On a class of quasi-exactly solvable systems: solutions and orthogonal polynomials

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Outline:

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- Biconfluent Heun equation
- A Quasi-exactly solvable model
- A doubly anharmonic oscillators model
- Conclusion

1 Introduction:

The Shrödinger equation for a quasi-exactly solvable model

$$H\psi = E\psi \tag{1}$$

with the class of Hamiltonian first discussed by A.Turbiner:

$$H = -\frac{d^2}{dx^2} + \frac{(4s-1)(4s-3)}{4x^2} - (4s+4J-2)x^2 + x^6. \tag{2}$$

Rewrite the above Schrödinger equation as

$$\psi'' + \left(-x^6 + (4s+4J-2)x^2 + E - \frac{(4s-1)(4s-3)}{4x^2}\right)\psi = 0. \tag{3}$$

The solution

$$\psi(x) = \exp(-\frac{1}{4}x^4)x^{2s-1/2}\sum_{n=0}^{\infty}(-\frac{1}{4})^n\frac{P_n(E)}{n!\Gamma(n+2s)}x^{2n}. \tag{4}$$

$P_n(E)$ satisfies the three-term recursion relation

$$P_n(E) = EP_{n-1}(E) + 16(n-1)(n-J-1)(n+2s-2)P_{n-2}(E), \quad (n \geq 2). \tag{5}$$

2 Biconfluent Heun equation (BHE)

The canonical form of Heun's (general) equation

$$\frac{d^2y}{dz^2} + (\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-a})\frac{dy}{dz} + \frac{\alpha\beta z - q}{z(z-1)(z-a)}y = 0. \tag{6}$$

with the condition

$$\gamma + \delta + \epsilon = \alpha + \beta + 1. \tag{7}$$

Singularities

$$z = \{0, 1, a, \infty\}.$$

The representative equation of the class $(0, 1, 1_4)$ (BHE)

$$x^2y'' + xy' + (A_0 + A_1x + A_2x^2 + A_3x^3 - x^4)y = 0. \tag{8}$$

The equivalent Normal form

$$y''(x) + (Ax^2 + Bx + C + \frac{D}{x} + \frac{E}{x^2})y(x) = 0. \tag{9}$$

The Radial Schrödinger equation of a three-dimensional anharmonic oscillator is

$$y''(x) + \{E - \frac{\nu}{x^2} - \mu x^2 - \lambda x^4 - \eta x^6\}y(x) = 0, \tag{10}$$

where $\nu = l(l+1), \nu > 0, \eta > 0$ and E is the energy. (3) is a special case of (10).

3 A quasi-exactly solvable model (3)

Variable transformation

$$y(x) = z^{\frac{1}{4}} Y(z), \quad z = \left(\frac{1}{4}\right)^{\frac{1}{4}} x^2 \tag{11}$$

$$\Downarrow (3)$$

$$z^2 Y''(z) + zY'(z) + \left\{ -\frac{(2s-1)^2}{4} + \frac{E}{2\sqrt{2}} z + (2s + 2J - 1) z^2 - z^4 \right\} Y(z) = 0 \tag{12}$$

Through the Lommel transformation $z = \alpha t^{\beta}, Y(z) = t^{\gamma} u(t)$

$$\Downarrow (12)$$

$$t^2 u''(t) + (2\gamma + 1) t u'(t) + \left(\gamma^2 + \beta^2 \sum_{j=0}^{4} \alpha^j k_j t^{\beta j} \right) u(t) = 0, \tag{13}$$

a further variable transformation

$$t = e^{mz}, f(z) = u(t) \tag{14}$$

leads to an equation of the form

$$f'' + 2\gamma m f' + m^2(\gamma^2 + \beta^2 \sum_{j=0}^{4} \alpha^j k_j e^{m\beta j z}) f = 0. \tag{15}$$

Take $\gamma = 0$ and $m\beta = \alpha = 1$, then (15) has the form

$$f''(z) + (k_0 + k_1 e^z + k_2 e^{2z} + k_4 e^{4z}) f(z) = 0, \tag{16}$$

$$k_0 = -\frac{(2s-1)^2}{4}, k_1 = \frac{E}{2\sqrt{2}}, k_2 = 2s + 2J - 1, k_3 = 0, k_4 = -1. \tag{17}$$

It's the periodic second order differential equation studied by Bank, Laine and Langley.

From Bank, Laine and Langley's result, if we require the solution f of (16) satisfies

$$\lambda(f) = \limsup_{r \to +\infty} \frac{\log n(r, f)}{\log r} < +\infty, \tag{18}$$

say, the exponent of convergence of zeros is finite. Here $n(r, f)$ denotes the number of zeros of f in the domain $|z| < r$, then there exist complex constants d, d_j and a polynomial $\psi(\zeta)$ with only simple roots, such that

$$f(z) = \psi(e^z) \exp(P(e^z) + dz). \tag{19}$$

where

$$P(\zeta) = \sum_{j=1}^{2} d_j \zeta^j, \quad \psi(\zeta) = c_n \zeta^n + \cdots + c_0 \quad (c_n \neq 0). \tag{20}$$

$$d_2 = -1/2, \qquad d_1 = 0, \qquad d = s - 1/2. \tag{21}$$

$\psi(\zeta)$ satisfies the equation

$$\zeta\psi''(\zeta) + \left(-2\zeta^2 + 2s\right)\psi'(\zeta) + \left((2J-2)\zeta + \frac{\sqrt{2}E}{4}\right)\psi = 0. \quad (22)$$

Eq.(12) could be changed into Eq.(22) by the variable transformation

$$Y(z) = z^{s-\frac{1}{2}}\exp(-\frac{z^2}{2})\psi(z), \quad z = \zeta. \quad (23)$$

Theorem 1 *when* $n = J - 1$, $s > 0$, *there exist* $n + 1$ *real numbers* $E_0 < E_1 < \cdots < E_n$ *so that the differential Eq.*

$$x f''(x) + \left(-2x^2 + 2s\right) f'(x) + \left((2J - 2)x + \frac{\sqrt{2}E_i}{4}\right) f(x) = 0. \quad (24)$$

has a polynomial solution of the degree n, *for* $i = 0, 1, \cdots n$.

Examples:

$$f_0(x) = 1, \quad E = 0,$$

$$f_1(x) = 1 - \frac{\sqrt{2}}{8s}Ex, \quad E = \pm\sqrt{32s},$$

$$f_2(x) = 1 - \frac{\sqrt{2}}{8s}Ex - \frac{64s - E^2}{32s(2s+1)}x^2, \quad E = 0, \pm\sqrt{128s+32},$$

$$f_3(x) = 1 - \frac{\sqrt{2}E}{8s}x + \frac{E^2 - 96s}{32s(2s+1)}x^2 + \frac{E(64+224s-E^2)}{768s(s+1)(2s+1)}x^3,$$

$$E = \pm 4\sqrt{5 + 10s \pm \sqrt{64s^2 + 64s + 25}}.$$

. . . .

Remark: When $E = 0, n = J-1$ Eq.(24) is the generalized Hermite polynomial equation.

Theorem 2 *For the n order polynomial solutions $y_{n,\mu}$ and $y_{n,\nu}$ of Eq.(24), there exists the orthogonality relation*

$$\int_{-\infty}^{+\infty} x^{2s-1} e^{-x^2} y_{n,\mu} y_{n,\nu} dx = 0, \quad \textit{for}\, \mu \neq \nu, 0 \leq \mu, \nu \leq n, \tag{25}$$

provided that $s \geq 0$. Here the polynomials $y_{n,\mu}, y_{n,\nu}$ correspond to the eigenvalues $E_{n,\mu}, E_{n,\nu}$ respectively.

Theorem 3 *If $f(x)$, $\hat{f}(x)$ are polynomial solutions of (24) corresponding to different values of s and n, then*

$$\int_{-\infty}^{0}\int_{0}^{+\infty} f(x)f(y)\hat{f}(x)\hat{f}(y)(xy)^{2s-1}e^{-x^2-y^2}(y-x)dxdy = 0. \tag{26}$$

Reverse Polynomial Solutions:

Through the variable transformation

$$y(x) = x^n f(1/x)$$

Eq.(24) could be changed into

$$x^4 y'' + [(3-2n-2s)x^3 + 2x]y' + \left(\frac{\sqrt{2}E}{4}x + n(n+2s-2)x^2\right) y = 0. \quad (27)$$

when $J + s = K$,K is an arbitrary constant, Eq.(27) could be rewritten as

$$x^3 y'' + (ax^2 + 2)y' + \left(\frac{\sqrt{2}E}{4} + n(1-a-n)x\right) y = 0, \quad (28)$$

where $a = 5 - 2K$.

Eigenvalues and corresponding eigenfunctions for Eq.(28)

$$y_0(x) = 1, \quad E = 0,$$
$$y_1(x) = 1 - \frac{\sqrt{-2a}}{2}x, \quad E = \pm 4\sqrt{-a},$$
$$y_2(x) = 1 + \frac{4\sqrt{2}E(1+a)}{E^2+64+32a}x + \frac{16(2+3a+a^2)}{E^2+64+32a}x^2, \quad E = 0, \pm 4\sqrt{-6-4a},$$
$$y_3(x) = 1 + \frac{6\sqrt{2}(384+2E^2+aE^2+288a+48a^2)}{E(112a+E^2+384)}x + \frac{48(a^2+5a+6)}{112a+E^2+384}x^2$$
$$+\frac{96\sqrt{2}(26a+a^3+9a^2+24)}{E(112a+E^2+384)}x^3,$$
$$E = \pm 4\sqrt{-15-5a \pm \sqrt{16a^2+96a+153}}.$$

. . . .

The weight function $\rho(x)$ for Eq.(28) is

$$\rho(x) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \frac{\Gamma(\frac{1}{2} + \frac{a}{2})}{\Gamma(n + \frac{a}{2} - \frac{1}{2})} \left(-\frac{1}{x^2}\right)^n . \tag{29}$$

The series converges for all x except zero. Expanding (29) gives

$$\rho(x) = \frac{1}{2\pi i} \left[\frac{a-1}{2} + \left(-\frac{1}{x^2}\right) + \frac{2}{a+1}\left(-\frac{1}{x^2}\right)^2 + \frac{4}{(a+1)(a+3)}\left(-\frac{1}{x^2}\right)^3 \right.$$
$$\left. + \frac{8}{(a+5)(a+3)(a+1)}\left(-\frac{1}{x^2}\right)^4 + \cdots \right] . \tag{30}$$

The function $\rho(x)$ differs, except when $a = 1$ or 3, from the function $\sigma(x)$ given by

$$\sigma(x) = x^{a-3} e^{-1/x^2} / 2\pi i, \tag{31}$$

which is the factor needed to make equation (28) self-adjoint and which is therefore a natural candidate for a weight function. However, $\sigma(x)$ is multiple-valued when a is not an integer, and this is inconvenient if we wish to integrate around the point $x = 0$. The function $\sigma(x)$ satisfies the differential equation:

$$(x^3\sigma)' = (ax^2 + 2)\sigma, \tag{32}$$

while $\rho(x)$ satisfies the related nonhomogeneous equation

$$(x^3\rho)' = (ax^2 + 2)\rho - \frac{(a-1)(a-3)}{2\pi i} x^2. \tag{33}$$

Theorem 4 *The n-order polynomial solutions of Eq.(28) for different eigenvalues form an orthogonal system, with path of integration an arbitrary curve U surrounding the zero point and with the weight function $\rho(x)$ above, that is*

$$\int_U y_{n,\mu} y_{n,\nu} \rho dx = 0, \tag{34}$$

4 doubly anharmonic oscillators

The Schrödinger equation for the system of interest is

$$\frac{d^2\psi}{dx^2} + \left(2E - \omega^2 x^2 - \frac{\lambda}{2}x^4 - \frac{\eta}{3}x^6\right)\psi = 0, \tag{35}$$

where E is the energy eigenvalue and $\eta > 0$. By the variable transformation

$$\psi(x) = z^{\frac{1}{4}}Y(z), \quad z = (\frac{\eta}{12})^{\frac{1}{4}}x^2, \tag{36}$$

Eq.(35)can be changed into

$$z^2Y''(z) + zY'(z) + \{-\frac{1}{16} + \frac{E}{2\alpha_1}z - \frac{\omega^2}{4\alpha_1^2}z^2 - \frac{\lambda}{8\alpha_1^3}z^3 - z^4\}Y(z) = 0, \tag{37}$$

where we have denoted $\alpha_1 = (\frac{\eta}{12})^{\frac{1}{4}}$ for simplicity.

variable transformation

$$t = e^{mz}, \quad f(z) = u(t),$$

$$z^2 Y''(z) + zY'(z) + \{-\frac{1}{16} + \frac{E}{2\alpha_1} z - \frac{\omega^2}{4\alpha_1^2} z^2 - \frac{\lambda}{8\alpha_1^3} z^3 - z^4\} Y(z) = 0. \quad (38)$$

$$\Downarrow$$

$$f'' + 2\gamma m f' + m^2 (\gamma^2 + \beta^2 \sum_{j=0}^{4} \alpha^j k_j e^{m\beta j z}) f = 0. \quad (39)$$

$$\Downarrow \gamma = 0 \text{ and } m\beta = \alpha = 1$$

$$f'' + (k_0 + k_1 e^z + k_2 e^{2z} + k_3 e^{3z} + k_4 e^{4z}) f = 0. \quad (40)$$

$$k_0 = -\tfrac{1}{16}, k_1 = \tfrac{E}{2\alpha_1}, k_2 = -\tfrac{\omega^2}{4\alpha_1^2}, k_3 = -\tfrac{\lambda}{8\alpha_1^3}, k_4 = -1.$$

$$\lambda(f) < +\infty,$$

$$\Downarrow$$

$$f(z) = \phi(e^z) \exp(P(e^z) + dz), \tag{41}$$

where

$$P(\varsigma) = d_1\varsigma + d_2\varsigma^2, \quad \phi(\varsigma) = c_n\varsigma^n + \cdots + c_0(c_n \neq 0). \tag{42}$$

$$4d_2^2 + k_4 = 0, \quad 4d_1d_2 + k_3 = 0, \quad d^2 + k_0 = 0, \tag{43}$$

considering that the solution should decay exponentially when $|x| \to +\infty$, so

$$d_2 = -\frac{1}{2}, \quad d_1 = -\frac{\lambda}{16\alpha_1^3}, \quad d = \pm\frac{1}{4}. \tag{44}$$

The solution $Y(z)$ for Eq.(38) is

$$Y(z) = z^d \exp(d_1 z + d_2 z^2)\phi(z), \tag{45}$$

and correspondingly the solution for Eq.(35) is

$$\psi(x) = z^{\frac{1}{4}} Y(z) = x^v \exp(-\frac{\alpha}{4}x^4 + \frac{\beta}{2}x^2)\phi(\alpha_1 x^2), \tag{46}$$

($v = 0$ or 1 for states of even (odd) parity) with $\alpha = \sqrt{\eta/3}, \beta = -\sqrt{\frac{3}{\eta}}\frac{\lambda}{4}$.

$$\varsigma\phi'' + \left(\upsilon + \frac{1}{2} - \frac{\lambda}{8\alpha_1^3}\varsigma - 2\varsigma^2\right)\phi'$$
$$+\left(-(\upsilon+\frac{1}{2})\frac{\lambda}{16\alpha_1^3} + \frac{E}{2\alpha_1} + [-1 + \frac{\lambda^2}{256\alpha_1^6} - (\upsilon+\frac{1}{2}) - \frac{\omega^2}{4\alpha_1^2}]\varsigma\right)\phi = 0. \quad (47)$$

the polynomial solution $\phi(\varsigma) = c_n\varsigma^n + \cdots + c_0(c_n \neq 0)$ and

$$A_k c_{k+1} + B_k c_k + C_k c_{k-1} = 0 \quad (48)$$

where

$$A_k = (\upsilon + \frac{1}{2} + k)(k+1), \quad (49)$$
$$B_k = -\frac{\lambda}{8\alpha_1^3}k + \frac{E}{2\alpha_1} - (\upsilon + \frac{1}{2})\frac{\lambda}{16\alpha_1^3}, \quad (50)$$
$$C_k = 2(n - k + 1). \quad (51)$$
$$\sqrt{\frac{3}{\eta}}(\frac{3\lambda^2}{16\eta} - \omega^2) = 4n + 2\upsilon + 3. \quad (52)$$

eigenvalues and corresponding eigenfunctions:
$3+2\upsilon = \sqrt{\frac{3}{\eta}}(\frac{3\lambda^2}{16\eta} - \omega^2), n = 0, c_0 \neq 0, c_1 = 0 = c_2 = \ldots = c_n$. The eigenvalues and exact eigenfunctions are

$$E = (\upsilon + \frac{1}{2})\frac{\lambda}{4}\sqrt{\frac{3}{\eta}}, \tag{53}$$

$$\psi(x) = c_0 x^{\upsilon} \exp(-\frac{\alpha}{4}x^4 + \frac{\beta}{2}x^2). \tag{54}$$

$7 + 2\upsilon = \sqrt{\frac{3}{\eta}}(\frac{3\lambda^2}{16\eta} - \omega^2), n = 1, c_0 \neq 0, c_1 \neq 0, c_2 = 0 = ... = c_n$. The eigenvalues and corresponding exact eigenfunctions are

$$E = -\frac{1}{2}\beta(2\upsilon + 3) \pm \sqrt{\beta^2 + (2 + 4\upsilon)\alpha}. \quad (55)$$

$$\phi(z) = 1 + \frac{\beta \mp \sqrt{\beta^2 + 2\alpha}}{\alpha_1}z, \quad (\upsilon = 0), \quad (56)$$

$$\phi(z) = 1 + \frac{\beta \mp \sqrt{\beta^2 + 6\alpha}}{3\alpha_1}z, \quad (\upsilon = 1). \quad (57)$$

$$\psi(x) = c_0 \exp(-\frac{\alpha}{4}x^4 + \frac{\beta}{2}x^2)\left(1 + x^2(\beta \mp \sqrt{\beta^2 + 2\alpha})\right), \qquad \upsilon = 0. \quad (58)$$

$$\psi(x) = c_0 x \exp(-\frac{\alpha}{4}x^4 + \frac{\beta}{2}x^2)\left(1 + \frac{x^2}{3}(\beta \mp \sqrt{\beta^2 + 6\alpha})\right), \quad \upsilon = 1. \quad (59)$$

5 Discussions

- moments for orthogonal polynomials
- generating functions
- Rodrigues' formula

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Thank you!