Chords and Solitons:
C & G of the KP Equation

Yuji Kodama

Ohio State University

Joint work with

S. Chakravarty, C-Y. Kao, M. Oikawa and H. Tsuji

International Workshop on

Nonlinear and Modern Mathematical Physics

Beijing Xiedao Group, Beijing, China, July 15-21, 2009

The KP Equation

The KP equation:

$$\frac{\partial}{\partial x}\underbrace{\left(4\frac{\partial u}{\partial t}+\frac{\partial^3 u}{\partial x^3}+6u\frac{\partial u}{\partial x}\right)}_{KdV}+3\frac{\partial^2 u}{\partial y^2}=0\,.$$

The solution $u(x,y,t)$ in terms of the τ-function:

$$u(x,y,t)=2\frac{\partial^2}{\partial x^2}\ln\tau(x,y,t).$$

The τ-function given by the Wronskian determinant:

$$\tau_N=\mathrm{Wr}(f_1,\ldots,f_N).$$

The KP Equation

The linearly independent set $\{f_i(x,y,t) : i = 1, \dots, N\}$:

$$\underbrace{\frac{\partial f_i}{\partial y} = \frac{\partial^2 f_i}{\partial x^2}}_{Heat\ equation}, \qquad \frac{\partial f_i}{\partial t} = -\frac{\partial^3 f_i}{\partial x^3}.$$

Finite dimensional solutions:

$$f_i(x,y,t) = \sum_{j=1}^{M} a_{ij} E_j(x,y,t), \qquad i = 1, \dots, N < M,$$

$$E_j(x,y,t) = \exp(k_j x + k_j^2 y - k_j^3 t), \qquad j = 1, \dots, M.$$

with the ordering $k_1 < k_2 < \cdots < k_M$.

The KP Equation

Note that we have a Grassmannian picture, i.e. Gr(N, M):

- Span$_{\mathbb{R}}\{E_j : j = 1, \ldots, M\} \cong \mathbb{R}^M$.
- Span$_{\mathbb{R}}\{f_i : i = 1, \ldots, N\}$ forms an N-dimensional subspace in $\mathbb{R}^M$,

$$(f_1, \ldots, f_N) = (E_1, \ldots, E_M)A^T,$$

where A-matrix is defined by

$$A = \begin{pmatrix} a_{11} & \cdots & \cdots & a_{1M} \\ \vdots & \ddots & \ddots & \vdots \\ a_{N1} & \cdots & \cdots & a_{NM} \end{pmatrix} \in M_{N\times M}(\mathbb{R}).$$

Each solution can be parametrized by the A-matrix.

The KP Equation

- For $\forall H \in GL_N(\mathbb{R})$, $(g_1, \dots, g_N) = (f_1, \dots, f_N)H$ gives the same solution, i.e. $\tau(g) = |H|\tau(f)$. This implies that the τ-function is identified as a point on the Grassmannian Gr(N, M), i.e.

$$\mathrm{Gr}(N, M) \cong GL_N(\mathbb{R}) \backslash M_{N\times M}(\mathbb{R}),$$

  with dim Gr$(N, M) = NM - N^2 = N(M - N)$.

- $H \in GL_N(\mathbb{R})$ gives a row reduction of the A-matrix. For example, a generic A can be written in the form (RREF),

$$A = \begin{pmatrix} 1 & \cdots & 0 & * & \cdots & * \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & * & \cdots & * \end{pmatrix}$$

The KP Equation

- $\mathrm{Gr}(N,M)$ has a Schubert decomposition,

$$\mathrm{Gr}(N,M)=\bigsqcup_{1\le j_1<\cdots<j_N\le M} W(j_1,\ldots,j_N),$$

where $(j_1,\ldots,j_N)$ is a Schubert symbol representing the pivot indices.

- The set of the Schubert symbols forms a partially ordered set (POSET) with a weak Bruhat order, i.e.

$$(j_1,\ldots,j_N)\iff \exists!\sigma\in S_M/P_N,$$

where S_M is the permutation group of order M, and P_N is a parabolic subgroup generated by the simple reflections (transposition) $s_k=(k,k+1)$ without s_{M-N}.

The KP Equation

Example:

- $\mathrm{Gr}(1,2) = W(1) \sqcup W(2)$ where $W(1) = \{(1,*)\}$ and $W(2) = \{(0,1)\}$. In terms of the permutation S_2, we have

$$(1\ 2) \xrightarrow{s_1^{-1}} (2\ 1).$$

- $\mathrm{Gr}(1,3) = W(1) \sqcup W(2) \sqcup W(3)$, and in terms of the permutation S_3/P_1 with $P_1 = \langle s_2 \rangle$,

$$(1\ 2\ 3) \xrightarrow{s_2^{-1}} (1\ 3\ 2) \xrightarrow{s_1^{-1}} (2\ 3\ 1).$$

- $\mathrm{Gr}(2,4) = W(1,2) \sqcup W(1,3) \sqcup \cdots \sqcup W(3,4)$, and

$$(1\ 2\ 3\ 4) \xrightarrow{s_2^{-1}} (1\ 3\ 2\ 4) \cdots (2\ 4\ 1\ 3) \xrightarrow{s_2^{-1}} (3\ 4\ 1\ 2).$$

The KP Equation

Example: For $N = 1$, the function $w = \frac{\partial}{\partial x} \ln \tau_1$ satisfies

$$\frac{\partial w}{\partial y} = 2w \frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2} \quad (\text{The Burgers equation}).$$

A shock solution is given by $\tau_1 = f_1 = E_1 + aE_2$, ($A = (1, a)$),

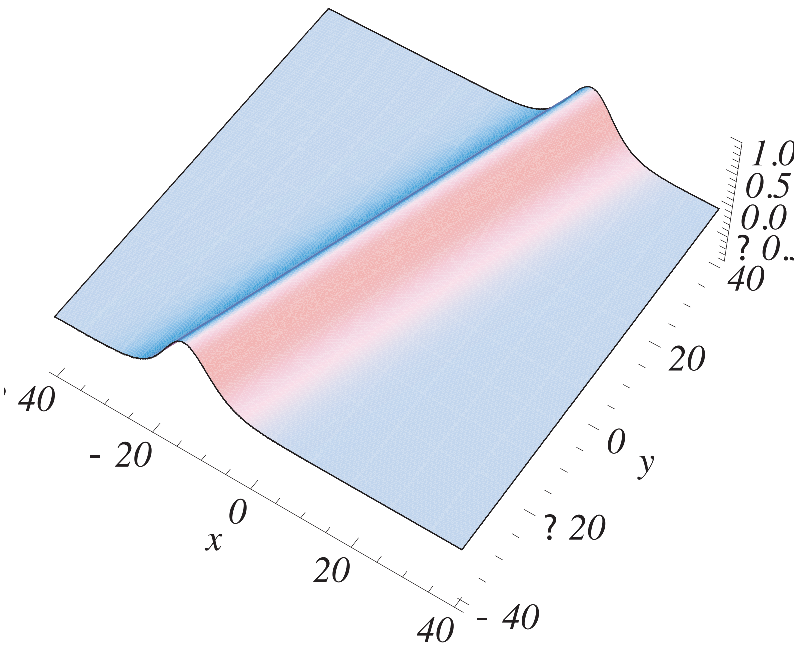
$$w = \frac{1}{2}(k_1 + k_2) + \frac{1}{2}(k_2 - k_1) \tanh \frac{1}{2}(\theta_2 - \theta_1 + \ln a),$$

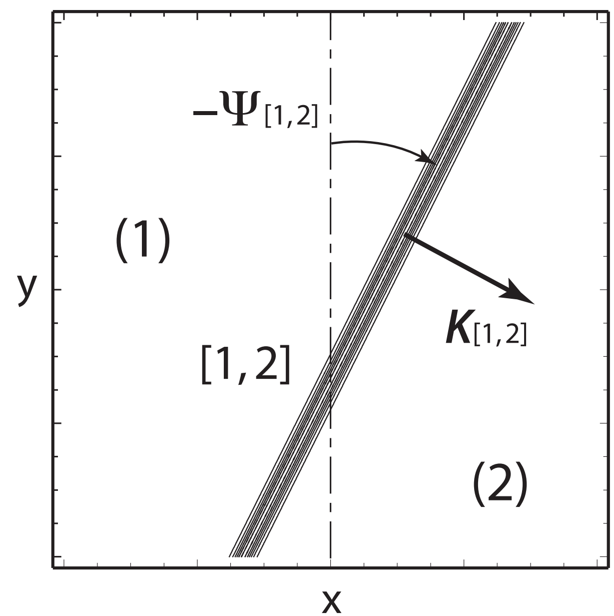
where $\theta_j = k_j x + k_j^2 y - k_j^3 t$. Notice that for $k_1 < k_2$,

$$w \longrightarrow \begin{cases} k_1 & x \to -\infty \\ k_2 & x \to \infty \end{cases}$$

The KP Equation

Example 1: One line-soliton solution with $\tau = E_1 + aE_2$.





3D figure of $u = 2\frac{\partial w}{\partial x}$, and the contour plot. The numbers (i) represent the dominant exponential term in the τ-function. We denote this $[1,2]$-soliton.

The KP Equation

One-solton solution is given by a balance between two exponential terms, and in general it is expressed with the parameters $\{k_i, k_j\}$,

$$u = A_{[i,j]}\mathrm{sech}^2\Theta_{[i,j]}\ ,$$

where the amplitude $A[i,j]$ and the phase $\Theta[i,j]$ are

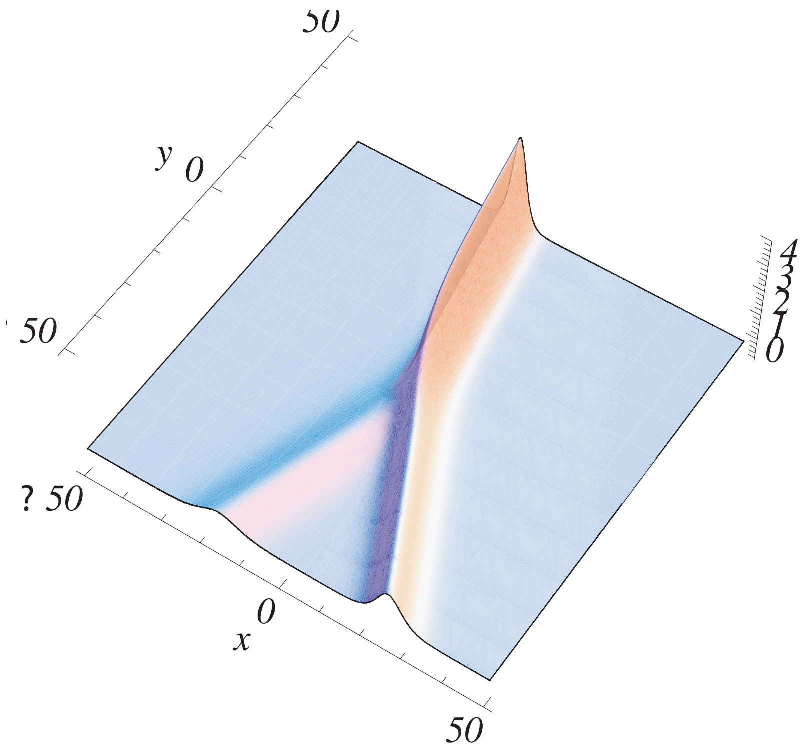
$$A_{[i,j]} = \frac{1}{2}(k_i - k_j)^2, \qquad \Theta_{[i,j]} = \frac{1}{2}(\theta_j - \theta_i)\ .$$

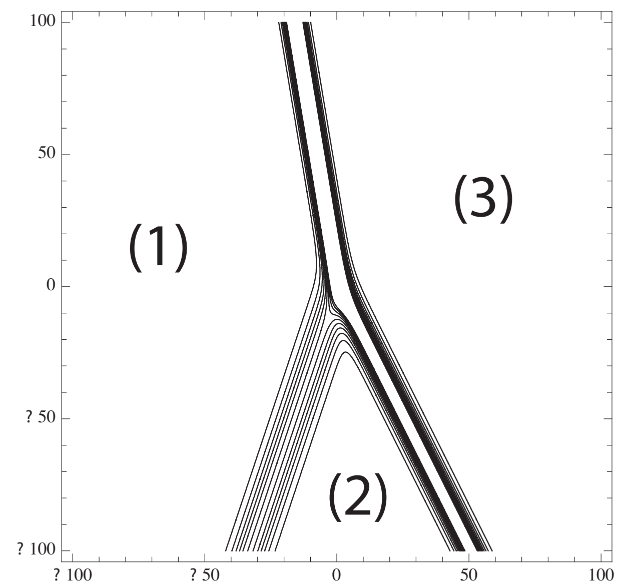
The slope of the soliton in the xy-plane is given by

$$\tan\Psi_{[i,j]} = \frac{K^y_{[i,j]}}{K^x_{[i,j]}} = k_i + k_j\ .$$

The KP Equation

Example 2: Y-type solution with $\tau_1 = f_1 = E_1 + aE_2 + bE_3$,

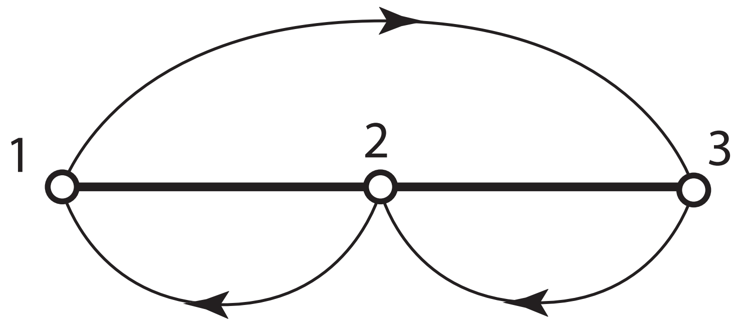




In each region, one of the exponential terms is dominant. Each line-soliton is given by the balance between two exponential terms, E_i and E_j, denoted as (i) and (j).

The KP Equation

Chord diagrams: One can express each soliton solution in a chord diagram (= permutation). For example, Y-type soliton with the parameters $\{k_1, k_2, k_3\}$ is given by



$$\pi = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

- The upper part represents $[1, 3]$-soliton in $y > 0$.
- The lower part represents $[1, 2]$- and $[2, 3]$-solitons in $y < 0$.

Classification Theorem

The τ-function is given by

$$\tau(x,y,t) = \det\left(K D(x,y,t) A^T\right),$$

where $D = \mathrm{diag}(E_1,\ldots,E_M)$ with $E_j = \exp(k_j x + k_j^2 y + k_j^3 t)$, and K is the $N \times M$ matrix given by

$$K = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ k_1 & k_2 & \cdots & k_M \\ \vdots & \vdots & \ddots & \vdots \\ k_1^{N-1} & k_2^{N-1} & \cdots & k_M^{N-1} \end{pmatrix}$$

Recall that $\tilde{A} = HA$ with $H \in GL_N(\mathbb{R})$ gives the same solution, i.e. A can be written in RREF.

Classification Theorem

Lemma: (Binet-Cauchy) The τ-function can be expanded as

$$\tau_N = \sum_{1 \le i_1 < \cdots < i_N \le M} \xi(i_1, \ldots, i_N) E(i_1, \ldots, i_N)\,,$$

where $\xi(i_1, \ldots, i_N)$ is the $N \times N$ minor of the A-matrix, and $E(i_1, \ldots, i_N)$ is given by

$$E(i_1, \ldots, i_N) = \left(\prod_{1 \le j < l \le N} (k_{i_j} - k_{i_l}) \right) E_{i_1} \cdots E_{i_N} > 0\,,$$

Note that if all the $N \times N$ minors of the A-matrix are non-negative (i.e. A is totally non-negative), then the τ-function is positive definite. Namely, u is non-singular.

Classification Theorem

We say that the A-matrix is irreducible, if

- in each column, there is at least one nonzero element,
- in each raw, there is at least one more nonzero element in addition to the pivot.

Example: For $N = 2$ and $M = 4$, there are only two types of irreducible A-matrices in RREF:

$$\begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{pmatrix}, \qquad \begin{pmatrix} 1 & * & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix}.$$

Note that other cases can be expressed by a smaller matrix of $N' \times M'$ with either $N' < N$ or $M' < M$.

Classification Theorem

Example: For $N = 2, M = 4$, there are seven types of the A-matrices in RREF which are both irreducible and totally non-negative:

$$\begin{pmatrix} 1 & 0 & -c & -d \\ 0 & 1 & a & b \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & -b & -c \\ 0 & 1 & a & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & -c \\ 0 & 1 & a & b \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & -b \\ 0 & 1 & a & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & a & 0 & -c \\ 0 & 0 & 1 & b \end{pmatrix} \quad \begin{pmatrix} 1 & a & 0 & 0 \\ 0 & 0 & 1 & b \end{pmatrix}$$

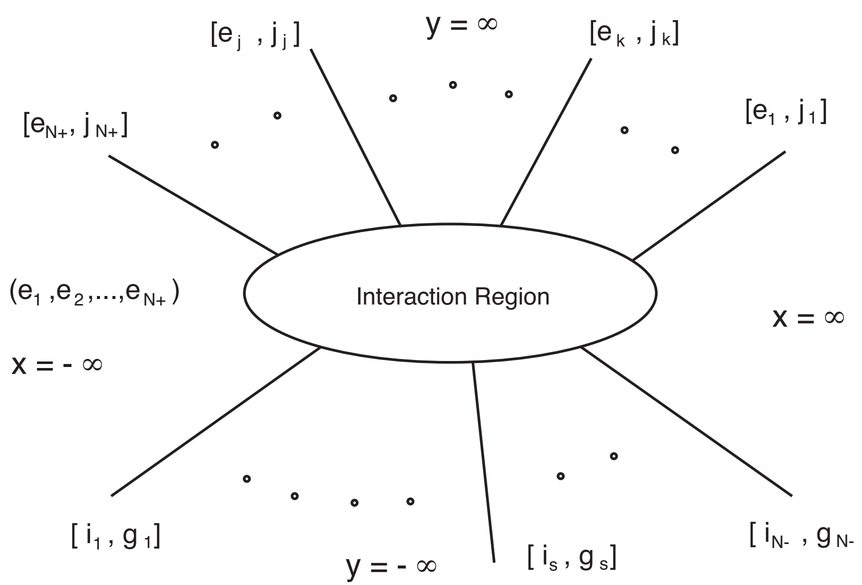
Here a, b, c and d are positive numbers, and for the first one, either $ad - cb > 0$ or $= 0$. The total number of nonzero minors is at least four, and the maximal number is six.

Classification Theorem

Theorem 1: Let $\{e_1, \ldots, e_N\}$ be the pivot indices, and let $\{g_1, \ldots, g_{M-N}\}$ be the non-pivot indices for an irreducible and totally non-negative A-matrix. Then the soliton solution associated with the A-matrix has

(a) N line-solitons of $[e_n, j_n]$-type for $n = 1, \ldots, N$ as $y \to \infty$,

(b) $M - N$ line-solitons of $[i_m, g_m]$-type for $m = 1, \ldots, M - N$ as $y \to -\infty$.



Classification Theorem

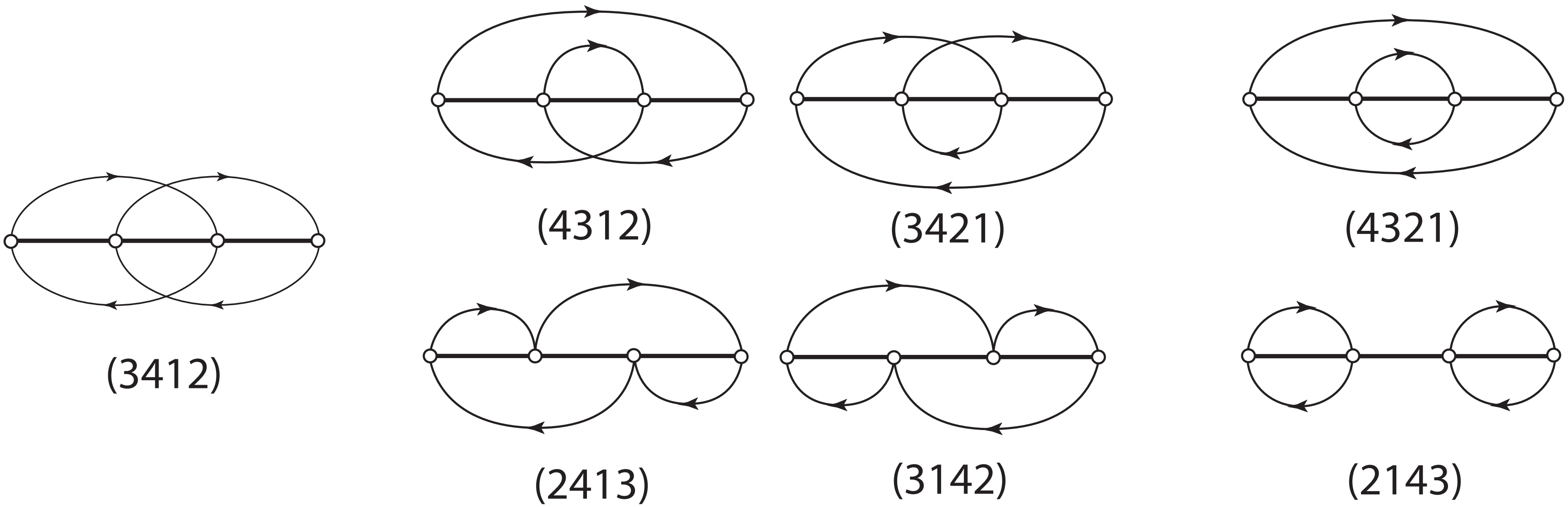
Theorem 2: The set of those solitons $[e_n, j_n]$ and $[i_m, g_m]$ are expressed by a unique chord diagram which corresponds a derangement of the permutation group $\mathcal{S}_M$, i.e.

$$\begin{pmatrix} e_1 & \cdots & e_N & g_1 & \cdots & g_{M-N} \\ j_1 & \cdots & j_N & i_1 & \cdots & i_{M-N} \end{pmatrix}$$

Theorem 3: Conversely, for each chord diagram associated with the derangement, one can construct an A-matrix, and the corresponding τ-function gives the solution of the KP equation having line-solitons expressed by the chord diagram. The entries of the A-matrix give the scattering data, i.e. the locations of those line-solitons and their interaction properties.

Classification Theorem

Example: $N = 2, M = 4$. We have seven different types of $(2,2)$-soliton solution, which are parametrized by the permutation group $\mathcal{S}_4$:

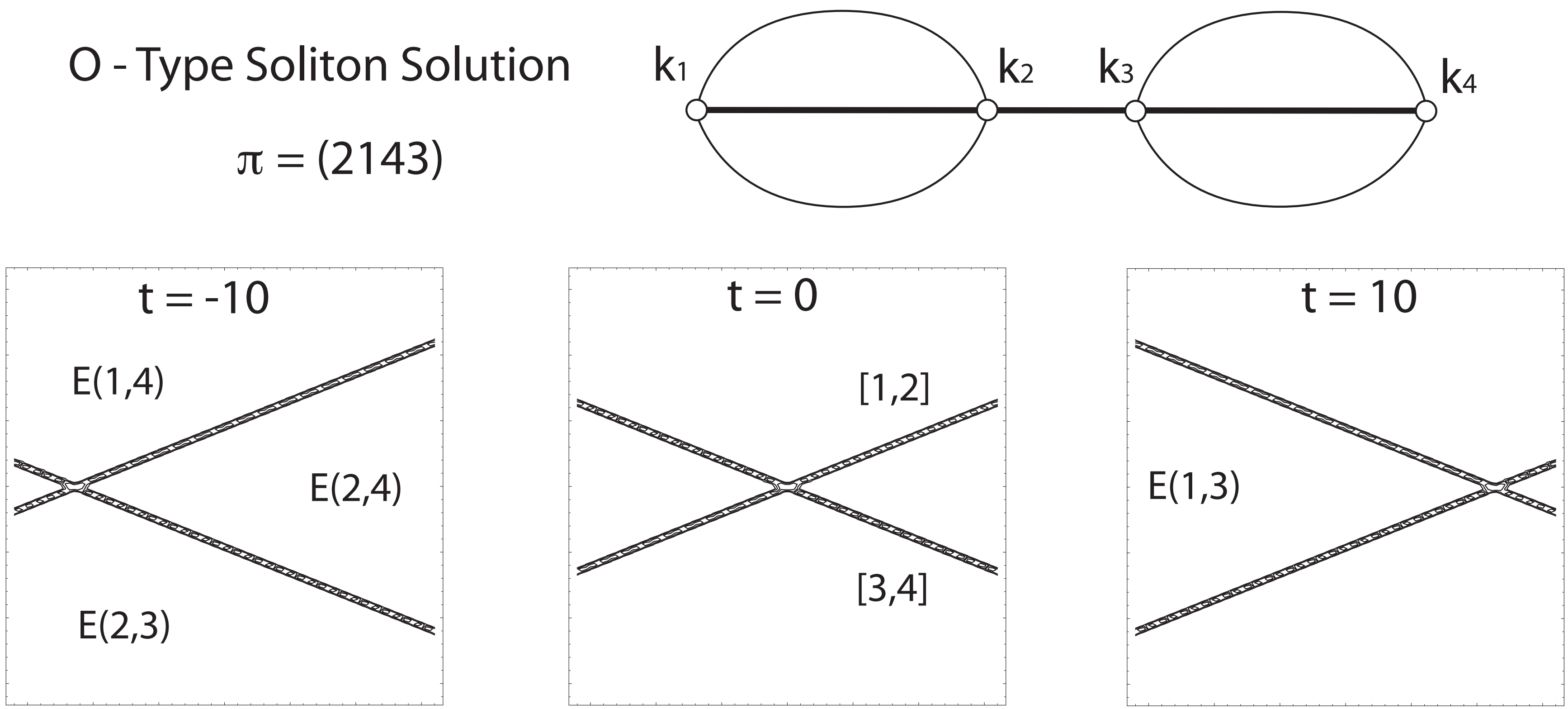


The 4-tuples of the diagrams represent the permutation,

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ \pi(1) & \pi(2) & \pi(3) & \pi(4) \end{pmatrix} = (\pi(1), \dots, \pi(4)).$$

Exact Solutions

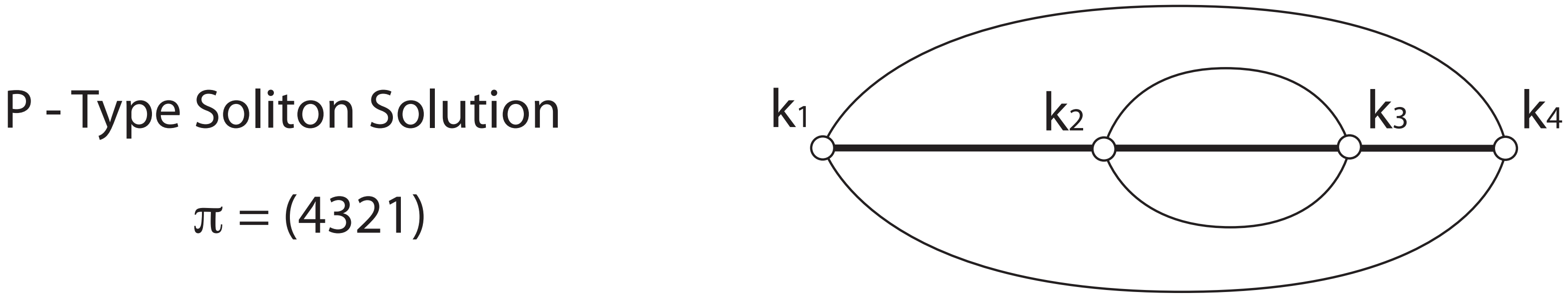
Example 1: O-type soliton solution.

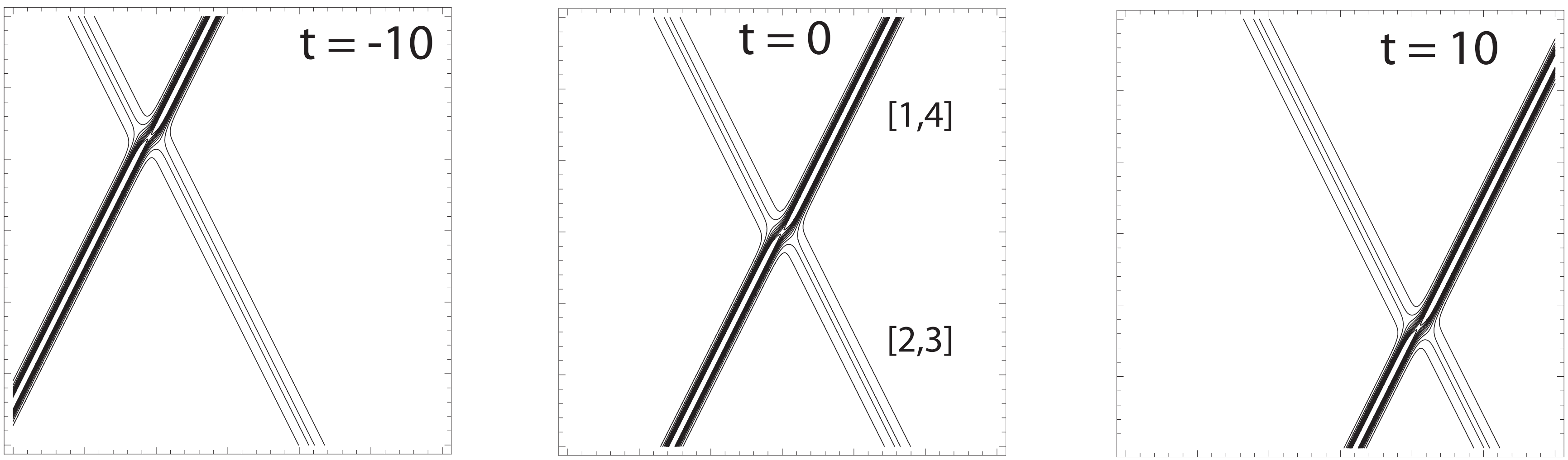


$$A_{[1,2]} + A_{[3,4]} < u_{\text{center}} < \left(\sqrt{A_{[1,2]}} + \sqrt{A_{[3,4]}}\right)^2 .$$

Exact Solutions

Example 2: P-type soliton solution.

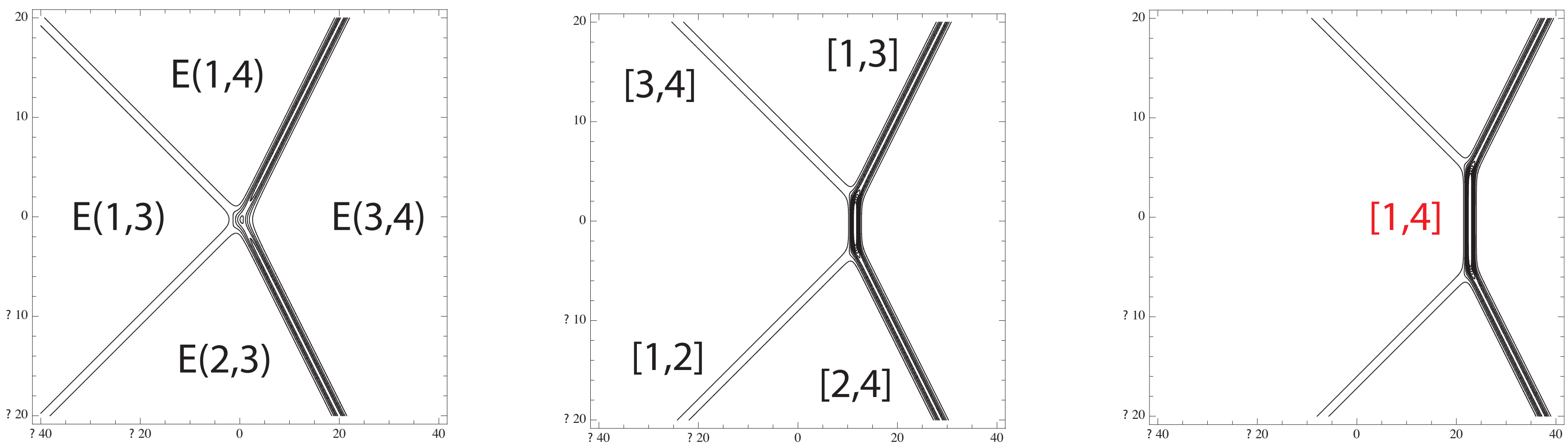




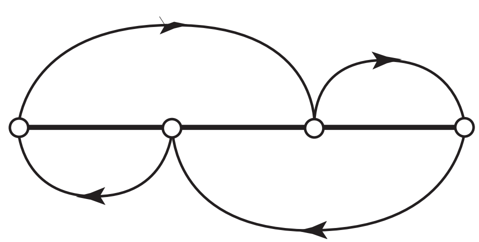
$$\left(\sqrt{A_{[1,4]}} - \sqrt{A_{[2,3]}}\right)^2 < u_{\text{center}} < A_{[1,4]} - A_{[2,3]}.$$

Exact Solutions

Example 3: (3142)-type soliton solution.



(3142)-type



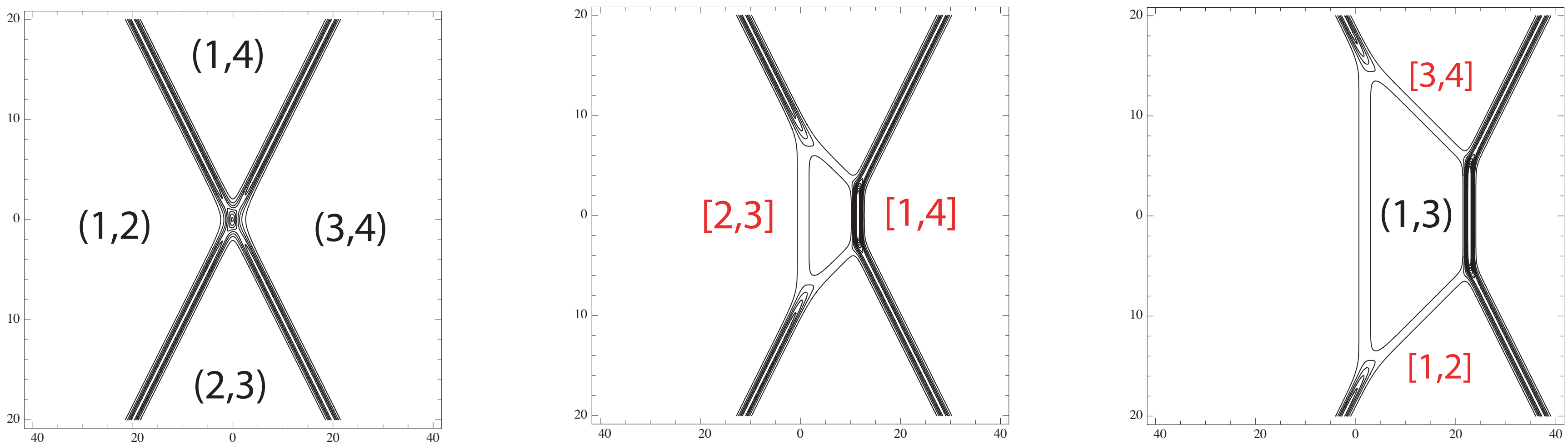
[1,3] and [3,4]-solitons

[1,2] and [2,4]-solitons

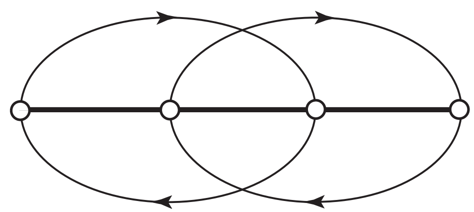
Note that $[1,4]$ gives the maximum amplitude $A = \frac{1}{2}(k_4 - k_1)^2$.

Exact Solutions

Example 4: T-type soliton solution (i.e. (3412)-type).



T - type soliton



[1,3] and [2,4]-solitons

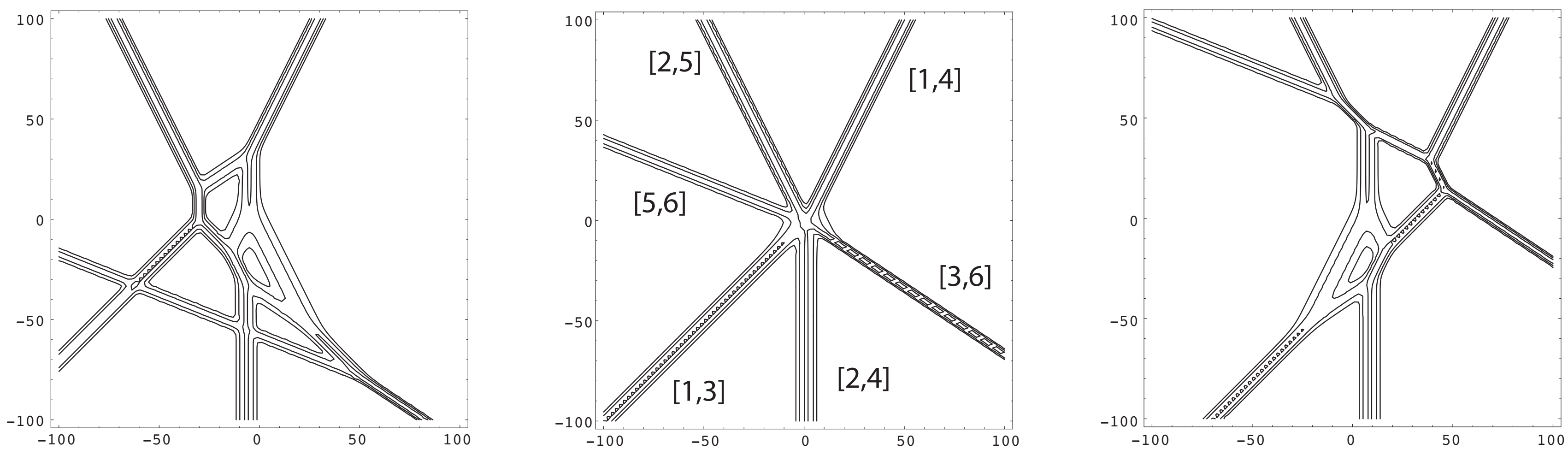
[1,3] and [2,4]-solitons

Notice that the front half is the same as (3142)-type.

Exact Solutions

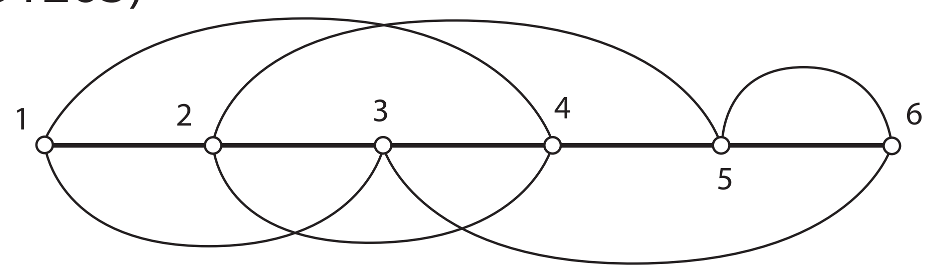
Example: $N = 3, M = 6$ (7-dimensional solution).

t = -30 t = 0 t = 30



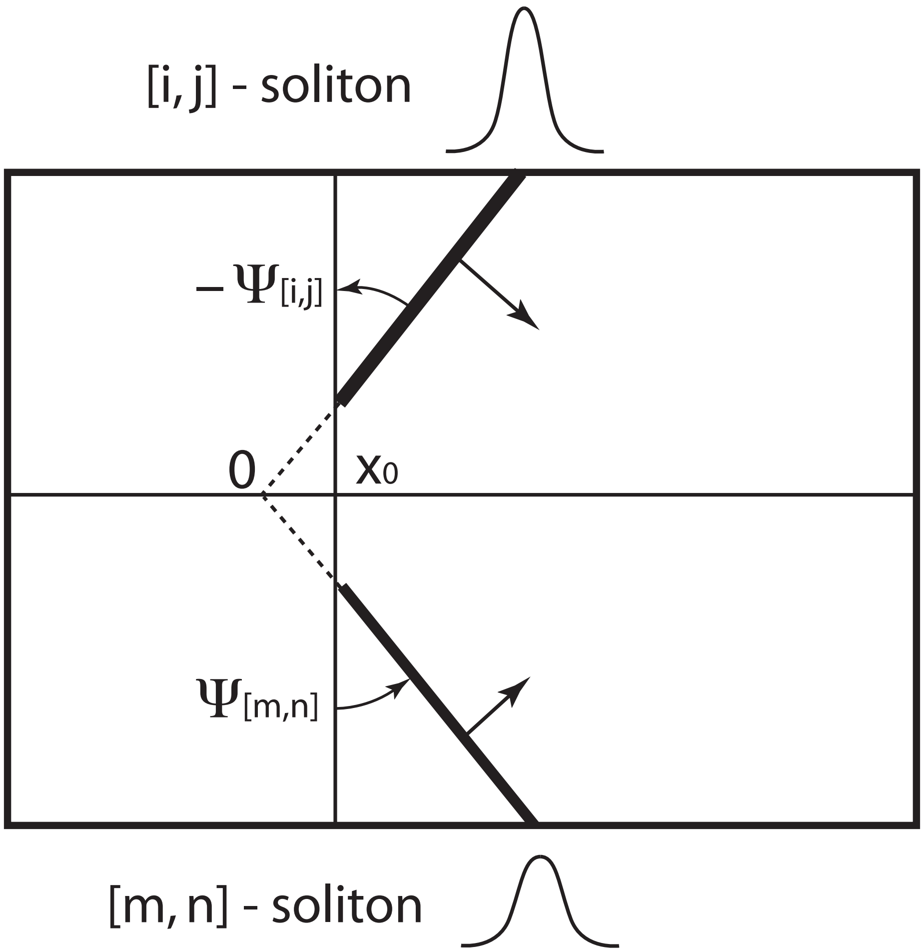
$$A = \begin{pmatrix} 1 & 0 & -a & -b & 0 & c \\ 0 & 1 & d & e & 0 & -f \\ 0 & 0 & 0 & 0 & 1 & g \end{pmatrix}$$

$\pi = (451263)$



Numerical Simulations

The initial wave profile:



$$A_{[i,j]} = \frac{1}{2}(k_i - k_j)^2$$

$$\tan \Psi_{[i,j]} = k_i + k_j$$

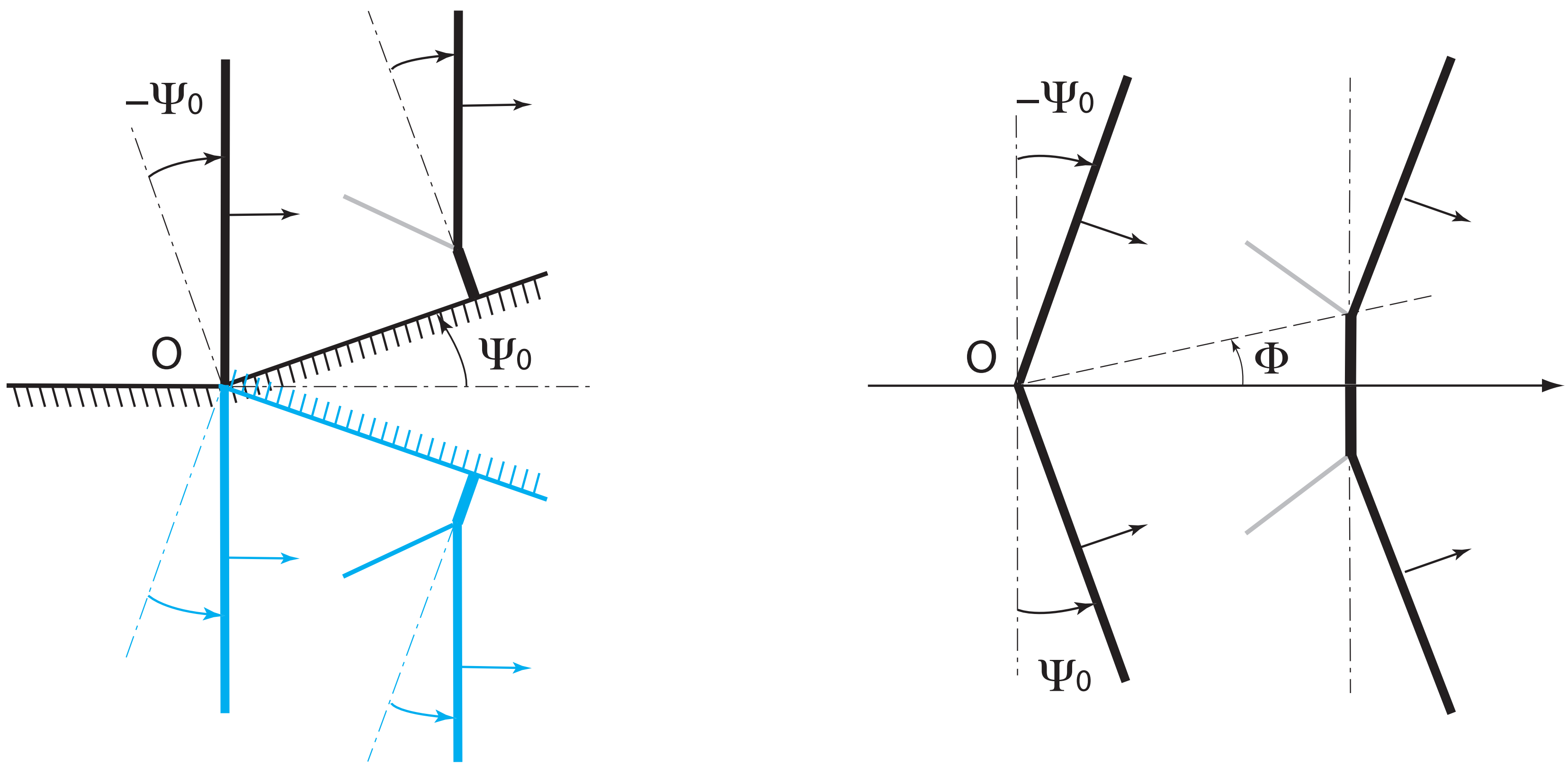
$$A_{[i,j]} = A_0$$

$$A_{[m,n]} = 2$$

$$\Psi_{[m,n]} = -\Psi_{[i,j]} = \Psi_0$$

Numerical Simulations

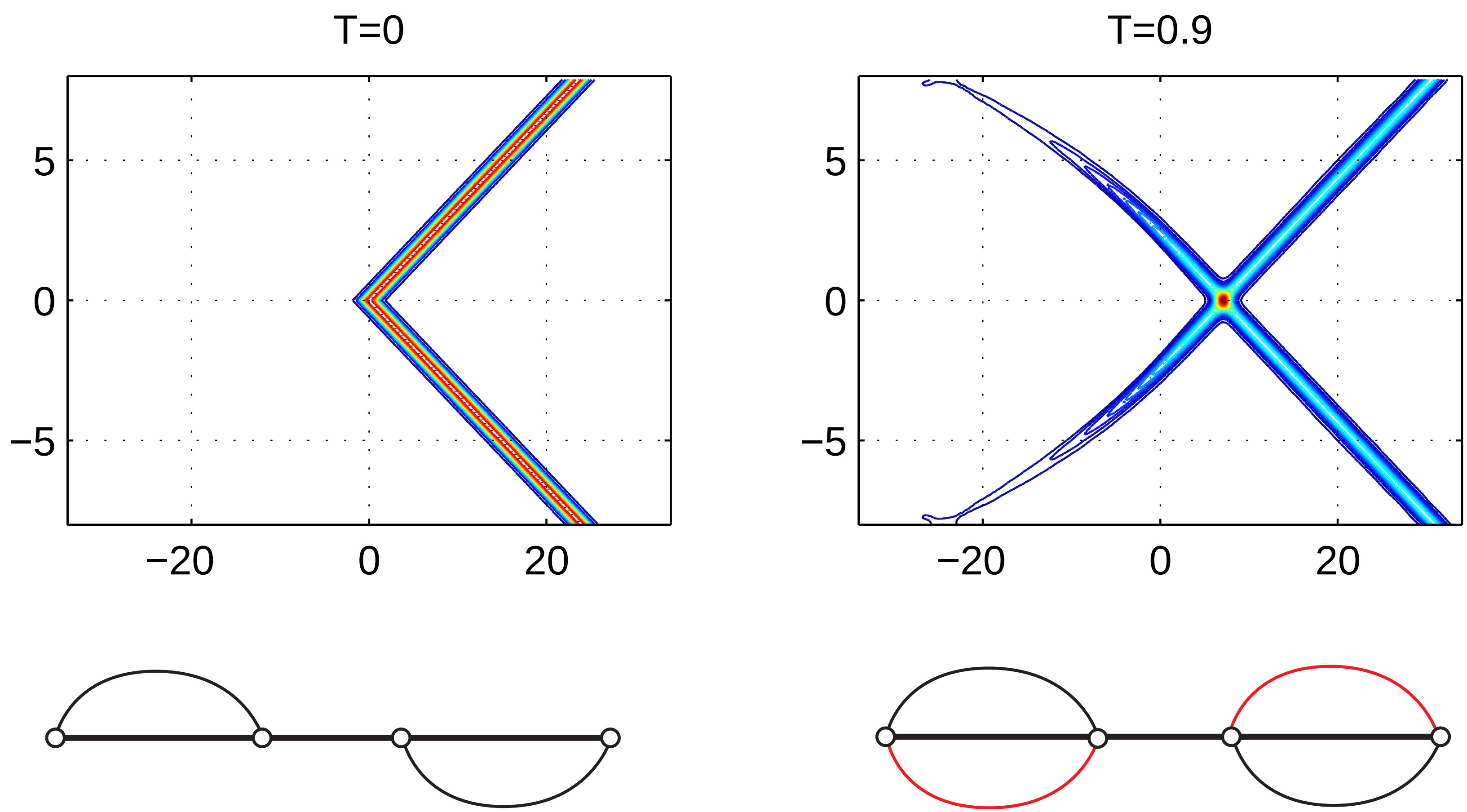
Physical example: The Mach reflection with a rigid wall:



Here $\Psi_0 < \Psi_c$. The right figure shows the equivalent system.

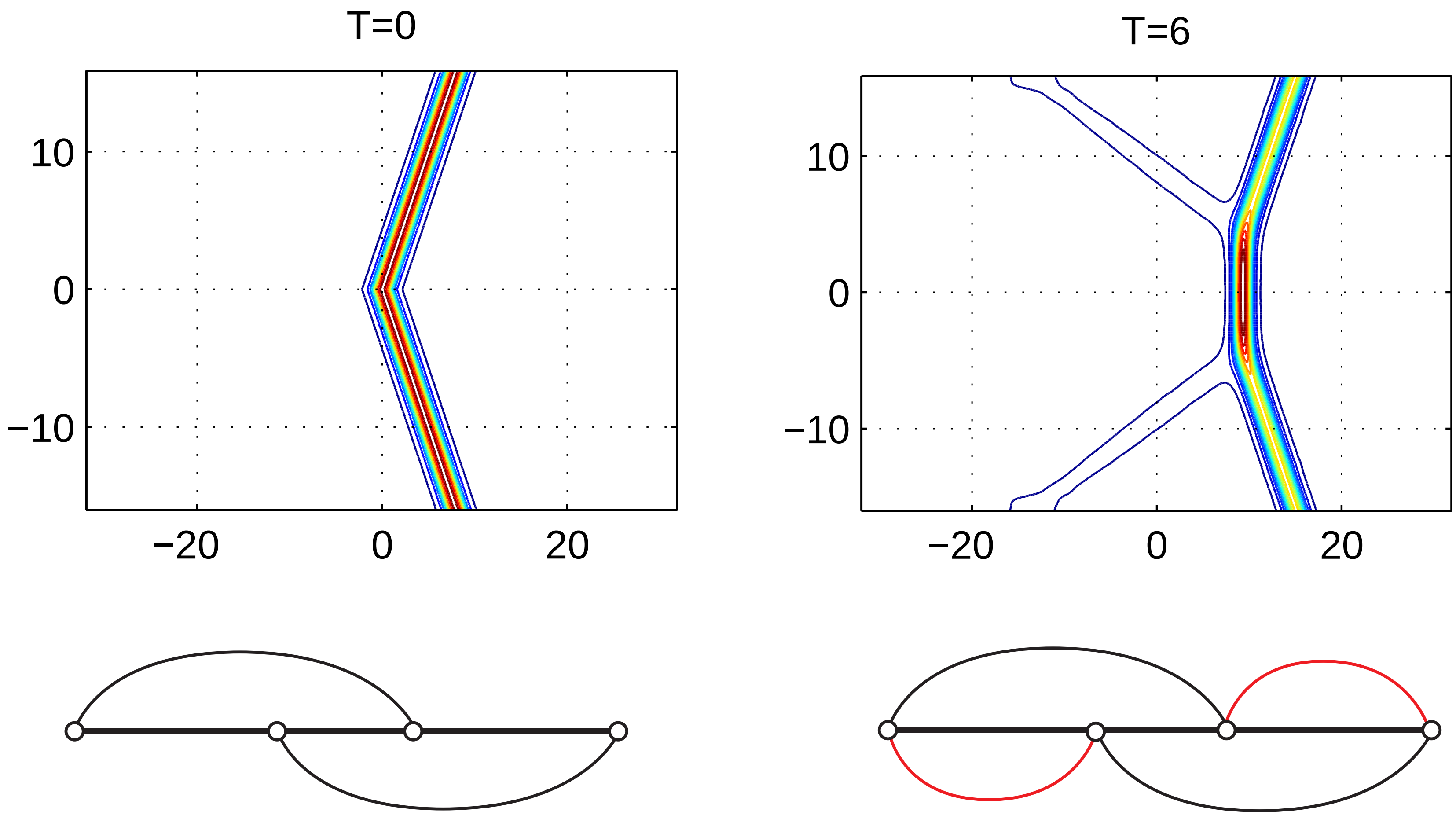
Numerical Simulations

V-shape initial data of O-type:



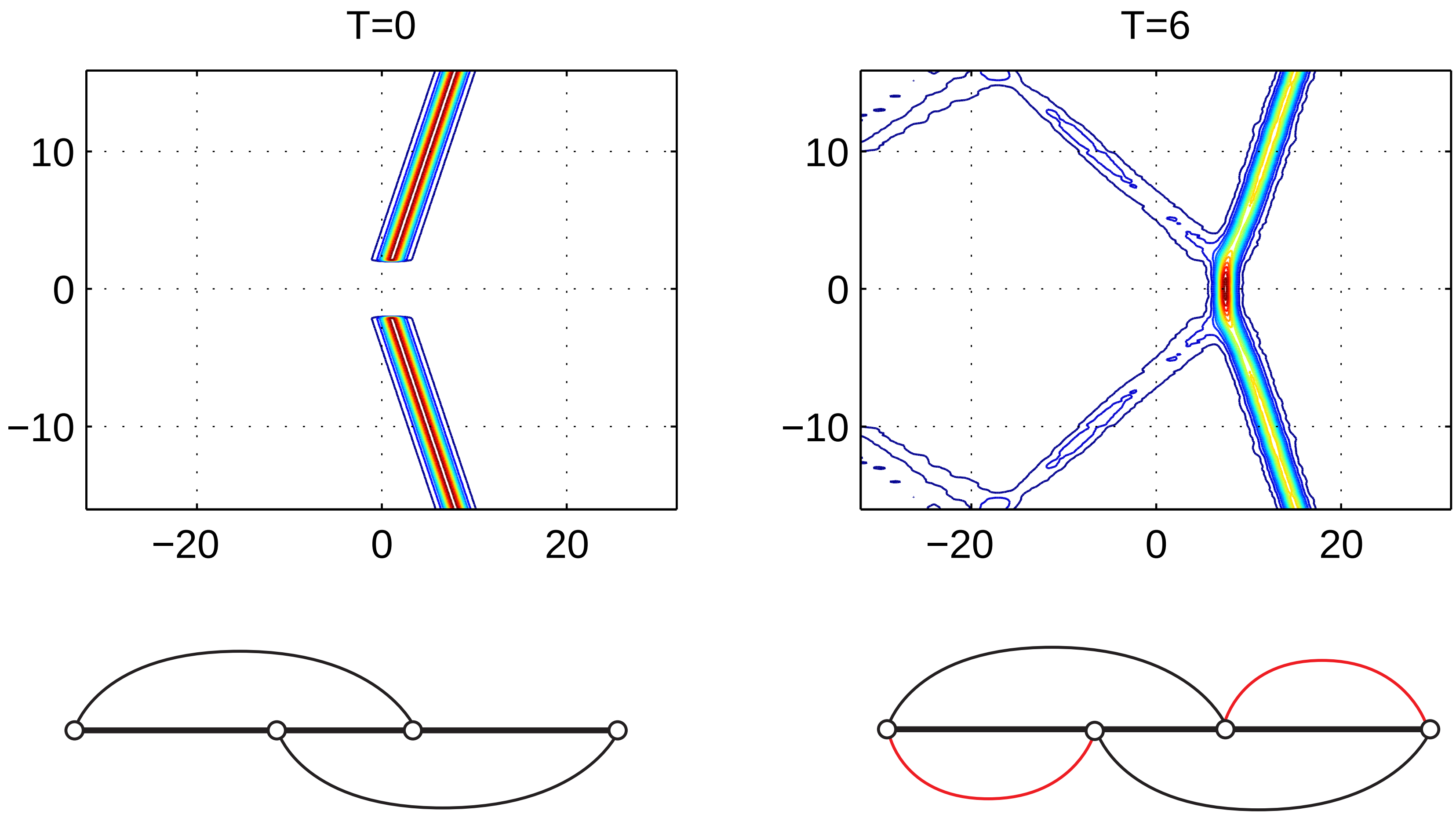
Numerical Simulations

(3142)-type:



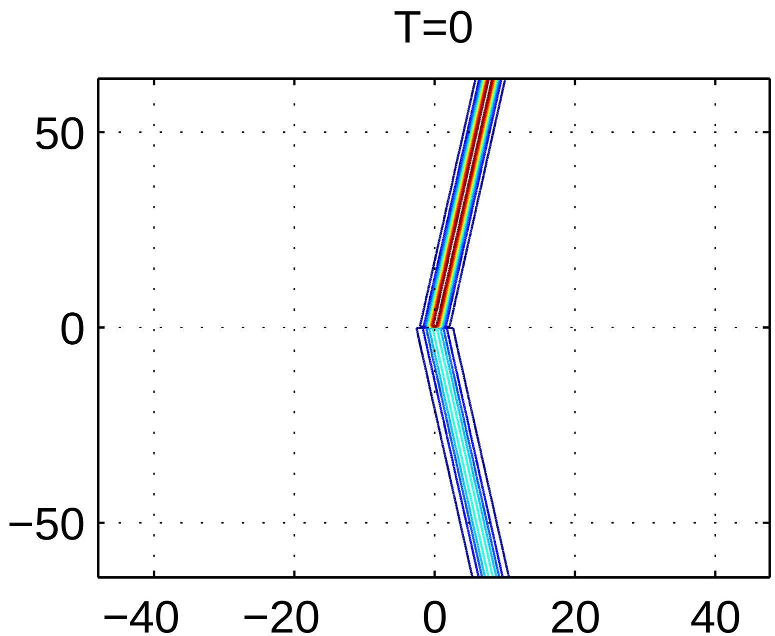
Numerical Simulations

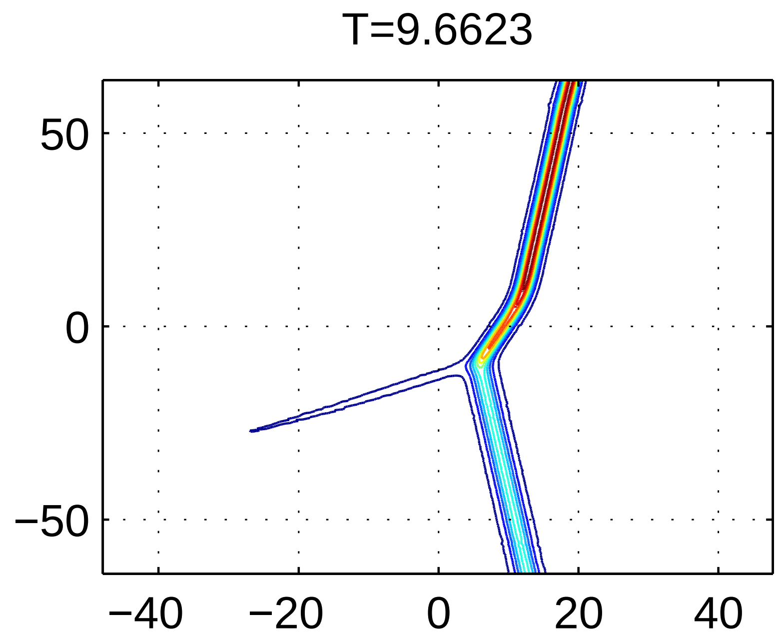
(3142)-type with a cut:

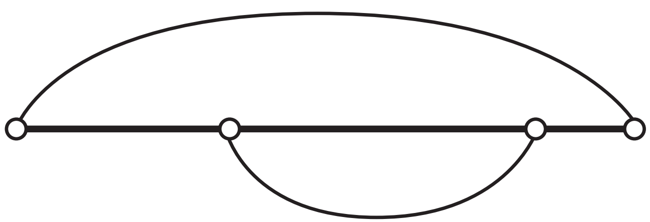


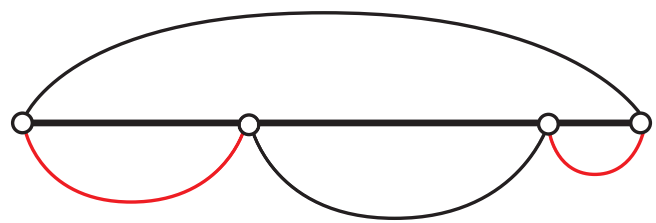
Numerical Simulations

P-type:



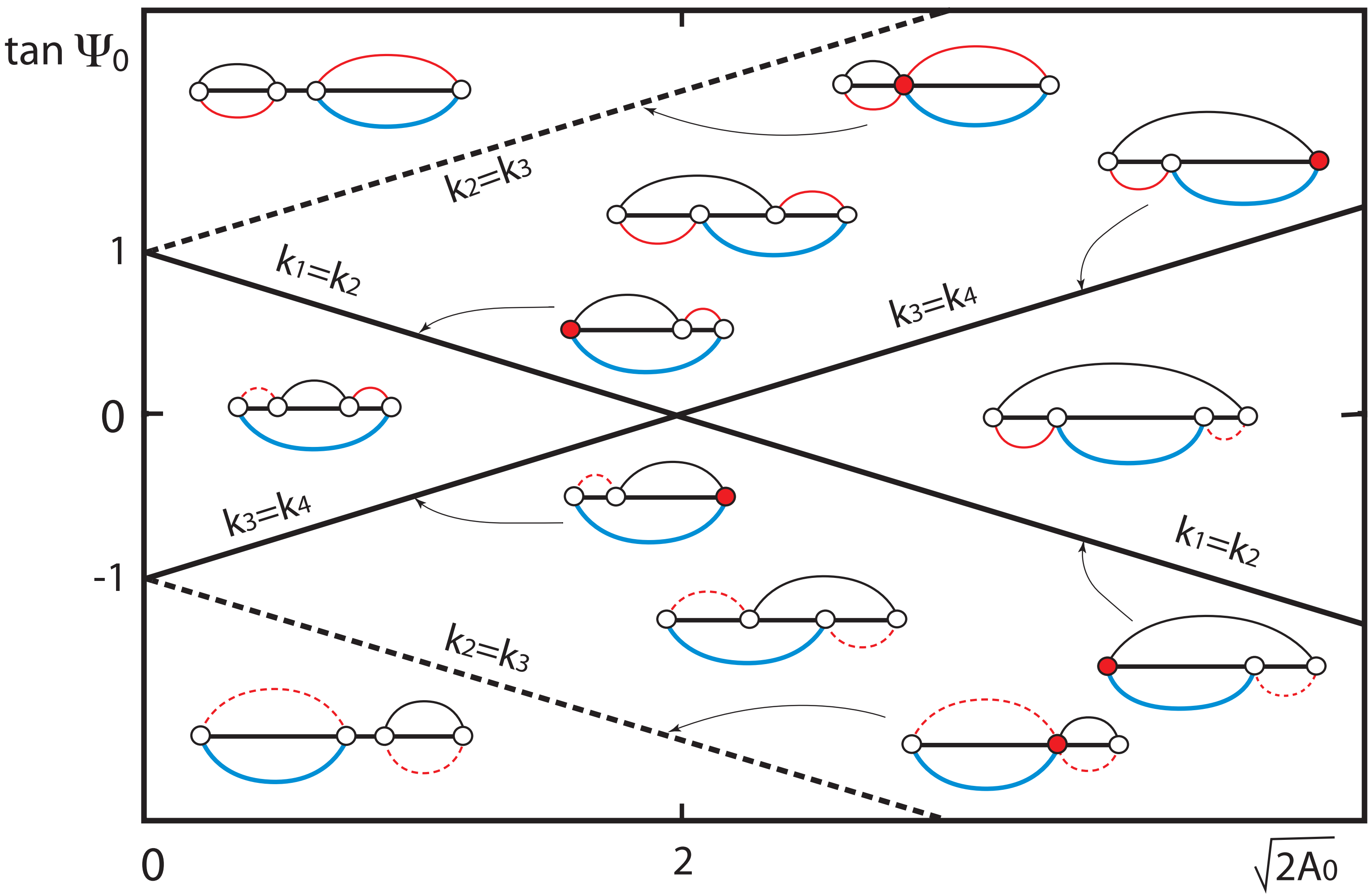






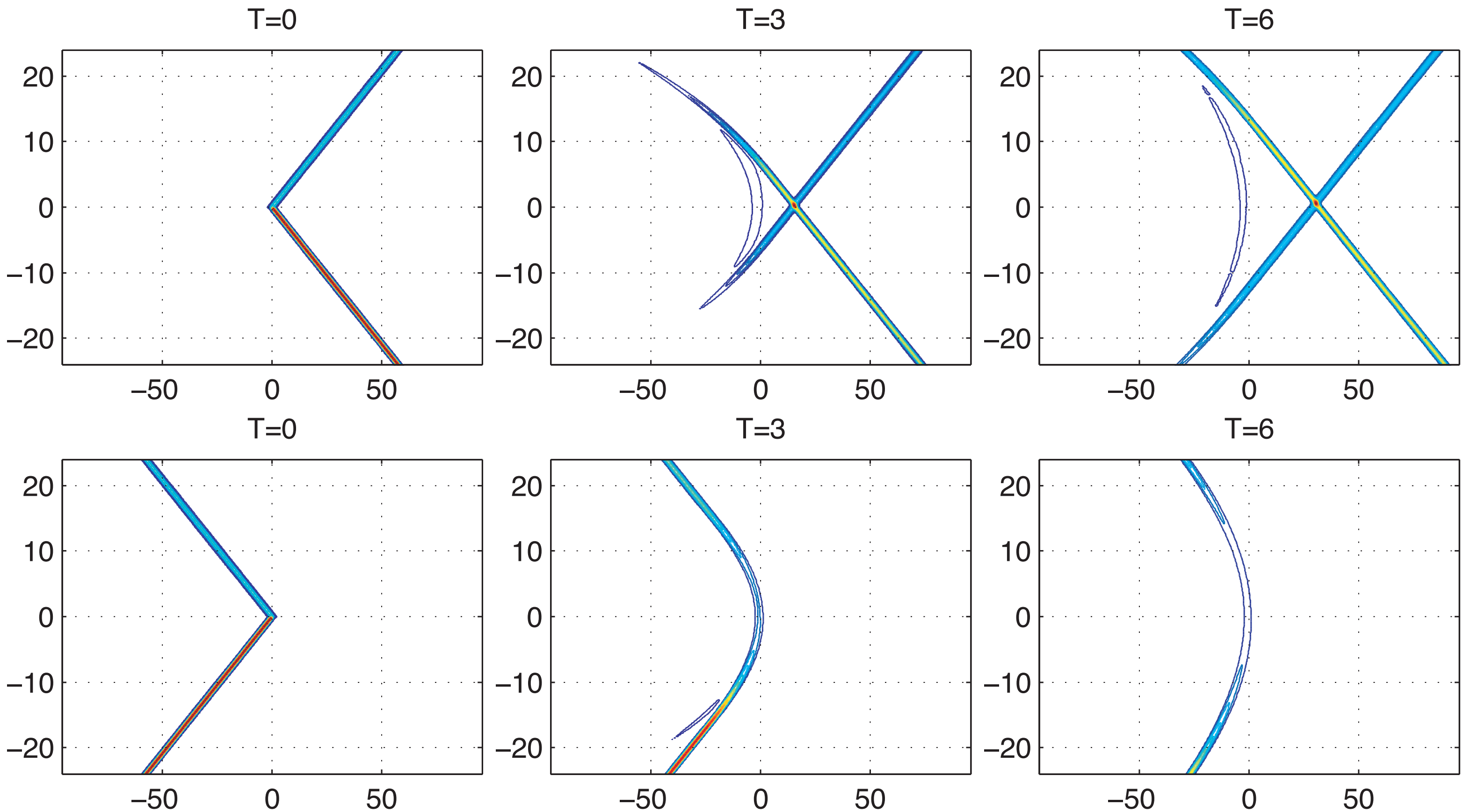
Summary for V-shape IWs

Chord diagrams for V-shape initial waves: A=2



Summary for V-shape IWs

Example 1 (O-type): $A_0 = 1,\ \Psi_0 \approx \pm 72^\circ$.

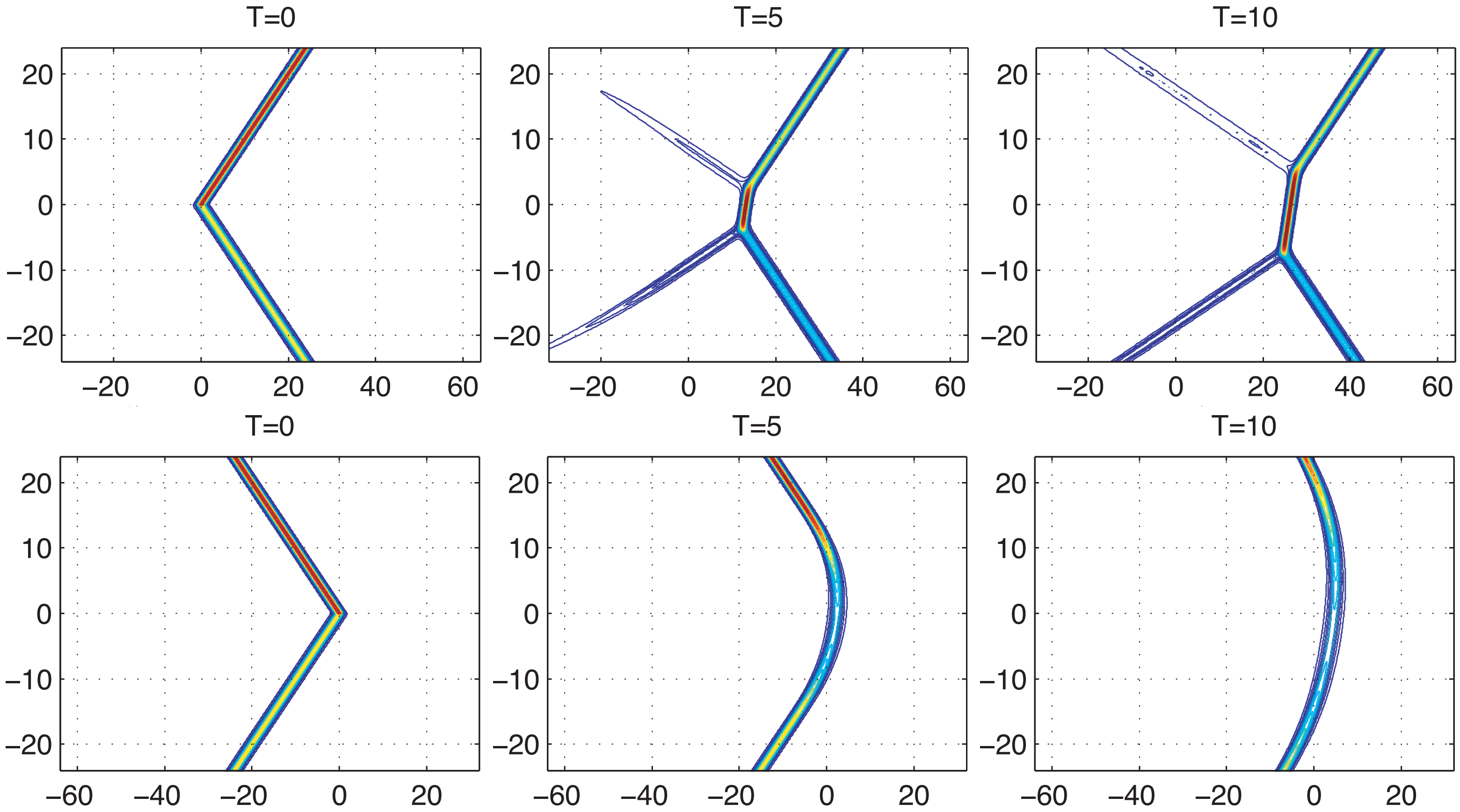


(a): $(k_1, k_2, k_3, k_4) = \frac{1}{4}(-(3+2\sqrt{2}),\ -(3-2\sqrt{2}),\ 2,\ 10)$.

(b): $(k_1, k_2, k_3, k_4) = \frac{1}{4}(-10,\ -2,\ 3-2\sqrt{2},\ 3+2\sqrt{2})$.

Summary for V-shape IWs

Example 2 ((3142) and dual): $A_0 = 3,\ \Psi_0 = \pm 45°$.

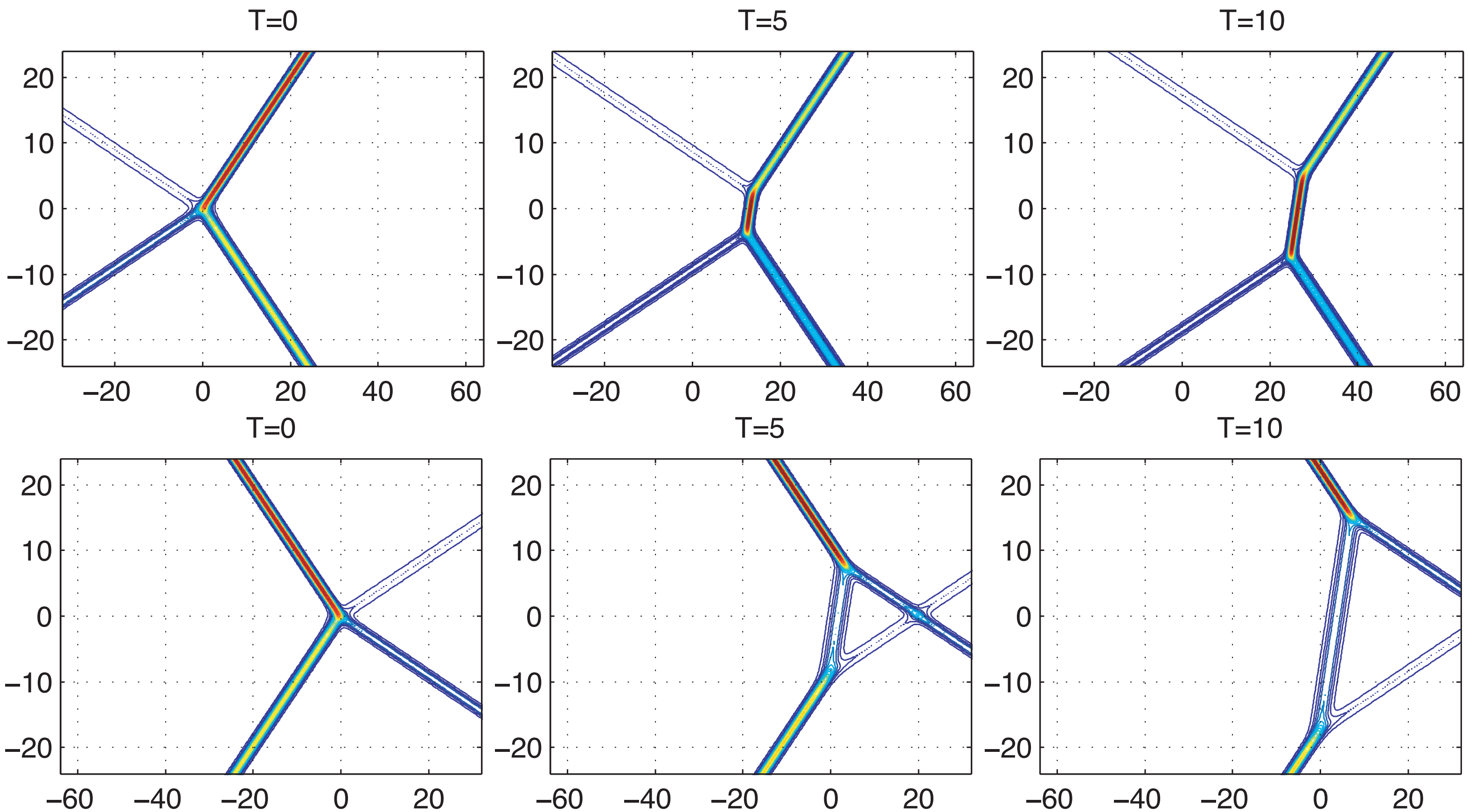


(a): $(k_1, k_2, k_3, k_4) = \frac{1}{2}(-(\sqrt{6}+1),\ -1,\ \sqrt{6}-1,\ 3)$.

(b): $(k_1, k_2, k_3, k_4) = \frac{1}{2}(-3,\ -(\sqrt{6}-1),\ 1,\ \sqrt{6}+1)$.

Summary for V-shape IWs

Example 2 Exact : $A_0 = 3,\ \Psi_0 = \pm 45°$.

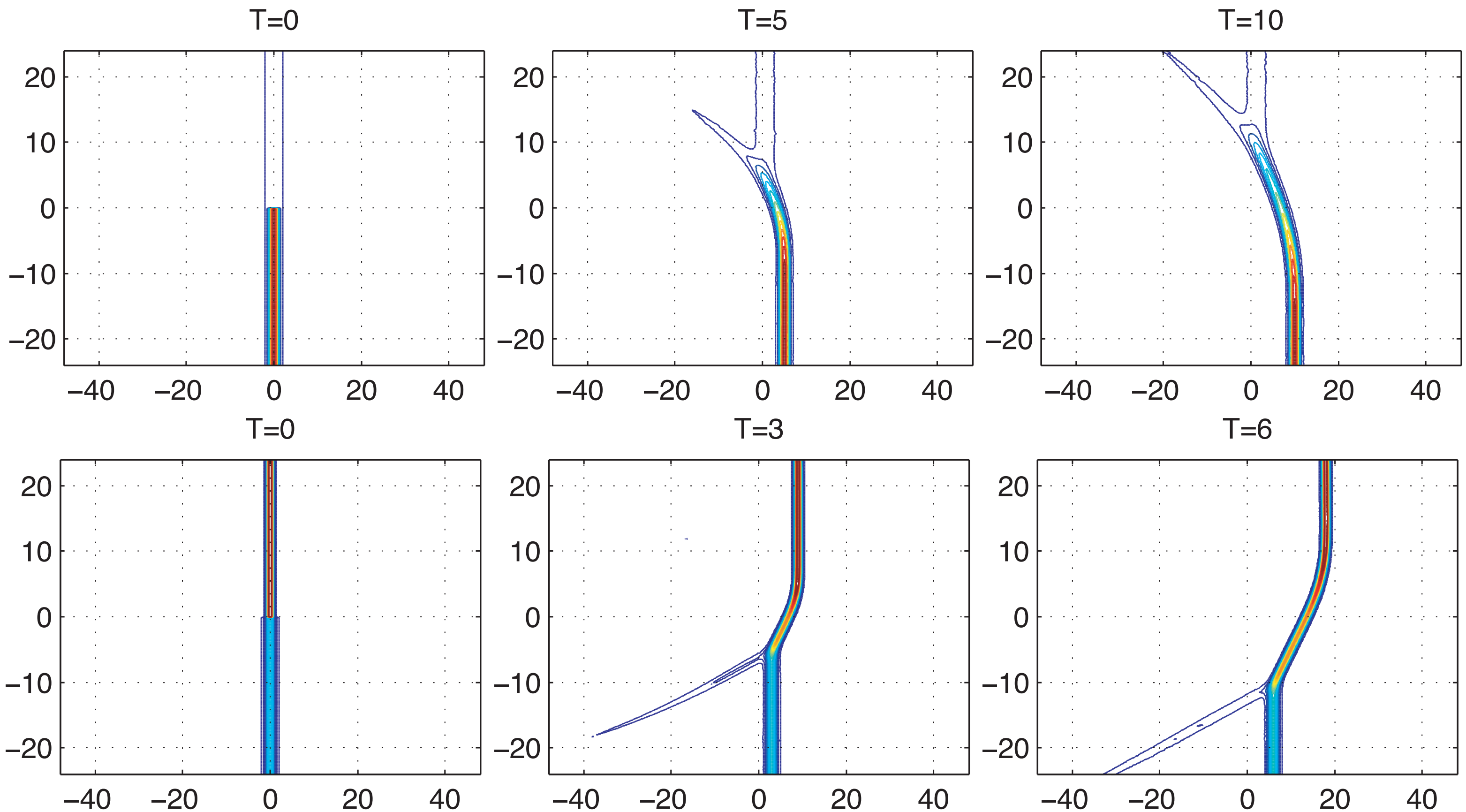


(a): $(k_1, k_2, k_3, k_4) = \frac{1}{2}(-(\sqrt{6}+1),\ -1,\ \sqrt{6}-1,\ 3)$.

(b): $(k_1, k_2, k_3, k_4) = \frac{1}{2}(-3,\ -(\sqrt{6}-1),\ 1,\ \sqrt{6}+1)$.

Summary for V-shape IWs

Example 3 ((1,3)- and dual): $\Psi_0 = 0°$.

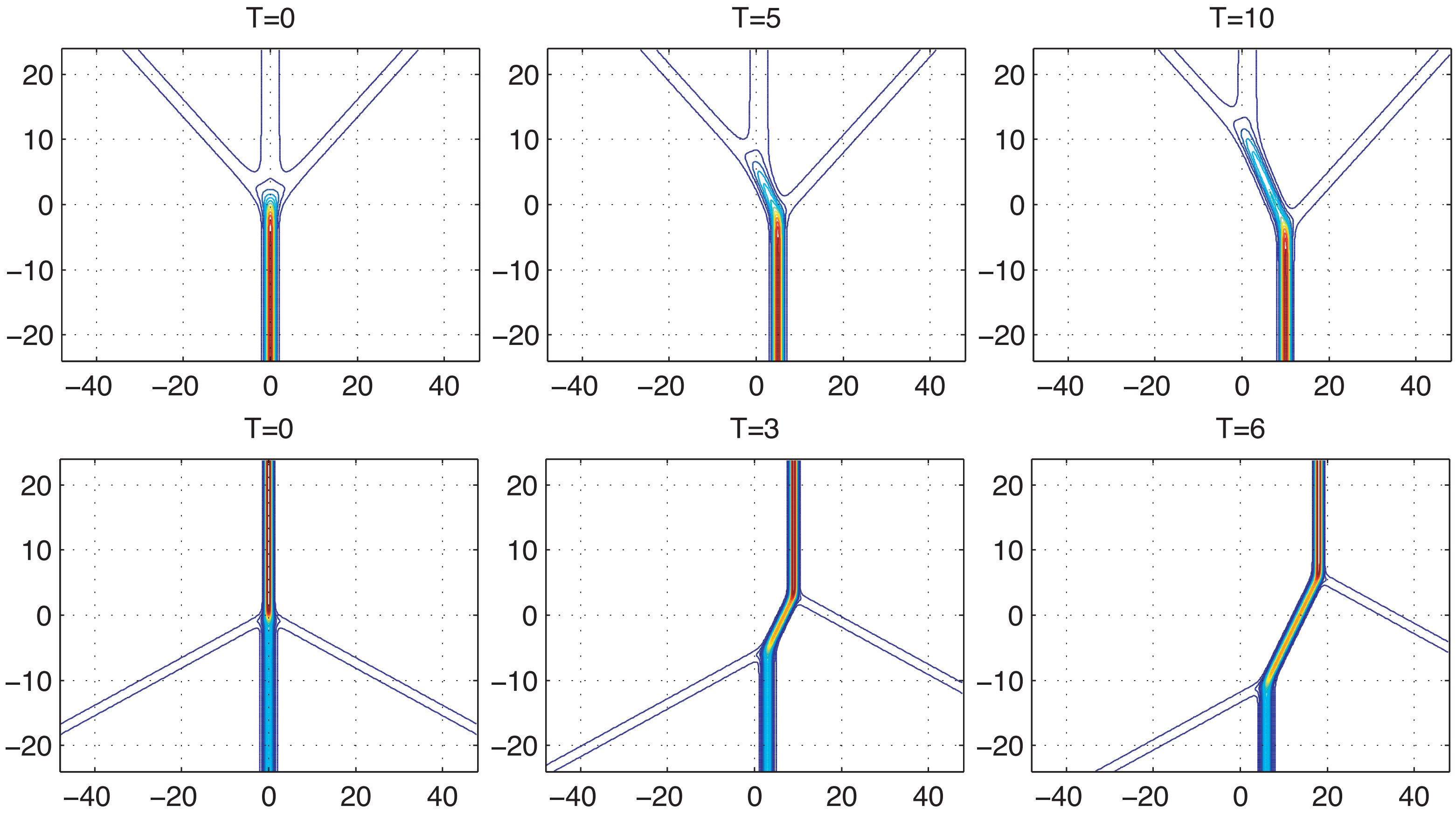


(a): $(k_1, k_2, k_3, k_4) = \frac{1}{2}(-2,\ -\sqrt{2},\ \sqrt{2},\ 2) \ \Rightarrow\ A_0 = 1$.

(b): $(k_1, k_2, k_3, k_4) = (-\sqrt{3},\ -1,\ 1,\ \sqrt{3}) \ \Rightarrow\ A_0 = 6$.

Summary for V-shape IWs

Example 3 Exact ((1,3)- and dual): $\Psi_0 = 0^\circ$.

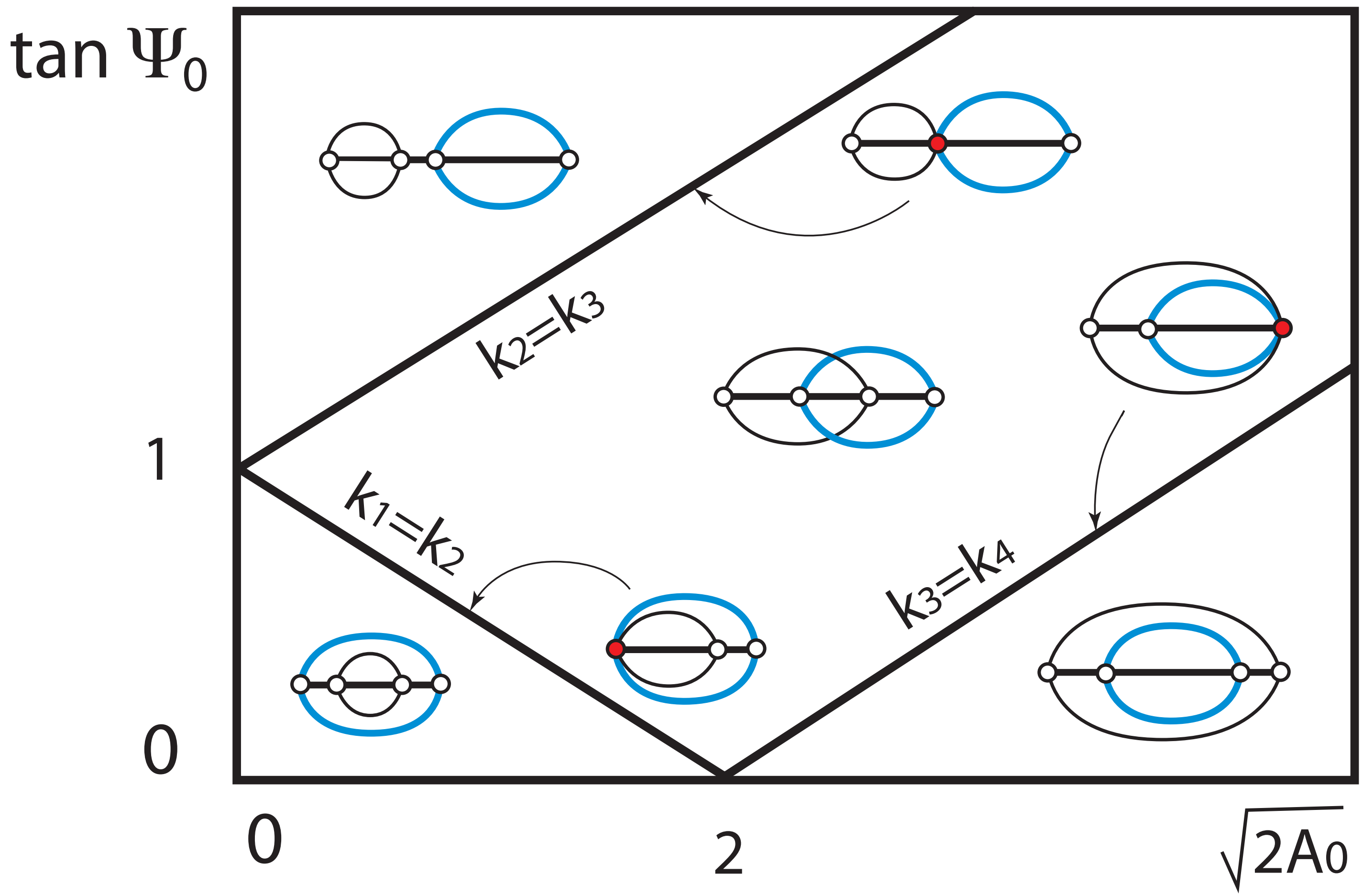


(a): $(k_1, k_2, k_3, k_4) = \frac{1}{2}(-2,\ -\sqrt{2},\ \sqrt{2},\ 2) \ \Rightarrow\ A_0 = 1$.

(b): $(k_1, k_2, k_3, k_4) = (-\sqrt{3},\ -1,\ 1,\ \sqrt{3}) \ \Rightarrow\ A_0 = 6$.

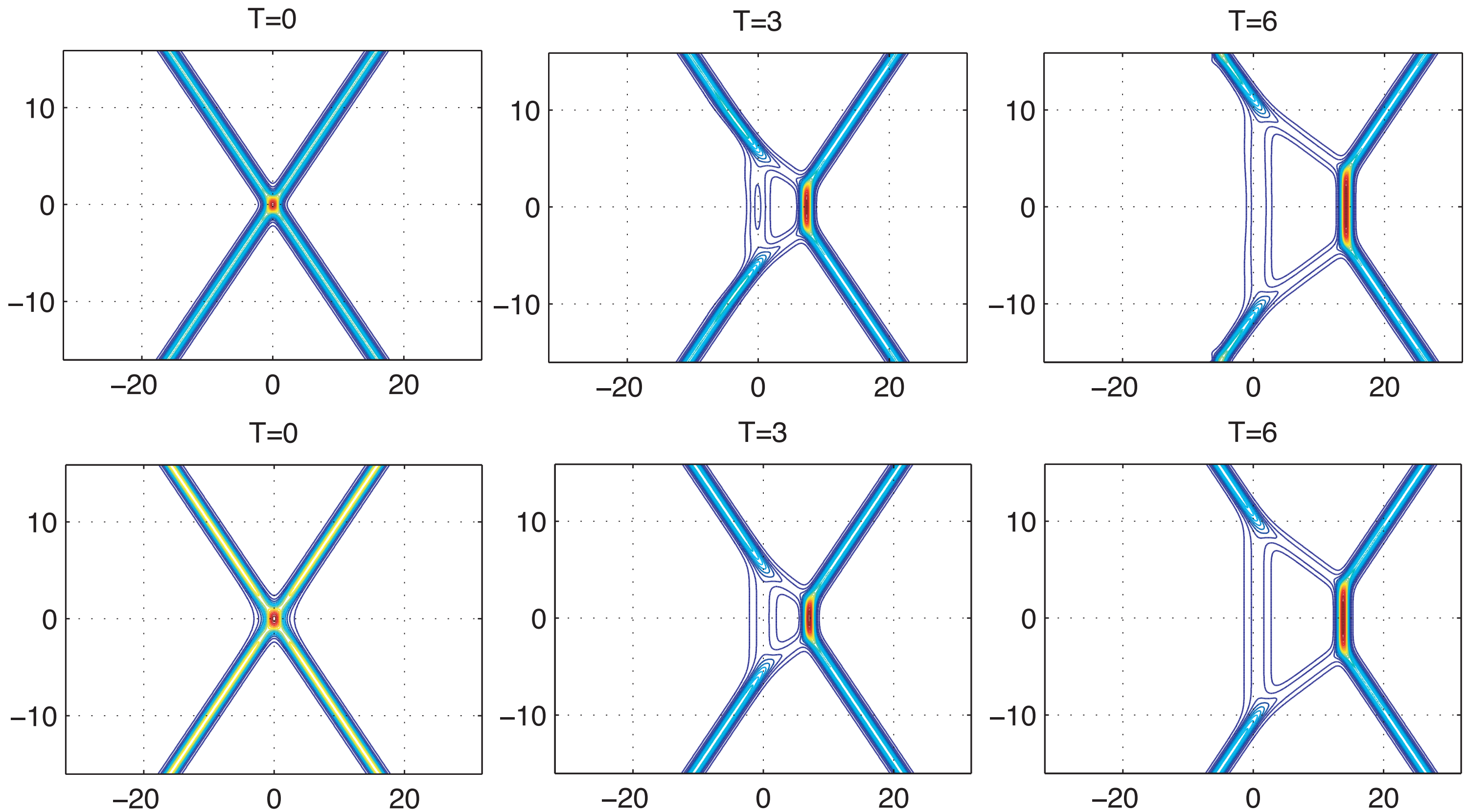
Summary for X-shape IWs

Chord diagrams for X-shpae initial waves:



Summary for X-shape IWs

T-type solution: $A_0 = 8,\ \Psi_0 = 45^\circ\ (\Psi_c \approx 63.4^\circ)$.

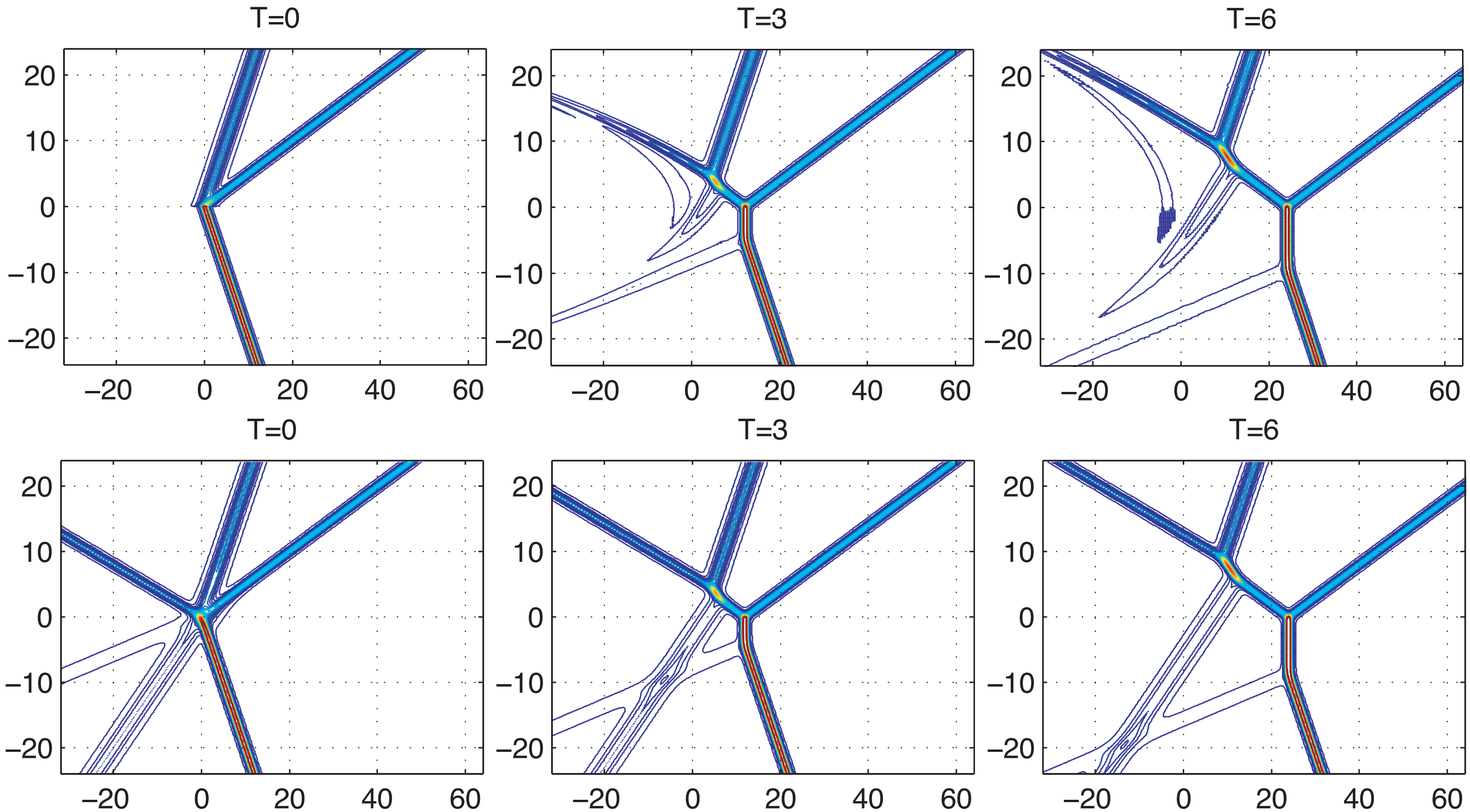


(a): Simulation for sum of two line-solitons with $A_0 = 2$.

(b): Exact solution with $(k_1, k_2, k_3, k_4) = \frac{1}{2}(-3, -1, 1, 3)$.

Example of 3 half-waves

(415362)-type solution (one of (3,3)-type solitons):



(a) Initial wave with $[1,4],[3,5]$ for $y>0$ and $[2,6]$ for $y<0$.

(b) Exact solution with $(k_1,\ldots,k_6)=\frac{1}{2}(-4,-3,-1,0,1,4)$.

References

- Y. K., *Young diagrams and N-soliton solutions of the KP equation*, J. Phys. A: Math. Gen. **37** (2004) 11169-11190.
- S. Chakravarty and Y. K., *Classification of the line-solitons of KPII*, J. Phys. A: Math. Theor. **41** (2008) 275209 (33pp).
- S. Chakravarty and Y. K., *A generating function for the N-soliton solutions of the Kadomtsev-Petviashvili II equation*, Contemporary Math. **471** (2008) 47-67.
- S. Chakravarty and Y. K., *Soliton solutions of the KP equation and application to shallow water waves*, Stud. in Appl. Math. **123** (2009) 83-151. (arXiv: 0902.4433)

References

- S. Chakravarty and Y. K., *Combinatorics and geometry of soliton solutions of the KP equation*, coming soon.
- C-Y. Kao and Y. K., *Numerical study on the Initial value problems of the KP equation*, coming soon.
- Y. K., M. Oikawa and H. Tsuji, *Soliton solutions of the KP equation with V-shape initial waves*, arXiv:0904.2620.
- B.-F. Feng, Y. K. and K.-I. Maruno, *Tow dimensional solitary wave solutions in the Boussinesq approximation of shallow water wave equation*, coming soon.
- Y. K. and H. Yeh, *Experimental study of soliton solutions of the KP equation in shallow water*, coming soon.