Orbifolds, the $A, D, E$ Classification, and Gravitational Lensing

Amir B. Aazami

Duke University

April 2, 2010
V. Arnold (1972) classified simple degenerate critical points, using the fact that a smooth function can be put in a polynomial form similar to the Morsian normal form near a nondegenerate critical point.
V. Arnold (1972) classified simple degenerate critical points, using the fact that a smooth function can be put in a polynomial form similar to the Morsian normal form near a nondegenerate critical point.

In doing so, he obtained a correspondence with the Coxeter-Dynkin diagrams of type $A_n \ (n \geq 2),\ D_n \ (n \geq 4),\ E_6,\ E_7,\ E_8$. 
V. Arnold (1972) classified simple degenerate critical points, using the fact that a smooth function can be put in a polynomial form similar to the Morsian normal form near a nondegenerate critical point.

In doing so, he obtained a correspondence with the Coxeter-Dynkin diagrams of type $A_n \ (n \geq 2)$, $D_n \ (n \geq 4)$, $E_6$, $E_7$, $E_8$.

We have already encountered a few of these: the fold ($A_2$) and the cusp ($A_3$).
V. Arnold (1972) classified simple degenerate critical points, using the fact that a smooth function can be put in a polynomial form similar to the Morsian normal form near a nondegenerate critical point.

In doing so, he obtained a correspondence with the Coxeter-Dynkin diagrams of type $A_n$ ($n \geq 2$), $D_n$ ($n \geq 4$), $E_6$, $E_7$, $E_8$.

We have already encountered a few of these: the fold ($A_2$) and the cusp ($A_3$).

Higher-order examples include the “swallowtail” ($A_4$) and the “parabolic umbilic” ($D_5$).
To each such singularity is associated a mapping $f_c : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ (analogous to the lensing map $\eta_c$).
To each such singularity is associated a mapping \( f_c : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) (analogous to the lensing map \( \eta_c \)).

Some terminology: for \( f_c(x) = s \), call \( x \in \mathbb{R}^2 \) the \textit{pre-image} of the \textit{target point} \( s \in \mathbb{R}^2 \).
To each such singularity is associated a mapping $f_c : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ (analogous to the lensing map $\eta_c$).

Some terminology: for $f_c(x) = s$, call $x \in \mathbb{R}^2$ the pre-image of the target point $s \in \mathbb{R}^2$.

In analogy with gravitational lensing, we call

$$M(x; s) \equiv \frac{1}{\det(Jac f_c)(x)}$$

the magnification of the pre-image $x$. 
To each such singularity is associated a mapping $f_c : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ (analogous to the lensing map $\eta_c$).

Some terminology: for $f_c(x) = s$, call $x \in \mathbb{R}^2$ the pre-image of the target point $s \in \mathbb{R}^2$.

In analogy with gravitational lensing, we call

$$M(x; s) \equiv \frac{1}{\det(\text{Jac} f_c)(x)}$$

the magnification of the pre-image $x$.

It was recently shown (Aazami & Petters 2009, 2010) that each such $f_c$ satisfies a magnification relation of the form

$$\sum_{i=1}^{n} M(x_i; s) = 0,$$

for any non-caustic target point $s$. 
Take any $f_c: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, with a given pre-image $x_0 = (x_0, y_0)$ of a non-caustic target point $s = (s_1, s_2)$. 
Take any $f_c: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, with a given pre-image $x_0 = (x_0, y_0)$ of a non-caustic target point $s = (s_1, s_2)$.

Let $f_c^{(1)}$ and $f_c^{(2)}$ denote the two components of $f_c$, with degrees $d_1$ and $d_2$, respectively.
Take any \( f_c : \mathbb{R}^2 \to \mathbb{R}^2 \), with a given pre-image \( x_0 = (x_0, y_0) \) of a non-caustic target point \( s = (s_1, s_2) \).

Let \( f_c^{(1)} \) and \( f_c^{(2)} \) denote the two components of \( f_c \), with degrees \( d_1 \) and \( d_2 \), respectively.

**STEP 1:**

\[
P_1(x, y) \equiv f_c^{(1)}(x, y) - s_1, \quad P_2(x, y) \equiv f_c^{(2)}(x, y) - s_2.
\]

Note that

\[
J(x_0) \equiv \det \begin{bmatrix} \frac{\partial}{\partial x} P_1 & \frac{\partial}{\partial y} P_1 \\ \frac{\partial}{\partial x} P_2 & \frac{\partial}{\partial y} P_2 \end{bmatrix}(x_0) = \det(\text{Jac } f_c)(x_0) = \frac{1}{\mathcal{M}(x_0; s)}.
\]
Take any $f_c : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, with a given pre-image $x_0 = (x_0, y_0)$ of a non-caustic target point $s = (s_1, s_2)$.

Let $f_c^{(1)}$ and $f_c^{(2)}$ denote the two components of $f_c$, with degrees $d_1$ and $d_2$, respectively.

**STEP 1:**

$$P_1(x, y) \equiv f_c^{(1)}(x, y) - s_1, \quad P_2(x, y) \equiv f_c^{(2)}(x, y) - s_2.$$ 

Note that

$$J(x_0) \equiv \det \begin{bmatrix} \frac{\partial}{\partial x} P_1 & \frac{\partial}{\partial y} P_1 \\ \frac{\partial}{\partial x} P_2 & \frac{\partial}{\partial y} P_2 \end{bmatrix}_{(x_0)} = \det(\text{Jac } f_c)(x_0) = \frac{1}{M(x_0; s)}.$$ 

**STEP 2:** treat the pre-image coordinates $x = (x, y)$ as complex variables, so that $x \in \mathbb{C}^2$, and consider the following meromorphic two-form defined on $\mathbb{C}^2$:

$$\omega = \frac{dx \, dy}{P_1(x, y)P_2(x, y)}.$$
At points where $J \neq 0$, it can be shown that the residue of $\omega$ is given by
\[
\text{Res} \omega = \frac{1}{J(x, y)} = M(x; s).
\]
Thus we have expressed the magnification $M(x; s)$ as the residue of a meromorphic two-form defined on $\mathbb{C}^2$. 
At points where \( J \neq 0 \), it can be shown that the residue of \( \omega \) is given by
\[
\text{Res} \omega = \frac{1}{J(x, y)} = M(x; s).
\]

Thus we have expressed the magnification \( M(x; s) \) as the residue of a meromorphic two-form defined on \( \mathbb{C}^2 \).

STEP 3: using homogeneous coordinates \([X : Y : U]\), where \( x = X/U \) and \( y = Y/U \), extend the \( P_i(x, y) \) to \( \mathbb{CP}^2 \):

\[
P_1(X, Y, U)_\text{hom} \equiv U^{d_1} f_c^{(1)}(X/U, Y/U) - s_1 U^{d_1}
\]
\[
P_2(X, Y, U)_\text{hom} \equiv U^{d_2} f_c^{(2)}(X/U, Y/U) - s_2 U^{d_2}.
\]

Affine space corresponds to \( U = 1 \).
STEP 4: extend $\omega$ to a form on $\mathbb{CP}^2$ that is homogeneous of degree zero:

$$
\omega = \frac{d(X/U)d(Y/U)}{P_1(X/U, Y/U)P_2(X/U, Y/U)}
U^{d_1+d_2-3}(UdXdY - XdUdY - YdXdU)
= \frac{P_1(X, Y, U)_{\text{hom}}P_2(X, Y, U)_{\text{hom}}}{P_1(X, Y, U)_{\text{hom}}P_2(X, Y, U)_{\text{hom}}}.
$$
STEP 4: extend $\omega$ to a form on $\mathbb{CP}^2$ that is homogeneous of degree zero:

$$
\omega = \frac{d(X/U)d(Y/U)}{P_1(X/U, Y/U)P_2(X/U, Y/U)}
- \frac{U^{d_1+d_2-3}(UdXdY - XdUdY - YdXdU)}{P_1(X, Y, U)_{\text{hom}}P_2(X, Y, U)_{\text{hom}}}.
$$

STEP 5: the Global Residue Theorem states that the sum of the residues of $\omega$, on $\mathbb{CP}^2$, is identically zero.
STEP 4: extend $\omega$ to a form on $\mathbb{CP}^2$ that is homogeneous of degree zero:

$$
\omega = \frac{d(X/U)d(Y/U)}{P_1(X/U, Y/U)P_2(X/U, Y/U)}
U^{d_1+d_2-3}(UdXdY - XdUdY - YdXdU)
= \frac{P_1(X, Y, U)_{\text{hom}}P_2(X, Y, U)_{\text{hom}}}{P_1(X, Y, U)_{\text{hom}}P_2(X, Y, U)_{\text{hom}}}.
$$

STEP 5: the Global Residue Theorem states that the sum of the residues of $\omega$, on $\mathbb{CP}^2$, is identically zero.

- all the poles of $\omega$ in affine space correspond to pre-images of $f_c$ and vice-versa, so the sum of their residues is the total signed magnification $M_{\text{tot}}(s)$. 
STEP 4: extend $\omega$ to a form on $\mathbb{CP}^2$ that is homogeneous of degree zero:

$$\omega = \frac{d(X/U)d(Y/U)}{P_1(X/U, Y/U)P_2(X/U, Y/U)}U^{d_1+d_2-3}(UdXdY - XdUdY - YdXdU)P_1(X, Y, U)_{\text{hom}}P_2(X, Y, U)_{\text{hom}}.$$

STEP 5: the Global Residue Theorem states that the sum of the residues of $\omega$, on $\mathbb{CP}^2$, is identically zero.

- all the poles of $\omega$ in affine space correspond to pre-images of $f_c$ and vice-versa, so the sum of their residues is the total signed magnification $M_{\text{tot}}(s)$.
- $M_{\text{tot}}(s)$ is thus precisely equal to minus the sum of the residues of $\omega$ at infinity ($U = 0$).
In conclusion, we arrive at the following:

**Dalal & Rabin (2001)**

The total signed magnification $M_{\text{tot}}(s)$ corresponding to a non-caustic target point $s$ of a mapping $f_c$ reflects the behavior of $f_c$ at infinity when it is extended to $\mathbb{C}P^2$. 

---

Amir B. Aazami
Orbifolds, the $A$, $D$, $E$ Classification, and Gravitational Lensing
In conclusion, we arrive at the following:

**Dalal & Rabin (2001)**

The total signed magnification $\mathcal{M}_{\text{tot}}(s)$ corresponding to a non-caustic target point $s$ of a mapping $f_c$ reflects the behavior of $f_c$ at infinity when it is extended to $\mathbb{CP}^2$.

**So what happens if a particular mapping $f_c$ has images at infinity?**
In conclusion, we arrive at the following:

**Dalal & Rabin (2001)**

The total signed magnification $M_{\text{tot}}(s)$ corresponding to a non-caustic target point $s$ of a mapping $f_c$ reflects the behavior of $f_c$ at infinity when it is extended to $\mathbb{CP}^2$.

- So what happens if a particular mapping $f_c$ has images at infinity?
- Then $\omega$ has poles at infinity, so their residues must be calculated (in general, this is not easy!).
Pick a mapping $f_c : \mathbb{R}^2 \rightarrow \mathbb{R}^2$. 
Pick a mapping $f_c : \mathbb{R}^2 \to \mathbb{R}^2$.

Let $s = (s_1, s_2)$ be a non-caustic target point.
Pick a mapping $f_c : \mathbb{R}^2 \rightarrow \mathbb{R}^2$.
Let $s = (s_1, s_2)$ be a non-caustic target point.
We know it satisfies:

$$M_{\text{tot}}(s) = \sum_{i=1}^{n} M(x_i; s) = 0.$$
Pick a mapping $f_c: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$.

Let $s = (s_1, s_2)$ be a non-caustic target point.

We know it satisfies:

$$M_{\text{tot}}(s) = \sum_{i=1}^{n} M(x_i; s) = 0.$$

Because r.h.s. is identically zero, we want there to be NO images at infinity.
Pick a mapping \( f_c : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \).

Let \( s = (s_1, s_2) \) be a non-caustic target point.

We know it satisfies:

\[
M_{\text{tot}}(s) = \sum_{i=1}^{n} M(x_i; s) = 0.
\]

Because r.h.s. is identically zero, we want there to be NO images at infinity.

Is this always the case when we extend to \( \mathbb{CP}^2 \)?
Pick a mapping \( f_c : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \).

Let \( \mathbf{s} = (s_1, s_2) \) be a non-caustic target point.

We know it satisfies:

\[
M_{\text{tot}}(\mathbf{s}) = \sum_{i=1}^{n} M(x_i; \mathbf{s}) = 0.
\]

Because r.h.s. is identically zero, we want there to be NO images at infinity.

Is this always the case when we extend to \( \mathbb{CP}^2 \)?

Answer: NO!
Take, for example, the “parabolic umbilic” singularity ($D_5$)…
Take, for example, the “parabolic umbilic” singularity \((D_5)\)…

Its induced mapping is

\[
\mathbf{f}_c(x, y) = (2xy, x^2 \pm 4y^3 + 3c_3y^2 + 2c_2y) = (s_1, s_2)
\]
Take, for example, the “parabolic umbilic” singularity ($D_5$)...

Its induced mapping is

$$f_c(x, y) = (2xy, x^2 \pm 4y^3 + 3c_3y^2 + 2c_2y) = (s_1, s_2)$$

Now extend $f_c$ to $\mathbb{CP}^2$

$$\begin{cases} 2XY - s_1U^2 \\ X^2U \pm 4Y^3 + 3c_3Y^2U + 2c_2YU^2 - s_2U^3. \end{cases}$$
Take, for example, the “parabolic umbilic” singularity \((D_5)\)...

Its induced mapping is

\[
f_c(x, y) = (2xy, x^2 \pm 4y^3 + 3c_3y^2 + 2c_2y) = (s_1, s_2)
\]

Now extend \(f_c\) to \(\mathbb{CP}^2\)

\[
\begin{align*}
2XY - s_1U^2 \\
X^2U \pm 4Y^3 + 3c_3Y^2U + 2c_2YU^2 - s_2U^3.
\end{align*}
\]

In affine space \((U = 1)\), this is just \(f_c\).
At infinity \((U = 0)\), these equations reduce to

\[
\begin{cases}
2XY \\
\pm 4Y^3.
\end{cases}
\]
At infinity \((U = 0)\), these equations reduce to

\[
\left\{
\begin{array}{l}
2XY \\
\pm 4Y^3.
\end{array}
\right.
\]

These have a nonzero common root: \([1 : 0 : 0]\).
At infinity \((U = 0)\), these equations reduce to
\[
\begin{align*}
2XY \\
\pm 4Y^3.
\end{align*}
\]

- These have a nonzero common root: \([1 : 0 : 0]\).
- CONCLUSION: the total signed magnification is equal to minus the residue of \(\omega\) at this point.
At infinity \((U = 0)\), these equations reduce to

\[
\begin{cases}
2XY \\
\pm 4Y^3.
\end{cases}
\]

These have a nonzero common root: \([1 : 0 : 0]\).

CONCLUSION: the total signed magnification is equal to minus the residue of \(\omega\) at this point.

Can we “get rid” of this pole at infinity?
At infinity \((U = 0)\), these equations reduce to

\[
\begin{cases}
2XY \\
\pm 4Y^3.
\end{cases}
\]

- These have a nonzero common root: \([1 : 0 : 0]\).
- CONCLUSION: the total signed magnification is equal to minus the residue of \(\omega\) at this point.
- Can we “get rid” of this pole at infinity?
- Answer: YES, but we need “weighted” projective space...
The rough idea... 

Whereas a manifold locally looks like an open subset of $\mathbb{R}^n$, an orbifold locally looks like the quotient of an open subset of $\mathbb{R}^n$ by a finite group action.
The rough idea...

Whereas a manifold locally looks like an open subset of $\mathbb{R}^n$, an orbifold locally looks like the quotient of an open subset of $\mathbb{R}^n$ by a finite group action.

The formal definition

Let $X$ be a paracompact Hausdorff space.

An *n-dimensional orbifold chart* is a connected open subset $\tilde{U} \subset \mathbb{R}^n$ and a continuous mapping $\phi: \tilde{U} \rightarrow \phi(\tilde{U}) \equiv U \subset X$, together with a finite group $G$ of diffeomorphisms of $\tilde{U}$ such that $\phi$ is $G$-invariant ($\phi \circ g = \phi$ for all $g \in G$) and induces a homeomorphism $\tilde{U}/G \cong U$.

There is a compatibility condition that two overlapping orbifolds charts will satisfy (details omitted).
For any $x \in X$, pick an orbifold chart $(\tilde{U}, G, \phi)$ containing it and pick a point $y$ in the fiber $\phi^{-1}(x) \subset \tilde{U} \subset \mathbb{R}^n$. 
For any \( x \in X \), pick an orbifold chart \((\tilde{U}, G, \phi)\) containing it and pick a point \( y \) in the fiber \( \phi^{-1}(x) \subset \tilde{U} \subset \mathbb{R}^n \).

Define the local group of \( x \) to be

\[ G_x = \{ g \in G : g(y) = y \}. \]
For any \( x \in X \), pick an orbifold chart \((\tilde{U}, G, \phi)\) containing it and pick a point \( y \) in the fiber \( \phi^{-1}(x) \subset \tilde{U} \subset \mathbb{R}^n \).

Define the *local group of* \( x \) to be

\[
G_x = \{ g \in G : g(y) = y \}.
\]

\( G_x \) is uniquely determined up to conjugacy (proof not easy).
For any $x \in X$, pick an orbifold chart $(\tilde{U}, G, \phi)$ containing it and pick a point $y$ in the fiber $\phi^{-1}(x) \subset \tilde{U} \subset \mathbb{R}^n$.

Define the local group of $x$ to be

$$G_x = \{g \in G : g(y) = y\}.$$

$G_x$ is uniquely determined up to conjugacy (proof not easy).

If $G_x \neq 1$, then $x$ is singular. If $X$ has no singular points, then the local actions are all free, so $X$ is a smooth manifold.
For any $x \in X$, pick an orbifold chart $(\tilde{U}, G, \phi)$ containing it and pick a point $y$ in the fiber $\phi^{-1}(x) \subset \tilde{U} \subset \mathbb{R}^n$.

Define the local group of $x$ to be

$$G_x = \{g \in G : g(y) = y\}.$$

$G_x$ is uniquely determined up to conjugacy (proof not easy).

If $G_x \neq 1$, then $x$ is singular. If $X$ has no singular points, then the local actions are all free, so $X$ is a smooth manifold.

Singular points will play an important role for us below: namely, we want to make sure to avoid them!
For any $x \in X$, pick an orbifold chart $(\tilde{U}, G, \phi)$ containing it and pick a point $y$ in the fiber $\phi^{-1}(x) \subset \tilde{U} \subset \mathbb{R}^n$.

Define the \textit{local group of} $x$ to be

$$G_x = \{g \in G : g(y) = y\}.$$ 

$G_x$ is uniquely determined up to conjugacy (proof not easy).

If $G_x \neq 1$, then $x$ is \textit{singular}. If $X$ has no singular points, then the local actions are all free, so $X$ is a smooth manifold.

Singular points will play an important role for us below: namely, we want to make sure to \textit{avoid} them!

The orbifold we’ll be interested in is a space that is a generalization of $\mathbb{C}P^n$...
$\mathbb{CP}^n$ as a Lie group action

$$\mathbb{S}^1 \times \mathbb{S}^{2n+1} \rightarrow \mathbb{S}^{2n+1}$$

$$(z, (w_0, \ldots, w_n)) \mapsto (zw_0, \ldots, zw_n)$$
\( \mathbb{C}P^n \) as a Lie group action

\[ S^1 \times S^{2n+1} \longrightarrow S^{2n+1} \]
\[ (z, (w_0, \ldots, w_n)) \longmapsto (zw_0, \ldots, zw_n) \]

- \( S^1 \subset \mathbb{C}, \ S^{2n+1} \subset \mathbb{C}^{n+1} \), so \( z, w_i \in \mathbb{C} \)
- This action satisfies the following properties:
\(\mathbb{CP}^n\) as a Lie group action

\[
S^1 \times S^{2n+1} \longrightarrow S^{2n+1}
\]

\[
(z, (w_0, \ldots, w_n)) \longmapsto (zw_0, \ldots, zw_n)
\]

- \(S^1 \subset \mathbb{C}, S^{2n+1} \subset \mathbb{C}^{n+1}\), so \(z, w_i \in \mathbb{C}\)
- This action satisfies the following properties:
  - it is smooth
$\mathbb{CP}^n$ as a Lie group action

$$\mathbb{S}^1 \times \mathbb{S}^{2n+1} \longrightarrow \mathbb{S}^{2n+1}$$

$$(z, (w_0, \ldots, w_n)) \longmapsto (zw_0, \ldots, zw_n)$$

- $\mathbb{S}^1 \subset \mathbb{C}$, $\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$, so $z, w_i \in \mathbb{C}$
- This action satisfies the following properties:
  - it is smooth
  - it is free: $(zw_0, \ldots, zw_n) = (w_0, \ldots, w_n) \iff z = 1 \in \mathbb{S}^1$
$\mathbb{CP}^n$ as a Lie group action

$$\mathbb{S}^1 \times \mathbb{S}^{2n+1} \longrightarrow \mathbb{S}^{2n+1}$$

$$(z, (w_0, \ldots, w_n)) \longmapsto (zw_0, \ldots, zw_n)$$

- $\mathbb{S}^1 \subset \mathbb{C}$, $\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$, so $z, w_i \in \mathbb{C}$
- This action satisfies the following properties:
  - it is smooth
  - it is free: $(zw_0, \ldots, zw_n) = (w_0, \ldots, w_n) \iff z = 1 \in \mathbb{S}^1$
  - it is “proper”: i.e., the map

$$\mathbb{S}^1 \times \mathbb{S}^{2n+1} \longrightarrow \mathbb{S}^{2n+1} \times \mathbb{S}^{2n+1}$$

$$(z, (w_0, \ldots, w_n)) \longmapsto ((zw_0, \ldots, zw_n), (w_0, \ldots, w_n))$$

is proper (pre-images of compact sets are compact).
These three conditions guarantee that the quotient space
\[ S^{2n+1}/S^1 \]
is a smooth manifold, which is none other than \( \mathbb{CP}^n \) (to see this, just restrict the domain of the usual quotient map
\[ \pi: \mathbb{C}^{n+1}\setminus\{0\} \rightarrow \mathbb{CP}^n \text{ to } S^{2n+1} \subset \mathbb{C}^{n+1}\setminus\{0\}. \)
These three conditions guarantee that the quotient space 

$$\mathbb{S}^{2n+1}/\mathbb{S}^1$$

is a smooth manifold, which is none other than \(\mathbb{C}\mathbb{P}^n\) (to see this, just restrict the domain of the usual quotient map 

\(\pi: \mathbb{C}^{n+1}\setminus\{0\} \rightarrow \mathbb{C}\mathbb{P}^n \) to \(\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}\setminus\{0\} \).)

The importance of this alternative definition of \(\mathbb{C}\mathbb{P}^n\) is that it can be generalized...
Consider now the following Lie group action:

\[ S^1 \times S^{2n+1} \longrightarrow S^{2n+1} \]

\[ (z, (w_0, \ldots, w_n)) \longrightarrow (z^{a_0} w_0, \ldots, z^{a_n} w_n), \]

where each \( a_i \in \mathbb{Z}_+ \) (they are usually coprime).
Consider now the following Lie group action:

\[ S^1 \times S^{2n+1} \longrightarrow S^{2n+1} \]

\[ (z, (w_0, \ldots, w_n)) \longmapsto (z^{a_0} w_0, \ldots, z^{a_n} w_n), \]

where each \( a_i \in \mathbb{Z}_+ \) (they are usually coprime).

- This action is still smooth.
Consider now the following Lie group action:

\[ S^1 \times S^{2n+1} \longrightarrow S^{2n+1} \]
\[ (z, (w_0, \ldots, w_n)) \longmapsto (z^{a_0} w_0, \ldots, z^{a_n} w_n), \]

where each \( a_i \in \mathbb{Z}_+ \) (they are usually coprime).

- This action is still smooth.
- It is still proper.
Consider now the following Lie group action:

$$\mathbb{S}^1 \times \mathbb{S}^{2n+1} \rightarrow \mathbb{S}^{2n+1}$$

$$(z, (w_0, \ldots, w_n)) \mapsto (z^{a_0} w_0, \ldots, z^{a_n} w_n),$$

where each $a_i \in \mathbb{Z}_+$ (they are usually coprime).

- This action is still smooth.
- It is still proper.
- But it is *not* free: e.g.,

  $$(0, \ldots, z^{a_i} w_i, \ldots, 0) = (0, \ldots, w_i, \ldots, 0)$$

  for any $a_i^{th}$ root of unity, not just 1.
Rather this action is *almost free*: the stabilizer group

\[ \{ z \in S^1 : (z^{a_0} w_0, \ldots, z^{a_n} w_n) = (w_0, \ldots, w_n) \} \subset S^1 \]

is not trivial for every \((w_0, \ldots, w_n) \in S^{2n+1}\), but it is always finite.
Rather this action is *almost free*: the stabilizer group

\[ \{ z \in S^1 : (z^{a_0} w_0, \ldots, z^{a_n} w_n) = (w_0, \ldots, w_n) \} \subset S^1 \]

is not trivial for every \((w_0, \ldots, w_n) \in S^{2n+1}\), but it is always *finite*.

Denote the resulting quotient space by \( \mathbb{W}P(a_0, \ldots, a_n) \). Since it's not a smooth manifold, is it an orbifold?
Rather this action is *almost free*: the stabilizer group

\[ \{ z \in S^1 : (z^{a_0} w_0, \ldots, z^{a_n} w_n) = (w_0, \ldots, w_n) \} \subset S^1 \]

is not trivial for every \((w_0, \ldots, w_n) \in S^{2n+1}\), but it is always *finite*.

Denote the resulting quotient space by \( \mathbb{W}P(a_0, \ldots, a_n) \). Since it’s not a smooth manifold, is it an orbifold?

**Orbifolds as Quotients of Manifolds by Lie Groups**

Let \( G \times M \longrightarrow M \) be a smooth action of a compact Lie group \( G \) on a smooth manifold \( M \). If the action is effective and almost free, then the quotient space \( M/G \) will be an orbifold.
Rather this action is *almost free*: the stabilizer group

\[ \{ z \in S^1 : (z^{a_0} w_0, \ldots, z^{a_n} w_n) = (w_0, \ldots, w_n) \} \subset S^1 \]

is not trivial for every \((w_0, \ldots, w_n) \in S^{2n+1}\), but it is always finite.

Denote the resulting quotient space by \(\mathbb{WP}(a_0, \ldots, a_n)\). Since it's not a smooth manifold, is it an orbifold?

**Orbifolds as Quotients of Manifolds by Lie Groups**

Let \(G \times M \longrightarrow M\) be a smooth action of a compact Lie group \(G\) on a smooth manifold \(M\). If the action is effective and almost free, then the quotient space \(M/G\) will be an orbifold.

The orbifold \(\mathbb{WP}(a_0, \ldots, a_n)\) is called *weighted projective space*. 
Let’s use this machinery: consider the weighted projective space $\mathbb{WP}(3, 2, 1)$. 
Let’s use this machinery: consider the weighted projective space $\mathbb{WP}(3, 2, 1)$.

Recall: this is the action

$$\mathbb{S}^1 \times \mathbb{S}^5 \longrightarrow \mathbb{S}^5$$

$$(z, (X, Y, U)) \longmapsto (z^3 X, z^2 Y, zU).$$

So $\mathbb{WP}(3, 2, 1) = \mathbb{S}^5 / \mathbb{S}^1$. 
Let's use this machinery: consider the weighted projective space $\mathbb{WP}(3, 2, 1)$.

Recall: this is the action

$$
\mathbb{S}^1 \times \mathbb{S}^5 \longrightarrow \mathbb{S}^5 \\
(z, (X, Y, U)) \longmapsto (z^3 X, z^2 Y, zU).
$$

So $\mathbb{WP}(3, 2, 1) = \mathbb{S}^5 / \mathbb{S}^1$.

CONCLUSION: the homogeneous coordinates $X$ and $Y$ now have “weights” 3 and 2, respectively.
Let’s use this machinery: consider the weighted projective space $\mathbb{WP}(3, 2, 1)$.

Recall: this is the action

$$\mathbb{S}^1 \times \mathbb{S}^5 \longrightarrow \mathbb{S}^5$$

$$(z, (X, Y, U)) \longmapsto (z^3 X, z^2 Y, zU).$$

So $\mathbb{WP}(3, 2, 1) = \mathbb{S}^5 / \mathbb{S}^1$.

CONCLUSION: the homogeneous coordinates $X$ and $Y$ now have “weights” 3 and 2, respectively.

Their relation to the usual coordinates $x, y$ are given by

$$x = \frac{X}{U^3}, \quad y = \frac{Y}{U^2}.$$
Let’s go back to the parabolic umbilic \((D_5)\)…
Let’s go back to the parabolic umbilic ($D_5$)…

Recall that its induced mapping is

$$f_c(x, y) = (2xy, x^2 \pm 4y^3 + 3c_3y^2 + 2c_2y) = (s_1, s_2)$$
Let’s go back to the parabolic umbilic ($D_5$)…

Recall that its induced mapping is

$$f_c(x, y) = (2xy, x^2 \pm 4y^3 + 3c_3y^2 + 2c_2y) = (s_1, s_2)$$

This times let’s extend $f_c$ to $\mathbb{WP}(3, 2, 1)$

$$\begin{cases} 
2XY - s_1 U^5 \\
X^2 \pm 4Y^3 + 3c_3 Y^2 U^2 + 2c_2 YU^4 - s_2 U^6.
\end{cases}$$
Let’s go back to the parabolic umbilic ($D_5$)...

Recall that its induced mapping is

\[ f_c(x, y) = (2xy, x^2 \pm 4y^3 + 3c_3y^2 + 2c_2y) = (s_1, s_2) \]

This times let’s extend $f_c$ to $\mathbb{WP}(3, 2, 1)$

\[
\begin{cases}
2XY - s_1U^5 \\
X^2 \pm 4Y^3 + 3c_3Y^2U^2 + 2c_2YU^4 - s_2U^6.
\end{cases}
\]

Compare this with the extension to $\mathbb{CP}^2$

\[
\begin{cases}
2XY - s_1U^2 \\
X^2U \pm 4Y^3 + 3c_3Y^2U + 2c_2YU^2 - s_2U^3.
\end{cases}
\]
Once again in affine space \((U = 1)\), this is just \(f_c\).
Once again in affine space ($U = 1$), this is just $f_c$.

At infinity ($U = 0$), however, we now have

$$\begin{cases} 
2XY \\
X^2 \pm 4Y^3.
\end{cases}$$
Once again in affine space ($U = 1$), this is just $f_c$.

At infinity ($U = 0$), however, we now have

$$\begin{cases} 
2XY \\
X^2 \pm 4Y^3.
\end{cases}$$

The only common root is $[0 : 0 : 0]$, which is not a point in $\mathbb{W}P(3, 2, 1)$. 
Once again in affine space \((U = 1)\), this is just \(f_c\).

At infinity \((U = 0)\), however, we now have

\[
\begin{align*}
2XY \\
X^2 \pm 4Y^3.
\end{align*}
\]

The only common root is \([0 : 0 : 0]\), which is not a point in \(\mathbb{WP}(3, 2, 1)\).

CONCLUSION: there are NO poles at infinity, hence no residues at infinity (i.e., we “got rid” of the pole at infinity).
Once again in affine space \((U = 1)\), this is just \(f_c\).

At infinity \((U = 0)\), however, we now have

\[
\begin{align*}
2XY \\
X^2 \pm 4Y^3.
\end{align*}
\]

The only common root is \([0 : 0 : 0]\), which is not a point in \(\mathbb{P}(3, 2, 1)\).

CONCLUSION: there are NO poles at infinity, hence no residues at infinity (i.e., we “got rid” of the pole at infinity).

Also, there are NO singular points in affine space, because \(U\) has weight 1.
So by the Global Residue Theorem (for compact orbifolds), the parabolic umbilic satisfies

$$M_{\text{tot}}(s) = \sum_{i=1}^{5} M_i = 0$$

(1)

for any non-caustic target point $s = (s_1, s_2)$. 

Amir B. Aazami

Orbifolds, the A, D, E Classification, and Gravitational Lensing
So by the Global Residue Theorem (for compact orbifolds), the parabolic umbilic satisfies

\[ M_{\text{tot}}(s) = \sum_{i=1}^{5} M_i = 0 \]  

for any non-caustic target point \( s = (s_1, s_2) \).

This in fact works for ALL the singularities of the \( A, D, E \) family...
So by the Global Residue Theorem (for compact orbifolds), the parabolic umbilic satisfies

$$M_{\text{tot}}(s) = \sum_{i=1}^{5} M_i = 0$$  \hspace{1cm} (1)

for any non-caustic target point $s = (s_1, s_2)$.

This in fact works for ALL the singularities of the $A, D, E$ family...

What are the advantages to this approach?
So by the Global Residue Theorem (for compact orbifolds), the parabolic umbilic satisfies

$$\mathcal{M}_{\text{tot}}(s) = \sum_{i=1}^{5} \mathcal{M}_i = 0$$  \hspace{1cm} (1)

for any non-caustic target point $s = (s_1, s_2)$.

This in fact works for ALL the singularities of the $A, D, E$ family. . .

What are the advantages to this approach?

- NO residues to calculate (the answer is immediate),
So by the Global Residue Theorem (for compact orbifolds), the parabolic umbilic satisfies

\[ \mathcal{M}_{\text{tot}}(s) = \sum_{i=1}^{5} \mathcal{M}_i = 0 \]  

for any non-caustic target point \( s = (s_1, s_2) \).

This in fact works for ALL the singularities of the \( A, D, E \) family.

What are the advantages to this approach?

- NO residues to calculate (the answer is immediate),
- an understanding that \( \mathbb{C}P^2 \) is not the only space in which to work,
So by the Global Residue Theorem (for compact orbifolds), the parabolic umbilic satisfies

\[ M_{\text{tot}}(s) = \sum_{i=1}^{5} M_i = 0 \quad (1) \]

for any non-caustic target point \( s = (s_1, s_2) \).

This in fact works for ALL the singularities of the \( A, D, E \) family...

What are the advantages to this approach?
- NO residues to calculate (the answer is immediate),
- an understanding that \( \mathbb{CP}^2 \) is not the only space in which to work,
- therefore, an explanation of such magnification relations: “eqn. (1) is really saying that in the appropriate space, there are no images at infinity.”