + 10\Upsilon_xW_xW^2 + 5\Upsilon_xW^4 - 10\Upsilon_xW^2(\mathcal{D}\Upsilon) + 5\Upsilon_xW(\mathcal{D}\Upsilon_x) - 10\Upsilon W_{xxx}W - 20\Upsilon W_{xx}W_x \\
& \quad + 10\Upsilon W_{xx}W^2 + 30\Upsilon W_x^2W + 20\Upsilon W_xW^3 - 30\Upsilon W_xW(\mathcal{D}\Upsilon) - 10\Upsilon W^2(\mathcal{D}\Upsilon_x) \\
& \quad - 5(\mathcal{D}W_x)\Upsilon_x\Upsilon - 5(\mathcal{D}W)\Upsilon_{xx}\Upsilon - 10(\mathcal{D}W)\Upsilon_x\Upsilon W = 0.
\end{aligned}
$$

$$W = 0, \quad \Upsilon = \phi.$$

we have

$$\phi_t + \phi_{xxxxx} + 5\phi_{xxx}(D\phi) + 5\phi_{xx}(D\phi_x) + 5\phi_x(D\phi)^2 = 0$$

Let $\phi = \xi(x,t) + \theta u(x,t)$, we have

$$\begin{aligned} u_t + u_{xxxxx} + 5uu_{xxx} + 5u_x u_{xx} + 5u^2 u_x - 5\xi_{xxx}\xi_x &= 0 \\ \xi_t + \xi_{xxxxx} + 5u\xi_{xxx} + 5u_x\xi_{xx} + 5u^2\xi_x &= 0 \end{aligned}$$

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Lax operator:

$$L = (\mathcal{D}^3 + \phi)(\mathcal{D}^3 + \phi)$$

Proposition

$$\mathrm{sres} L^{\frac{n}{3}} \in \mathrm{Im}\mathcal{D} \ .$$

where sres *means taking the super residue of a super pseudodifferential operator.*

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$$\mathcal{D}^3 + \phi = \Lambda^3$$

Thus,

$$L = \Lambda^6$$

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$$\mathtt{sres}(L^{\frac{n}{3}}) = \mathtt{sres}\Lambda^{2n} = \frac{1}{2}\mathtt{sres}[\Lambda^{2n-1}, \Lambda]$$

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We now turn to $L^{\frac{n}{6}}$

It can be proved that

$$\frac{\partial}{\partial t} L^{\frac{n}{6}} = [9(L^{\frac{5}{3}})_+, L^{\frac{n}{6}}].$$

So, the super residue of $L^{\frac{n}{6}}$ is conserved.

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The first two nontrivial conserved quantities

$$\begin{aligned}
\int \mathrm{sres} L^{\frac{7}{6}} \mathrm{d}x\mathrm{d}\theta &= -\frac{1}{9}\int [2(\mathcal{D}\phi_{xx}) + (\mathcal{D}\phi)^2 - 6\phi_x\phi]\mathrm{d}x\mathrm{d}\theta \\
\int \mathrm{sres} L^{\frac{11}{6}} \mathrm{d}x\mathrm{d}\theta &= -\frac{1}{81}\int [6(\mathcal{D}\phi_{xxxx}) + 18(\mathcal{D}\phi_{xx})(\mathcal{D}\phi) + 9(\mathcal{D}\phi_x)^2 \\
&\quad + 4(\mathcal{D}\phi)^3 - 18\phi_{xxx}\phi + 6\phi_{xx}\phi_x - 36\phi_x\phi(\mathcal{D}\phi)]\mathrm{d}x\mathrm{d}\theta
\end{aligned}$$

Remarks:

1. What is remarkable is that the conserved densities found in this way, unlike the supersymmetric KdV case, are local.
2. All those conserved quantities are fermioic. To our knowledge, this is the first supersymmetric integrable system whose only conserved quantities are fermioic.

Finally, we have the recursion operator

$$\begin{aligned}
\mathscr{R} =& \partial_x^6 + 6(\mathcal{D}\phi)\partial_x^4 + 9(\mathcal{D}\phi_x)\partial_x^3 + 6\phi_{xx}\partial_x^2\mathcal{D} + \{5(\mathcal{D}\phi_{xx}) + 9(\mathcal{D}\phi)^2\}\partial_x^2 \\
&+ \{9\phi_{xxx} + 12\phi_x(\mathcal{D}\phi)\}\partial_x\mathcal{D} + \{(\mathcal{D}\phi_{xxx}) + 9(\mathcal{D}\phi_x)(\mathcal{D}\Phi)\}\partial_x \\
&+ \{5\phi_{xxxx} + 12\phi_{xx}(\mathcal{D}\phi) + 6\phi_x(\mathcal{D}\phi_x)\}\mathcal{D} + \{4(\mathcal{D}\phi_{xx})(\mathcal{D}\phi) + 4(\mathcal{D}\phi)^3 - 3\phi_{xx}\phi_x\} \\
&+ \{\phi_{xxxxx} + 5\phi_{xxx}(\mathcal{D}\phi) + 5\phi_{xx}(\mathcal{D}\phi_x) + 2\phi_x(\mathcal{D}\phi_{xx}) + 6\phi_x(\mathcal{D}\phi)^2\}\mathcal{D}^{-1} \\
&- \{2(\mathcal{D}\phi_{xx}) + 2(\mathcal{D}\phi)^2\}\mathcal{D}^{-1}\phi_x - 4\phi_x(\mathcal{D}\phi)\partial_x^{-1}\phi_x - 2(\mathcal{D}\phi)\mathcal{D}^{-1}[\phi_{xxx} + 2\phi_x(\mathcal{D}\phi)] \\
&- 2\phi_x\mathcal{D}^{-1}\{(\phi_{xxx} + \phi_x(\mathcal{D}\phi))\mathcal{D}^{-1} - 3(\mathcal{D}\phi)\mathcal{D}^{-1}\phi_x - 2\phi_x\partial_x^{-1}\phi_x + 2\mathcal{D}^{-1}[\phi_{xxx} + 2\phi_x(\mathcal{D}\phi)]\}
\end{aligned}$$

Popowicz: Odd Hamiltonian structure for supersymmetric Sawada-Kotera equation. `arXiv:0902.2861v1`

"Probably it is the *first* nontrivial supersymmetric model with odd bi-Hamiltonian structure with the modified bosonic sector. So this model realizes in *fully* the idea of equal footing of fermions and bosons fields".

novel SUSY 5th KdV

$$\begin{aligned}\phi_t &= \phi_{xxxxx} + 2\alpha\phi_{xxx}(D\phi) + 3\alpha\phi_{xx}(D\phi_x) + \alpha\phi_x(D\phi_{xx}) \\ &\quad + \frac{3}{5}\alpha^2\phi_x(D\phi)^2 + \frac{3}{5}\alpha^2\phi(D\phi_x)(D\phi)\end{aligned}$$

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- Hamiltonian structure

$$\phi_t = \mathcal{D}\frac{\delta H_1}{\delta\phi}$$

where $H_1 = \frac{1}{2}\int\left[\phi_{xxxx}(\phi) + 5\phi_{xx}(\mathcal{D}\phi)^2 + 5\phi(\phi)^3\right]dxd\theta$

novel SUSY 5th KdV

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- Recursion operator

$$\begin{aligned}\mathscr{R} &= \partial_x^6 + 12(D\Phi)\partial_x^4 + 6\Phi_x\partial_x^3 D + 24(D\Phi_x)\partial_x^3 + \{21\Phi_{xx} + 18\Phi(D\Phi)\}\partial_x^2 D \\ &\quad + \{19(D\Phi_{xx}) + 30(D\Phi)^2\}\partial_x^2 + \{23\Phi_{xxx} + 57\Phi_x(D\Phi) + 27\Phi(D\Phi_x)\}\partial_x D \\ &\quad + \{7(D\Phi_x xx) + 60(D\Phi_x)(D\Phi)\}\partial_x \\ &\quad + \{11\Phi_{xxxx} + 54\Phi_{xx}(D\Phi) + 48\Phi_x(D\Phi_x) + 15\Phi(D\Phi_{xx}) + 36\Phi(D\Phi)^2\}D \\ &\quad + \{(D\Phi_{xxxx}) + 27(D\Phi xx)(D\Phi) + 20(D\Phi_x)^2 + 28(D\Phi)^3 - 5\Phi_{xx}\Phi_x + 18\Phi_x\Phi(D\Phi)\} \\ &\quad + \{2\Phi_{xxxxx} + 20\Phi_{xxx}(D\Phi) + 30\Phi_{xx}(D\Phi_x) + 14\Phi_x(D\Phi_{xx}) + 21\Phi_x(D\Phi)^2 \\ &\qquad + 3\Phi(D\Phi_{xxx}) + 51\Phi(D\Phi_x)(D\Phi)\}D^{-1} \\ &\quad + \{-2\Phi_{xxx} + 30\Phi_x(D\Phi) - 6\Phi(D\Phi_x)\}D^{-1}(D\Phi) + 3(D\Phi_{xxx})\partial_x^{-1}\{(D\Phi) - \Phi_x D^{-1}\} \\ &\quad - 3(D\Phi)D^{-1}\{\Phi_{xxx} + (-7\Phi_{xx} - 3\Phi(D\Phi))\partial_x^{-1}[(D\Phi) - \Phi_x D^{-1}]\} \\ &\quad - 2\Phi_x\partial_x^{-1}\{\Phi_{xxx} + 3\Phi(D\Phi) + [2(D\Phi_{xxx}) + 12(D\Phi_x)(D\Phi) - 3\Phi_{xx}\Phi]D^{-1} \\ &\qquad + [-6\Phi_{xx} - 9\Phi(D\Phi)]\partial_x^{-1}[(D\Phi) - \Phi_x D^{-1}]\}\end{aligned}$$

- A possible second structure (bi-hamiltonian ?)

$$\phi_t = \mathcal{R}\mathcal{D}\frac{\delta H_0}{\delta \phi}$$

where

$$H_0 = \frac{1}{2}\int \phi dx d\theta$$

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- A multi-linear form

$$f^2\mathcal{D}_x\left\{f^2 \cdot S(\mathcal{D}_t - \mathcal{D}_x^5)f \cdot f\right\} \\ +5\left\{(\mathcal{D}_x(\mathcal{D}_x^2 f \cdot f) \cdot f^2)(S\mathcal{D}_x^3 f \cdot f) - (\mathcal{D}_x(\mathcal{D}_x^4 f \cdot f) \cdot f^2)(S\mathcal{D}_x f \cdot f)\right\} = 0$$

THANK YOU !