Jacobi structures of evolutionary PDEs and their reciprocal transformations

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Jacobi structure (Lichnerowicz 1978)

Generalization of Poisson structure defined on a finite dimensional smooth manifold M^n, consists of a pair (Λ, X) of a bivector Λ and a vector field X on M, satisfy the conditions

$$[\Lambda, \Lambda] = 2X \wedge \Lambda, \quad [X, \Lambda] = 0.$$

It is equivalent to a local Lie algebra structure defined on the space of smooth functions on M via the following bracket:

$$\{f, g\} = \Lambda(df, dg) + fX(g) - gX(f), \quad \forall f, g \in C^\infty(M).$$

Although this bracket no longer satisfies the Leibniz rule, it still satisfies the Jacobi identity, and this is the main property that enable Jacobi structures to describe certain generalized Hamiltonian structures for dynamical systems.

Jacobi structure unifies several important mathematical structures which include Poisson manifolds, contact manifolds and locally conformal symplectic manifolds.

Infinite dimensional (local) Hamiltonian structures

$$\frac{\partial u^i}{\partial t} = \{u^i(x), H\} = P^{ij} \frac{\delta H}{\delta u^j(x)}, \quad i = 1, \ldots, n.$$

Here P^{ij} are differential operators in $\frac{\partial}{\partial x}$ whose coefficients are differential polynomials of $u^1(x), \ldots u^n(x)$, and H is a local functional $H = \int h(u, u_x, \ldots) dx$.

Poisson bracket

$$\{F, G\} = \int \frac{\delta F}{\delta u^i(x)} P^{ij} \frac{\delta G}{\delta u^j(x)} \, dx,$$

where F, G are local functionals.

An approach of classification of Integrable evolutionary PDEs:

Classification of bihamiltonian structures

$$\{\,,\,\}_2 + \lambda\{\,,\,\}_1$$

modulo Miura type transformations

$$u^i \mapsto \tilde{u}^i = F_0^i(u) + \varepsilon A_j^i(u) u_x^j + \varepsilon^2 \left(B_j^i(u) u_{xx}^j + C_{jl}^i(u) u_x^j u_x^l \right) + \dots .$$

With the condition that $\det\left(\frac{\partial F_0^i(u)}{\partial u^j}\right) \neq 0$.

Miura type transformations preserve the form of local Hamiltonian structures.

Reciprocal transformations

We consider reciprocal transformations of evolutionary PDEs

$$u^i_t = K^i(u; u_x, u_{xx}, \dots), \quad i = 1, \dots, n$$
$$u = (u^1, \dots, u^n) \in M^n.$$

Assume it possesses two conservation laws

$$\frac{\partial a(u; u_x, \dots)}{\partial t} = \frac{\partial b(u; u_x, \dots)}{\partial x}, \quad \frac{\partial p(u; u_x, \dots)}{\partial t} = \frac{\partial q(u; u_x, \dots)}{\partial x},$$

satisfying the condition

$$aq - pb \neq 0.$$

Here a, b, p, q are differential polynomials of $u^1, \dots, u^n$.

We can make a change of the independent variables

$$(x,t) \mapsto (y(x,t,u(x,t)), s(x,t,u(x,t))$$

by the following defining relations

$$\begin{aligned} dy &= a(u; u_x, \dots)\, dx + b(u; u_x, \dots)\, dt, \\ ds &= p(u; u_x, \dots)\, dx + q(u; u_x, \dots)\, dt. \end{aligned}$$

This is called a reciprocal transformations

Examples: Hodograph transformations

- When $n = 1$, we take

$$dy = u_x dx + u_t dt, \quad ds = dt$$

  Using u, t as independent and x as dependent variables.

- When $n = 2$ we can use u^1, u^2 as independent variables, while take x, t as dependent variables

$$dy = u^1_x dx + u^1_t dt, \quad ds = u^2_x dx + u^2_t dt,$$

They are used in fluid mechanics and gas dynamics, e.g. to transform a nonlinear systems to a linear one.

More Examples

- The Camassa-Holm equation

$$m_t + u\, m_x + 2m\, u_x = 0, \quad m = u - u_{xx} + \frac{1}{2}\kappa$$

is transformed to the first negative flow of the KdV hierarchy via the reciprocal transformation defined by

$$dy = \sqrt{m}\, dx - u\sqrt{m}\, dt, \quad ds = dt$$

- The reciprocal relation between the 2-component Camassa-Holm equations and the AKNS hierarchy, between the Toda hierarchy and the nonlinear Schrödinger hierarchy.

Classify bihamiltonian integrable hierarchies modulo Miura type transformation + reciprocal transformation.

Expected outcome: in the scalar case (i.e. with one unknown function), the bihmailtonian integrable hierarchies are equivalent to the Korteweg - de Vries hierarchy.

Problem: Reciprocal transformations in general destroy the locality of Hamiltonian structures.

Infinite dimensional Jacobi structure appears when we try to understand the non-local Hamiltonian structures that are obtained from local ones via reciprocal transformations.

2. Reciprocal transformations of Hamiltonian systems

Let us consider a special class of nonlinear reciprocal transformation acting on Hamiltonian systems of hydrodynamic type

$$u^i_t = V^i_j(u)u^j_x = P^{ij}\frac{\partial h}{\partial u^j}, \quad i = 1, \ldots, n.$$

Here $P^{ij} = g^{ij}(u)\partial_x + \Gamma^{ij}_k(u)u^k_x$, ($g^{ij}$ nondegenerate and symmetric.

Assume this system has a conservation law

$$\frac{\partial \rho}{\partial t} = \frac{\partial \sigma}{\partial x}$$

Then we can consider the reciprocal transformation defined by

$$dy = \rho(u)dx + \sigma(u)dt, \quad ds = dt$$

Theorem (Ferapontov & Pavlov, 2003)

The transformed system is still Hamiltonian w.r.t. a nonlocal Hamiltonian operator of the form

$$\widetilde{P}^{ij} = \tilde{g}^{ij}(u)\partial_y + \widetilde{\Gamma}^{ij}_k u^k_y + \omega^i_k u^k_y \partial_y^{-1} u^j_y + u^i_y \partial_y^{-1} \omega^j_k u^k_y$$

Here $\tilde{g}^{ij} = \rho(u)^2 g^{ij}$, $\omega^i_k = \rho\nabla^i\nabla_k\rho - \frac{1}{2}\delta^i_k(\nabla\rho)^2$. *The density of the Hamiltonian of the transformed system is given* $\tilde{h} = h/\rho$.

The metric $\tilde{g}^{ij}$ is no longer flat, it is conformally flat instead. And the Hamiltonian operator contains nonlocal terms.

2. Reciprocal transformations of Hamiltonian systems

To understand better the transformation rule of Hamiltonian and bihamiltonian structure of evolutionary PDEs under reciprocal transformations, we need to give a rigorous definition of the class of non-local Hamiltonian structures. To this end, we introduce a class of quasi-local multi-vectors on the infinite jet space of a n-dimensional manifold, and define generalized Hamiltonian structures via the definition of Schouten-Nijenhuis bracket among such multi-vectors, and the non-local Hamiltonian structures are defined as the infinite dimensional counterpart of Jacobi structures.

Note that we can eliminate the integral operator ∂^{-1} that appears in the nonlocal Hamiltonian structure by using the identity

$$\partial_x^{-1}\left(u_x^i \frac{\delta G}{\delta u^i(x)}\right) = -E(G),$$

where E is a linear operator defined on the space of local functionals $G = \int g(u, u_x, \dots)dx$ by

$$E(G) = \sum_{s\geq 0}\sum_{t\geq 1}(-1)^s u^{i,t}\partial_x^s \frac{\partial g}{\partial u^{i,s+t}} - g, \quad u^{i,t} = \partial_x^t u^i.$$

We call E the energy operator

The Poisson bracket defined by the above nonlocal Hamiltonian operator

$$P^{ij} = \alpha^{ij} + X^i \partial_x^{-1} u_x^j + u_x^i \partial_x^{-1} X^j,$$
$$\alpha^{ij} : \text{differential operators}, \quad X^i : \text{differential polynomials.}$$

is written as

$$\begin{aligned}\{F, G\} &= \int \frac{\delta F}{\delta u^i} \left(\alpha^{ij} + X^i \partial_x^{-1} u_x^j + u_x^i \partial_x^{-1} X^j\right) \frac{\delta G}{\delta u^j}\, dx \\ &= \int \left[\frac{\delta F}{\delta u^i} \alpha^{ij} \frac{\delta G}{\delta u^j} + X^i \left(E(F) \frac{\delta G}{\delta u^i} - \frac{\delta F}{\delta u^i} E(G)\right)\right] dx.\end{aligned}$$

- $\mathcal{A}$: the space differential polynomials on the jet space $J^\infty(M^n)$.
- $\mathcal{F} = \mathcal{A}/\partial_x\mathcal{A}$, the space of local functionals

$$\int f dx, \quad f \in \mathcal{A}.$$

- Local coordinates on $J^\infty(M) : u^1, \ldots, u^n, u^1_x, \ldots, u^n_x, u^1_{xx}, \ldots.$
- We denote the linear space of all alternating p-linear maps from Λ_x to Λ_x by $\mathcal{V}^p = \mathrm{Alt}^p(\Lambda_x, \Lambda_x)$, its elements are called generalized p-vectors. We also use the notations $\mathcal{V}^0 = \mathcal{F}$, $\mathcal{V}^{<0} = 0$, and $\mathcal{V} = \bigoplus_{p \geq 0} \mathcal{V}^p$.

Theorem.

There exists a unique bracket $[\ ,\ ] : \mathcal{V}^p \times \mathcal{V}^q \to \mathcal{V}^{p+q-1}$ *satisfying the following conditions:*

$$
\begin{aligned}
&[P, Q] = (-1)^{pq}[Q, P],\\
&(-1)^{pr}[[P, Q], R] + (-1)^{qp}[[Q, R], P] + (-1)^{rq}[[R, P], Q] = 0,\\
&[P, F_1](F_2, \cdots, F_p) = P(F_1, \cdots, F_p),
\end{aligned}
$$

for any $P \in \mathcal{V}^p$, $Q \in \mathcal{V}^q$, $R \in \mathcal{V}^r$, $F_1, F_2, \cdots, F_p \in \mathcal{F}$.

The bracket defined above is called the Nijenhuis-Richardson (1966, 67) bracket among generalized multi-vectors.

Definition.

Let $P \in \mathcal{V}^p$ be a generalized p-vector, we say that P is quasi-local if the action of P on $F_1, \cdots, F_p \in \mathcal{F}$ takes the following form

$$P(F_1, \cdots, F_p) = \int \Big(Q^{i_1 \cdots i_p}_{s_1 \cdots s_p} \partial^{s_1} \delta_{i_1}(F_1) \cdots \partial^{s_p} \delta_{i_p}(F_p)$$

$$+ R^{j_1 \cdots j_{p-1}}_{t_1 \cdots t_{p-1}} \sum_{k=1}^{p} (-1)^{k-1} E(F_k)\, \partial^{t_1} \delta_{j_1}(F_1) \cdots \hat{F}_k \cdots \partial^{t_{p-1}} \delta_{j_{p-1}}(F_p) \Big)\, dx$$

where $Q^{i_1 \cdots i_p}_{s_1 \cdots s_p}$, $R^{j_1 \cdots j_{p-1}}_{t_1 \cdots t_{p-1}} \in \mathcal{A}$. P is called local if the second term does not appear.

3. The space of quasi-local multi-vectors

Theorem.

The space of quasi-local multi-vectors is closed under the operation of Nijenhuis - Richardson bracket.

We call the restricted bracket the Nijenhuis-Schouten bracket.

3. The space of quasi-local multi-vectors

Representing the multi-vectors in terms of super variables

Super variables: $\theta_i^s,\ \zeta, \quad i = 1, \ldots, n,\ s \geq 0$

$$\mathcal{S} = \mathcal{A} \otimes \wedge^*(V),\ \text{where } V = \bigoplus_{i,s} (\mathbb{R}\theta_i^s) \oplus \mathbb{R}\zeta.$$

Extend the derivation ∂_x on $\mathcal{A}$ to a derivation on $\mathcal{S}$

$$\partial_x = \sum_{s\geq 0} \left(u^{i,s+1} \frac{\partial}{\partial u^{i,s}} + \theta_i^{s+1} \frac{\partial}{\partial \theta_i^s} \right) - \left(u_x^i \theta_i \right) \frac{\partial}{\partial \zeta} : \mathcal{S} \to \mathcal{S}$$

and define

$$\mathcal{E} = \mathcal{S}/\partial_x \mathcal{S}.$$

Theorem.

The space of quasi-local multi-vectors is isomorphic to $\mathcal{S}$.

The isomorphism is given by

$$\jmath(P)(F_1, \cdots, F_p) = \sum_{s_k \geq 0} \int \Big(\partial_{s_p}^{i_p} \cdots \partial_{s_1}^{i_1}(\alpha) \cdot \delta_{i_1}^{s_1}(F_1) \cdots \delta_{i_p}^{s_p}(F_p)$$

$$+ \partial_{s_{p-1}}^{i_{p-1}} \cdots \partial_{s_1}^{i_1} \partial_\zeta(\alpha) \cdot \sum_{k=1}^{p} (-1)^{k-1} E(F_k)\ \delta_{i_1}^{s_1}(F_1) \cdots \hat{F}_k \cdots \delta_{i_{p-1}}^{s_{p-1}}(F_p) \Big)\, dx,$$

here $\partial_\zeta = \frac{\partial}{\partial \zeta}$, $\partial_s^i = \frac{\partial}{\partial \theta_i^s}$, $\delta_i^s = \partial_x^s \frac{\delta}{\delta u^i}$, and $\alpha \in \mathcal{S}^p$ is a representative of P, i.e. $P = \int \alpha\, dx$, and $F_1, \cdots, F_p \in \mathcal{F}$.

Theorem.

The Nijenhuis-Schouten bracket is given by the following formula:

$$[P, Q] = \int \left(\frac{\delta \alpha}{\delta \theta_i} \frac{\delta \beta}{\delta u^i} + (-1)^p \frac{\delta \alpha}{\delta u^i} \frac{\delta \beta}{\delta \theta_i} + \frac{\partial \alpha}{\partial \zeta} \hat{E}(\beta) + (-1)^p \hat{E}(\alpha) \frac{\partial \beta}{\partial \zeta} \right) dx,$$

where $P = \int \alpha\, dx \in \mathcal{E}^p$, $Q = \int \beta\, dx \in \mathcal{E}^q$ *with* $\alpha \in \mathcal{S}^p$, $\beta \in \mathcal{S}^q$, *and the operator* $\hat{E}$ *is given by*

$$\hat{E} = \sum_{s \ge 0} \sum_{t \ge 1} (-1)^s \left(u^{i,t} \partial_x^s \frac{\partial}{\partial u^{i,s+t}} + \theta_i^t \partial_x^s \frac{\partial}{\partial \theta_i^{s+t}} \right) + \theta_i \frac{\delta}{\delta \theta_i} - 1.$$

Definition.

A quasi-local bivector $P \in \mathcal{E}^2$ is called a Jacobi structure if

$$[P, P] = 0.$$

One can always represent a Jacobi structure in the form

$$P = \int \left(\frac{1}{2}\theta_i \left(\alpha_s^{ij}\partial^s\right)\theta_j + \zeta X^i \theta_i \right),$$

where $\alpha_s^{ij},\ X^i \in \mathcal{A}$, and $\alpha = \left(\alpha^{ij}\right) = \left(\alpha_s^{ij}\partial^s\right)$ is skew-symmetric.

local part: $P_0 = \frac{1}{2}\theta_i\alpha^{ij}\theta_j$, The structure flow: $X = X^i\theta_i$.

We denote $P = (P_0, X)$, or $P \sim P_0 + \zeta X$.

Jacobi structures of degree zero

The simplest Jacobi structures are the ones with degree zero. Let $P = (P_0, X)$ be a Jacobi structure such that $\deg(P) = 0$, i.e.

$$\alpha^{ij} = \alpha_0^{ij}(u), \quad X^i = X^i(u),$$

then P_0 is just a bivector on M and X is vector field on M. The condition $[P, P] = 0$ implies

$$[P_0, P_0] + 2\,X \wedge P_0 = 0,\ [P_0, X] = 0,$$

so the pair $(P_0, -X)$ gives a classical Jacobi structure on M.

Jacobi structures of degree one

Let $P_0 = \frac{1}{2}\left(g^{ij}(u)\theta_i\theta_j^1 + \Gamma^{ij}_k u^{k,1}\theta_i\theta_j\right)$, $X = V^i_k(u)u^{k,1}\theta_i$, and assume that $\det\left(g^{ij}\right) \neq 0$.

Theorem. (Ferapontov 1995)

A bivector $P = (P_0, X)$ of the above form is a Jacobi structure if and only if the following conditions are satisfied

$$\Gamma^j_{kl} = \Gamma^j_{lk},\ \ V_{kj} = V_{jk},\ \ \nabla_k V_{lj} = \nabla_l V_{kj},$$
$$R_{ijkl} = g_{ik}V_{jl} + g_{jl}V_{ik} - g_{jk}V_{il} - g_{il}V_{jk},$$

where $\Gamma^j_{kl} = -g_{ki}\Gamma^{ij}_l$, $V_{kj} = g_{ki}V^i_j$, ∇ and R_{ijkl} are the Levi-Civita connection and Riemannian curvature tensor of g_{ij} respectively.

Let $\rho \in \mathcal{A}$ be an invertible element, it defines a derivation on $\mathcal{A}$

$$\partial_y = \rho^{-1}\partial_x$$

Denote

$$\tilde{\mathcal{F}} = \mathcal{A}/\partial_y\mathcal{A}, \quad \tilde{\mathcal{V}} = \mathrm{Alt}^*(\tilde{\mathcal{F}}, \tilde{\mathcal{F}})$$

We have an isomorphism

$$\Phi_0 : \mathcal{F} \to \tilde{\mathcal{F}}, \ \int f\, dx \mapsto \int \rho^{-1} f\, dy.$$

It induces an isomorphism $\Phi : \mathcal{V} \to \tilde{\mathcal{V}}$ as follows:

$$\Phi(P)(F_1, F_2, \cdots, F_p) = \Phi_0 \left(P \left(\Phi_0^{-1} F_1, \Phi_0^{-1} F_2, \cdots, \Phi_0^{-1} F_p \right) \right),$$

where $P \in \mathcal{V}^p$ and $F_1, F_2, \cdots, F_p \in \tilde{\mathcal{F}}$.

Definition.

The isomorphism $\Phi : \mathcal{V} \to \tilde{\mathcal{V}}$ is called a reciprocal transformation with respect to ρ.

Theorem.

1. *The restriction of the reciprocal transformation Φ to $\mathcal{E} \subset \mathcal{V}$ yields an isomorphism*

$$\Phi : \mathcal{E} \to \tilde{\mathcal{E}}.$$

2. *The reciprocal transformation of a Jacobi structure is also a Jacobi structure.*

5. Reciprocal transformations of Jacobi structures

The isomorphism is given by

$$\Phi\left(\int \alpha\, dx\right) = \int \rho^{-1}\hat{\Phi}(\alpha)\, dy, \quad \alpha \in \mathcal{S}^p,$$

where

$$\begin{aligned}
\hat{\Phi}(f) =& f,\ \ f \in \mathcal{A}, \\
\hat{\Phi}(\theta_i^s) =& \left(\rho\tilde{\partial}\right)^s \left(\rho\, \tilde{\theta}_i - \sum_{s\geq 0}(-\rho\tilde{\partial})^s \left(\partial_{i,s}(\rho)\tilde{\zeta}\right)\right), \\
\hat{\Phi}(\zeta) =& \rho\, \tilde{\zeta} - \sum_{s\geq 0}\sum_{\alpha\geq 1} u^{i,\alpha}(-\rho\tilde{\partial})^s \left(\partial_{i,s+\alpha}(\rho)\tilde{\zeta}\right).
\end{aligned}$$

Let $P \sim \frac{1}{2}\alpha^{ij}\theta_i\theta_j + \zeta\, X^i\, \theta_i$ be a Jacobi structure of degree zero, and ρ be a nowhere zero smooth function on M. One can define the reciprocal transformation Φ w.r.t. ρ. The isomorphism $\hat{\Phi}$ now reads

$$\hat{\Phi}(\theta_i) = \rho\,\tilde{\theta}_i - \partial_{i,0}(\rho)\tilde{\zeta},\ \hat{\Phi}(\zeta) = \rho\,\tilde{\zeta},$$

so we can obtain the reciprocal transformation of P

$$\Phi(P) \tilde{\sim} \frac{1}{2}\tilde{\alpha}^{ij}\tilde{\theta}_i\tilde{\theta}_j + \tilde{\zeta}\,\tilde{X}^i\,\tilde{\theta}_i,$$

where $\tilde{\alpha}^{ij} = \rho\,\alpha^{ij}$, $\tilde{X}^i = \rho\, X^i + \alpha^{ij}\partial_{j,0}(\rho)$. This is in fact a conformal change of classical Jacobi structures.

5. Reciprocal transformations of Jacobi structures

Let $\alpha^{ij} = g^{ij}\partial_x + \Gamma^{ij}_k u^{k,1}$ be a hydrodynamic Hamiltonian operator, then we have a local bivector $P = \frac{1}{2}\int \theta_i \alpha^{ij}\theta_j$.

Define the reciprocal transformation Φ w.r.t. a nowhere zero smooth function ρ on M, we obtain

$$\Phi(P) \tilde{\sim} \frac{1}{2}\left(\rho\,\tilde{\theta}_i - \partial_{i,0}(\rho)\tilde{\zeta}\right)\left(g^{ij}\partial_y + \Gamma^{ij}_k \tilde{u}^{k,1}\right)\left(\rho\,\tilde{\theta}_j - \partial_{j,0}(\rho)\tilde{\zeta}\right),$$

which is equivalent to Ferapontov and Pavlov's transformation formula.

6. Works to be done

1. Deformation theory of Jacobi structures of hydrodynamic type.
2. Properties of evolutionary PDEs possessing Jacobi structures.
3. Relations of integrability and multi-Jacobi structures of evolutionary PDEs.
4. Classify bihamiltonian integrable hierarchy with respect to Miura-type transformations and reciprocal transformations.

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