Orbifolds, the A, D, E Classification, and Gravitational Lensing

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April 2, 2010

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- We have already encountered a few of these: the fold (A_2) and the cusp (A_3).
- Higher-order examples include the "swallowtail" (A_4) and the "parabolic umbilic" (D_5).

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- In analogy with gravitational lensing, we call

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  the *magnification* of the pre-image $\mathbf{x}$.
- It was recently shown (Aazami & Petters 2009, 2010) that each such $\mathbf{f_c}$ satisfies a magnification relation of the form

$$\sum_{i=1}^{n} \mathfrak{M}(\mathbf{x}_i; \mathbf{s}) = 0,$$

  for any non-caustic target point $\mathbf{s}$.

- Take any $\mathbf{f_c}\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, with a given pre-image $\mathbf{x}_0 = (x_0, y_0)$ of a non-caustic target point $\mathbf{s} = (s_1, s_2)$.

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- STEP 1:

$$P_1(x,y) \equiv f_{\mathbf{c}}^{(1)}(x,y) - s_1 \qquad , \qquad P_2(x,y) \equiv f_{\mathbf{c}}^{(2)}(x,y) - s_2.$$

Note that

$$J(\mathbf{x}_0) \equiv \det\left[\begin{array}{cc} \partial_x P_1 & \partial_y P_1 \\ \partial_x P_2 & \partial_y P_2 \end{array}\right]_{(\mathbf{x}_0)} = \det(\mathrm{Jac}\,\mathbf{f_c})(\mathbf{x}_0) = \frac{1}{\mathfrak{M}(\mathbf{x}_0;\mathbf{s})}.$$

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- STEP 2: treat the pre-image coordinates $\mathbf{x} = (x, y)$ as complex variables, so that $\mathbf{x} \in \mathbb{C}^2$, and consider the following meromorphic two-form defined on $\mathbb{C}^2$:

$$\omega = \frac{dx\, dy}{P_1(x,y) P_2(x,y)}.$$

- At points where $J \neq 0$, it can be shown that the residue of ω is given by

$$\operatorname{Res}\omega = \frac{1}{J(x,y)} = \mathfrak{M}(\mathbf{x};\mathbf{s}).$$

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- STEP 3: using homogeneous coordinates $[X : Y : U]$, where $x = X/U$ and $y = Y/U$, extend the $P_i(x,y)$ to $\mathbb{CP}^2$:

$$\begin{aligned} P_1(X,Y,U)_{\text{hom}} &\equiv U^{d_1} f_{\mathbf{c}}^{(1)}(X/U, Y/U) - s_1 U^{d_1} \\ P_2(X,Y,U)_{\text{hom}} &\equiv U^{d_2} f_{\mathbf{c}}^{(2)}(X/U, Y/U) - s_2 U^{d_2}. \end{aligned}$$

Affine space corresponds to $U = 1$.

- STEP 4: extend ω to a form on $\mathbb{CP}^2$ that is homogeneous of degree zero:

$$
\begin{aligned}
\omega &= \frac{d(X/U)d(Y/U)}{P_1(X/U, Y/U)P_2(X/U, Y/U)} \\
&= \frac{U^{d_1+d_2-3}(UdXdY - XdUdY - YdXdU)}{P_1(X, Y, U)_{\text{hom}} P_2(X, Y, U)_{\text{hom}}}.
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  - $\mathfrak{M}_{\text{tot}}(\mathbf{s})$ is thus precisely equal to minus the sum of the residues of ω at infinity ($U = 0$).

- In conclusion, we arrive at the following:

Dalal & Rabin (2001)

The total signed magnification $\mathfrak{M}_{\text{tot}}(\mathbf{s})$ corresponding to a non-caustic target point $\mathbf{s}$ of a mapping $\mathbf{f_c}$ reflects the behavior of $\mathbf{f_c}$ at infinity when it is extended to $\mathbb{CP}^2$.

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- So what happens if a particular mapping $\mathbf{f_c}$ has images at infinity?
- Then ω has poles at infinity, so their residues must be calculated (in general, this is not easy!).

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- Answer: NO!

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- In affine space ($U = 1$), this is just $\mathbf{f_c}$.

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- Can we "get rid" of this pole at infinity?
- Answer: YES, but we need "weighted" projective space...

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The formal definition

Let X be a paracompact Hausdorff space.

An *n-dimensional orbifold chart* is a connected open subset $\widetilde{U} \subset \mathbb{R}^n$ and a continuous mapping $\phi\colon \widetilde{U} \longrightarrow \phi(\widetilde{U}) \equiv U \subset X$, together with a finite group G of diffeomorphisms of $\widetilde{U}$ such that ϕ is G-invariant ($\phi \circ g = \phi$ for all $g \in G$) and induces a homeomorphism $\widetilde{U}/G \cong U$.

There is a compatibility condition that two overlapping orbifolds charts will satisfy (details omitted).

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- The orbifold we'll be interested in is a space that is a generalization of $\mathbb{CP}^n$...

$\mathbb{CP}^n$ as a Lie group action

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  - it is "proper": i.e., the map

$$\begin{array}{ccc} \mathbb{S}^1 \times \mathbb{S}^{2n+1} & \longrightarrow & \mathbb{S}^{2n+1} \times \mathbb{S}^{2n+1} \\ (z, (w_0, \ldots, w_n)) & \longmapsto & ((zw_0, \ldots, zw_n), (w_0, \ldots, w_n)) \end{array}$$

    is proper (pre-images of compact sets are compact).

- These three conditions guarantee that the quotient space

$$\mathbb{S}^{2n+1}/\mathbb{S}^1$$

is a smooth manifold, which is none other than $\mathbb{CP}^n$ (to see this, just restrict the domain of the usual quotient map $\pi\colon \mathbb{C}^{n+1}\backslash\{0\} \longrightarrow \mathbb{CP}^n$ to $\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}\backslash\{0\}$.)

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- The importance of this alternative definition of $\mathbb{CP}^n$ is that it can be generalized...

. . . Consider now the following Lie group action:

$$\begin{aligned} \mathbb{S}^1 \times \mathbb{S}^{2n+1} &\longrightarrow \mathbb{S}^{2n+1} \\ (z, (w_0, \ldots, w_n)) &\longmapsto (z^{a_0} w_0, \ldots, z^{a_n} w_n), \end{aligned}$$

where each $a_i \in \mathbb{Z}_+$ (they are usually coprime).

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where each $a_i \in \mathbb{Z}_+$ (they are usually coprime).

- This action is still smooth.
- It is still proper.
- But it is *not* free: e.g.,

$$(0,\ldots,z^{a_i}w_i,\ldots,0) = (0,\ldots,w_i,\ldots,0)$$

for any a_i^{th} root of unity, not just 1.

- Rather this action is *almost free:* the stabilizer group

$$\{z \in \mathbb{S}^1 : (z^{a_0} w_0, \ldots, z^{a_n} w_n) = (w_0, \ldots, w_n)\} \subset \mathbb{S}^1$$

is not trivial for every $(w_0, \ldots, w_n) \in \mathbb{S}^{2n+1}$, but it is always *finite*.

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  is not trivial for every $(w_0, \dots, w_n) \in \mathbb{S}^{2n+1}$, but it is always *finite*.
- Denote the resulting quotient space by $\mathbb{WP}(a_0, \dots, a_n)$. Since it's not a smooth manifold, is it an orbifold?

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Let $G \times M \longrightarrow M$ be a smooth action of a compact Lie group G on a smooth manifold M. If the action is effective and almost free, then the quotient space M/G will be an orbifold.

- The orbifold $\mathbb{WP}(a_0, \dots, a_n)$ is called *weighted projective space*.

- Let's use this machinery: consider the weighted projective space $\mathbb{WP}(3, 2, 1)$.

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- CONCLUSION: the homogeneous coordinates X and Y now have "weights" 3 and 2, respectively.
- Their relation to the usual coordinates x, y are given by

$$x = \frac{X}{U^3} \quad , \quad y = \frac{Y}{U^2}.$$

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Compare this with the extension to $\mathbb{CP}^2$

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- CONCLUSION: there are NO poles at infinity, hence no residues at infinity (i.e., we "got rid" of the pole at infinity).
- Also, there are NO singular points in affine space, because U has weight 1.

- So by the Global Residue Theorem (for compact orbifolds), the parabolic umbilic satisfies

$$\mathfrak{M}_{\mathrm{tot}}(\mathbf{s}) = \sum_{i=1}^{5} \mathfrak{M}_i = 0 \tag{1}$$

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- What are the advantages to this approach?
  - NO residues to calculate (the answer is immediate),
  - an understanding that $\mathbb{CP}^2$ is not the only space in which to work,
  - therefore, an explanation of such magnification relations: "eqn. (1) is really saying that *in the appropriate space,* there are no images at infinity."

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