DARBOUX TRANSFORMATION FOR THE NONLINEAR SCHRÖDINGER EQUATION

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(joint with C. van der Mee, Cagliari)

Outline:

- Darboux transformation
- Bäcklund transformation
- Darboux transform for Zakharov-Shabat system/NLS equation
- general approach to Darboux transformations
- references

For details:

T. Aktosun and C. van der Mee, *A unified approach to Darboux transformations,* Inverse Problems (2009), to appear

Unperturbed problem: $\mathcal{L}\psi = \lambda\psi$

- $\mathcal{L}$ differential operator in x
- λ spectral parameter
- potential $u(x,t)$, t parameter
- $\psi(\lambda, x, t)$ wave function
- spectrum of $\mathcal{L}$, discrete and continuous spectra
- eigenvalues and eigenfunctions
- (generalized) eigenvalues and eigenfunctions

Perturbed problem: $\tilde{\mathcal{L}}\tilde{\psi} = \lambda\tilde{\psi}$

- $\tilde{\mathcal{L}}$ is a finite-rank perturbation of $\mathcal{L}$
- $\tilde{\mathcal{L}}$ is obtained from $\mathcal{L}$ by changing only the discrete spectrum
- perturbed potential $\tilde{u}(x,t)$
- perturbed $\tilde{\psi}(\lambda,x,t)$ wave function

Darboux transformation

- express, via unperturbed quantities and finite-rank perturbation:

  $\tilde{u}(x,t) - u(x,t)$ (DT at the potential level)

  $\tilde{\psi}(\lambda,x,t) - \psi(\lambda,x,t)$ (DT at the wave function level)

Bäcklund transformation

- differential equation(s) only involving:

  $\tilde{u}(x,t)$, $u(x,t)$, and their x- and t-derivatives

Zakharov-Shabat system: $\mathcal{L}\psi = \lambda\psi$

- $\mathcal{L} := i \begin{bmatrix} \dfrac{d}{dx} & -u(x,t) \\ -u(x,t)^* & -\dfrac{d}{dx} \end{bmatrix}$
- $\psi(\lambda, x, t) := \begin{bmatrix} \psi_1(\lambda, x, t) \\ \psi_2(\lambda, x, t) \end{bmatrix}$ Jost solution (from the left)
- $\psi(\lambda, x, t) \sim \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix}$ as $x \to +\infty$

Nonlinear Schrödinger (NLS) equation: $iu_t + u_{xx} + 2|u|^2 u = 0$

Inverse scattering transform

- integrable nonlinear PDE, linear spectral problem $\mathcal{L}\psi = \lambda\psi$

- KdV $\quad u_t - 6uu_x + u_{xxx} = 0$

  1-D Schrödinger eq $\qquad -\dfrac{d^2\psi}{dx^2} + u(x,t)\,\psi = \lambda\psi$

- mKdV $\qquad u_t + 6u^2u_x + u_{xxx} = 0, \qquad \begin{cases} \dfrac{d\xi}{dx} = -i\lambda\xi + u(x,t)\,\eta \\ \dfrac{d\eta}{dx} = i\lambda\eta - u(x,t)\,\xi \end{cases}$

- sine-Gordon $\qquad u_{xt} = \sin u, \qquad \begin{cases} \dfrac{d\xi}{dx} = -i\lambda\xi - \dfrac{1}{2}\,u_x(x,t)\,\eta \\ \dfrac{d\eta}{dx} = i\lambda\eta + \dfrac{1}{2}\,u_x(x,t)\,\xi \end{cases}$

Darboux transformation for ZS system/NLS

Add one bound state at $\lambda = \lambda_1$ with the norming constant $c_1(t)$. Then:

$$\tilde{u}(x,t) - u(x,t) = \frac{P_0}{|\Gamma_1|^2 + |c_1|^2\Gamma_2^2}$$

$$\begin{bmatrix} \tilde{\psi}_1(\lambda,x,t) \\ \tilde{\psi}_2(\lambda,x,t) \end{bmatrix} - \begin{bmatrix} \psi_1(\lambda,x,t) \\ \psi_2(\lambda,x,t) \end{bmatrix} = \frac{1}{|\Gamma_1|^2 + |c_1|^2\Gamma_2^2} \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix} \begin{bmatrix} P_5 \\ P_6 \end{bmatrix}$$

$$\Gamma_1 := 1 + ic_1 \left[\psi_1(\lambda_1, x)\, \dot{\psi}_2(\lambda_1, x) - \psi_2(\lambda_1, x)\, \dot{\psi}_1(\lambda_1, x)\right]$$

$$\Gamma_2 := \frac{|\psi_1(\lambda_1, x)|^2 + |\psi_2(\lambda_1, x)|^2}{2\,\mathrm{Im}[\lambda_1]}$$

$$P_0 := 2c_1\psi_1(\lambda_1, x)^2\Gamma_1^* - 2c_1^*[\psi_2(\lambda_1, x)^*]^2\Gamma_1 + 4|c_1|^2\psi_1(\lambda_1, x)\,\psi_2(\lambda_1, x)^*\,\Gamma_2$$

$$P_1 := -|c_1|^2\,\psi_2(\lambda_1, x)^*\,\Gamma_2 - c_1\psi_1(\lambda_1, x)\,\Gamma_1^*$$

$$P_2 := |c_1|^2\,\psi_1(\lambda_1, x)\,\Gamma_2(x) - c_1^*\,\psi_2(\lambda_1, x)^*\,\Gamma_1$$

$$P_3 := |c_1|^2\,\psi_1(\lambda_1, x)^*\,\Gamma_2(x) - c_1\psi_2(\lambda_1, x)\,\Gamma_1^*$$

$$P_4 := |c_1|^2\,\psi_2(\lambda_1, x)\,\Gamma_2(x) + c_1^*\,\psi_1(\lambda_1, x)^*\,\Gamma_1$$

$$P_5 := \frac{i}{\lambda - \lambda_1}\,[\psi_1(\lambda_1, x)\,\psi_2(\lambda, x) - \psi_2(\lambda_1, x)\,\psi_1(\lambda, x)]$$

$$P_6 := \frac{-i}{\lambda - \lambda_1^*}\,[\psi_1(\lambda_1, x)^*\,\psi_1(\lambda, x) + \psi_2(\lambda_1, x)^*\,\psi_2(\lambda, x)\,]$$

$$c_1 := c_1(t) = c_1(0)\,e^{4i\lambda_1^2 t}$$

$$\psi_1(\lambda_1, x) := \psi_1(\lambda_1, x, t), \qquad \psi_2(\lambda, x) := \psi_2(\lambda, x, t)$$

Darboux transformation for ZS system/NLS

• Add/remove N bound states at $\{\lambda_j\}_{j=1}^N$ with λ_j of multiplicity n_j and norming constants $c_{j1}, c_{j2}, \ldots, c_{jn_j}$

• determine $\tilde{u}(x,t) - u(x,t)$ and $\tilde{\psi}(\lambda,x,t) - \psi(\lambda,x,t)$

General approach to Darboux transformations

- unified approach
- applicable when Marchenko/GL methods are applicable
- multiple eigenvalues with multiplicities
- applicable to matrix versions of integrable equations
- no assumptions on extensions to both $\mathbf{C}^+$ and $\mathbf{C}^-$

General approach to Darboux transformations

- relate $\psi(\lambda, x, t)$ to $\alpha(x, y, t)$ via Fourier transform
- $\tilde{\psi}(\lambda, x, t)$ is related to $\tilde{\alpha}(x, y, t)$
- $\alpha(x, y, t)$ satisfies $\alpha + \omega + \alpha\Omega = 0$ (Marchenko/Gel'fand-Levitan eq)

$$\alpha(x, y, t) + \omega(x, y, t) + \int_x^\infty dz\, \alpha(x, z, t)\, \omega(z, y, t) = 0$$

- $\tilde{\alpha}(x, y, t)$ satisfies $\tilde{\alpha} + \tilde{\omega} + \tilde{\alpha}\tilde{\Omega} = 0$

$$\tilde{\Omega} - \Omega = FG, \qquad \tilde{\omega}(x, y, t) - \omega(x, y, t) = f(x, t)\, g(y, t)$$

- $\tilde{\alpha}[I + FG(I + R)] = \alpha - fg(I + R)$ equivalent separable eq

$I + R := (I + \Omega)^{-1}$ explicit evaluation in terms of α

- $\tilde{\alpha}(x, y, t) - \alpha(x, y, t)$ is obtained in terms of $\alpha(x, y, t)$, $f(x, t)$, $g(y, t)$
- $\tilde{\psi}(\lambda, x, t) - \psi(\lambda, x, t)$ is obtained from $\tilde{\alpha}(x, y, t) - \alpha(x, y, t)$
- $\tilde{u}(x, t) - u(x, t)$ is obtained from $\tilde{\alpha}(x, x, t) - \alpha(x, x, t)$

General approach to Darboux transformations

- $\alpha + \omega + \alpha\Omega = 0 \qquad \alpha = -\omega(I+\Omega)^{-1} = -\omega(I+R)$
- $R + \Omega + R\Omega = 0$

$$r(x;y,z) + \omega(y,z) + \int_x^\infty ds\, r(x;y,s)\,\omega(s,z) = 0, \quad x < \min\{y,z\}$$

$$r(x;y,z) = \begin{cases} \alpha(y,z) + \displaystyle\int_x^y ds\, J\,\alpha(s,y)^\dagger J\,\alpha(s,z), & x < y < z, \\ J\,\alpha(z,y)^\dagger J + \displaystyle\int_x^z ds\, J\,\alpha(s,y)^\dagger J\,\alpha(s,z), & x < z < y, \end{cases}$$

- $\tilde\alpha + \tilde\omega + \tilde\alpha\tilde\Omega = 0 \qquad \tilde\alpha + \omega + fg + \tilde\alpha(\Omega + FG) = 0$

$$[\tilde\alpha + \omega + fg + \tilde\alpha(\Omega + FG)](I+R) = 0$$

$$\tilde\alpha[I + FG(I+R)] = \alpha - fg(I+R) \qquad \text{separable integral eq}$$

$$\tilde\alpha - \alpha = -(f + \alpha F)[I + G(I+R)F]^{-1}G(I+R)$$

General approach applied to ZS/NLS

- $f(x,t) = \begin{bmatrix} 0 & B^\dagger e^{-A^\dagger x} \\ Ce^{-Ax-4iA^2t} & 0 \end{bmatrix}$

- $g(y,t) = \begin{bmatrix} e^{-Ay}B & 0 \\ 0 & -e^{-A^\dagger y+4i(A^\dagger)^2t}C^\dagger \end{bmatrix}$

- $\alpha(x,y,t) = [\,\bar{K}(x,y,t) \quad K(x,y,t)\,]$ $\qquad 2\times 2$ matrix

- $\omega(x,y,t) = \begin{bmatrix} 0 & -\Omega_l(x+y,t)^\dagger \\ \Omega_l(x+y,t) & 0 \end{bmatrix}$ $\qquad 2\times 2$ matrix

- $u(x,t) = -2\,[\,1 \quad 0\,]\,\alpha(x,x,t)\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

- $\begin{cases} \bar{K}(x,y,t) + \begin{bmatrix} 0 \\ \Omega_l(x+y,t) \end{bmatrix} + \displaystyle\int_x^\infty dz\, K(x,z,t)\,\Omega_l(z+y,t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ K(x,y,t) - \begin{bmatrix} \Omega_l(x+y,t)^\dagger \\ 0 \end{bmatrix} - \displaystyle\int_x^\infty dz\, \bar{K}(x,z,t)\,\Omega_l(z+y,t)^\dagger = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{cases}$

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