From the Fundamental Theorem of Algebra to Astrophysics: a "Harmonious" Journey.

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Fundamental Theorem of Algebra

Theorem 1. *Every complex polynomial $p(z) := a_n z^n + ... + a_0$, $a_n \neq 0$ of degree n has precisely n complex roots (counted with multiplicities).*

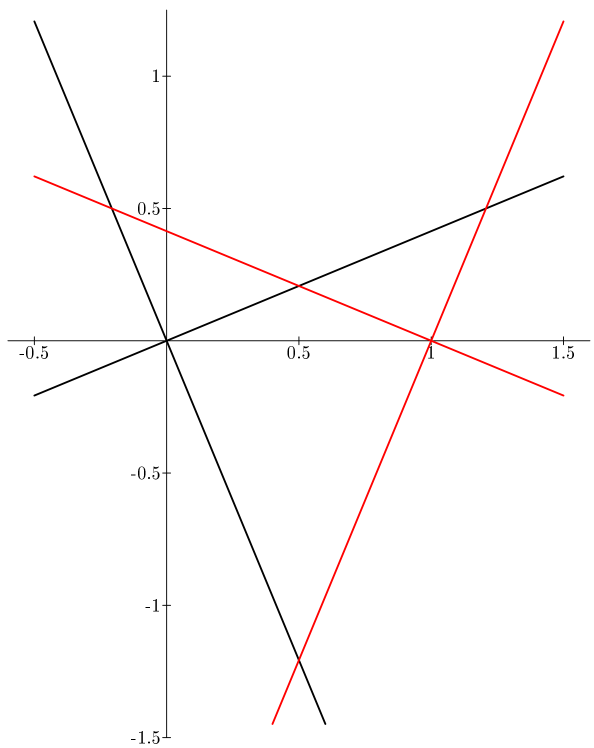
In the 1990s T. Sheil-Small, A. Wilmshurst proposed to extend FTA to a larger class of polynomials, harmonic polynomials.

$$h(z) := p(z) - \overline{q(z)}, n := deg\, p > m := deg\, q.$$

Theorem 2. *(A. Wilmshurst, '92)*

$$\sharp\{z : h(z) = 0\} \leq n^2.$$

Moreover, there exist $p, q : deg\, q = n - 1$ such that the upper bound n^2 is attained.



Wilmshurst's example for $n = 2$.

Wilmshurst's Example Continued.

$$h(z) := Im(e^{-\frac{i\pi}{4}} z^n) + iIm(e^{\frac{i\pi}{4}}(z-1)^n).$$

A little bit of algebra gives a more elegant example:

$$h(z) := z^n + (z-1)^n + i\bar{z}^n - i(\bar{z}-1)^n.$$

Note: m=n-1.

Question: If $m << n$, what is the precise upper bound for the number of zeros of the polynomial $p(z) - \overline{q(z)}$?

Conjecture 1. *(A. Wilmshurst, '92)*

$$\sharp\{z : p(z) - \overline{q(z)} = 0\} \leq m(m-1) + 3n - 2.$$

For $m = n - 1$, the above example shows that Conjecture 1 holds and is sharp. For $m = 1$, it becomes

Conjecture 2. *(T. Sheil-Small - A. Wilmshurst, '92)*

$$\sharp\{z : p(z) - \bar{z} = 0, n > 1\} \leq 3n - 2.$$

History of Conjecture 2

- In the 1990s D. Sarason and B. Crofoot and, independently, D. Bshouty, A. Lyzzaik and W. Hengartner verified it for $n = 2, 3$.

- In 2001, using elementary complex dynamics and the argument principle for harmonic mappings, G. Swiatek and DK proved Conjecture 2 for all $n > 1$.

- In 2003-2005 L. Geyer showed, using dynamics, that $3n - 2$ bound is sharp for all n.

Main Results

Theorem 3. *(G. Swiatek -DK, '01)*

$$\sharp\{z : p(z) - \bar{z} = 0, n > 1\} \leq 3n - 2.$$

The bound $3n-2$ is sharp for all n (L. Geyer, '03 -'05).

Let $r(z) := \frac{p(z)}{q(z)}$ be a rational function, $p(z), q(z)$ are polynomials. $deg\, r := max\{deg\, p, deg\, q\}$.

Theorem 4. *(G. Neumann -DK, '05)*

$$\sharp\{z : r(z) - \bar{z} = 0, n := deg\, r > 1\} \leq 5n - 5.$$

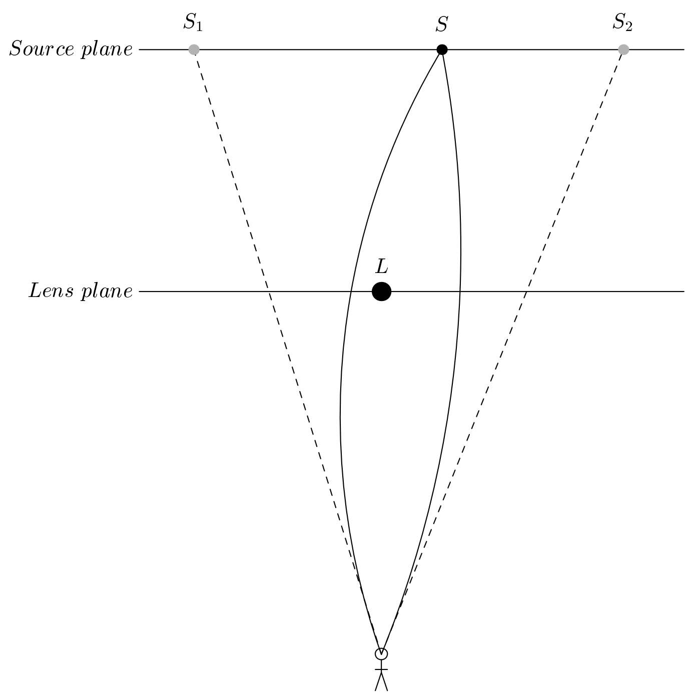
The bound $5n-5$ is sharp for all n (S. Rhie, '03).

Gravitational Microlensing

- n co-planar point-masses (e.g. condensed galaxies, black holes, etc.) in *lens plane* or *deflector plane*.

- Consider a light source in the plane parallel to the lens plane (*source plane*) and perpendicular to the line of sight from the observer.

- Due to deflection of light by masses multiple images of the source are formed. This phenomenon is known as *gravitational microlensing*.



Basics of gravitational lensing.

Lens Equation

Light source is located in the position w in the *source plane*. The lensed image is located at the position z in the *lens plane* while the masses are located at the positions z_j in the *lens plane*.

$$w = z - \sum_1^n \sigma_j / (\overline{z} - \overline{z_j}),$$

where $\sigma_j \neq 0$ are real constants.

Letting $r(z) = \sum\limits_1^n \sigma_j/(z - z_j) + \overline{w}$, the lens equation becomes

$$z - \overline{r(z)} = 0, \, deg\, r = n.$$

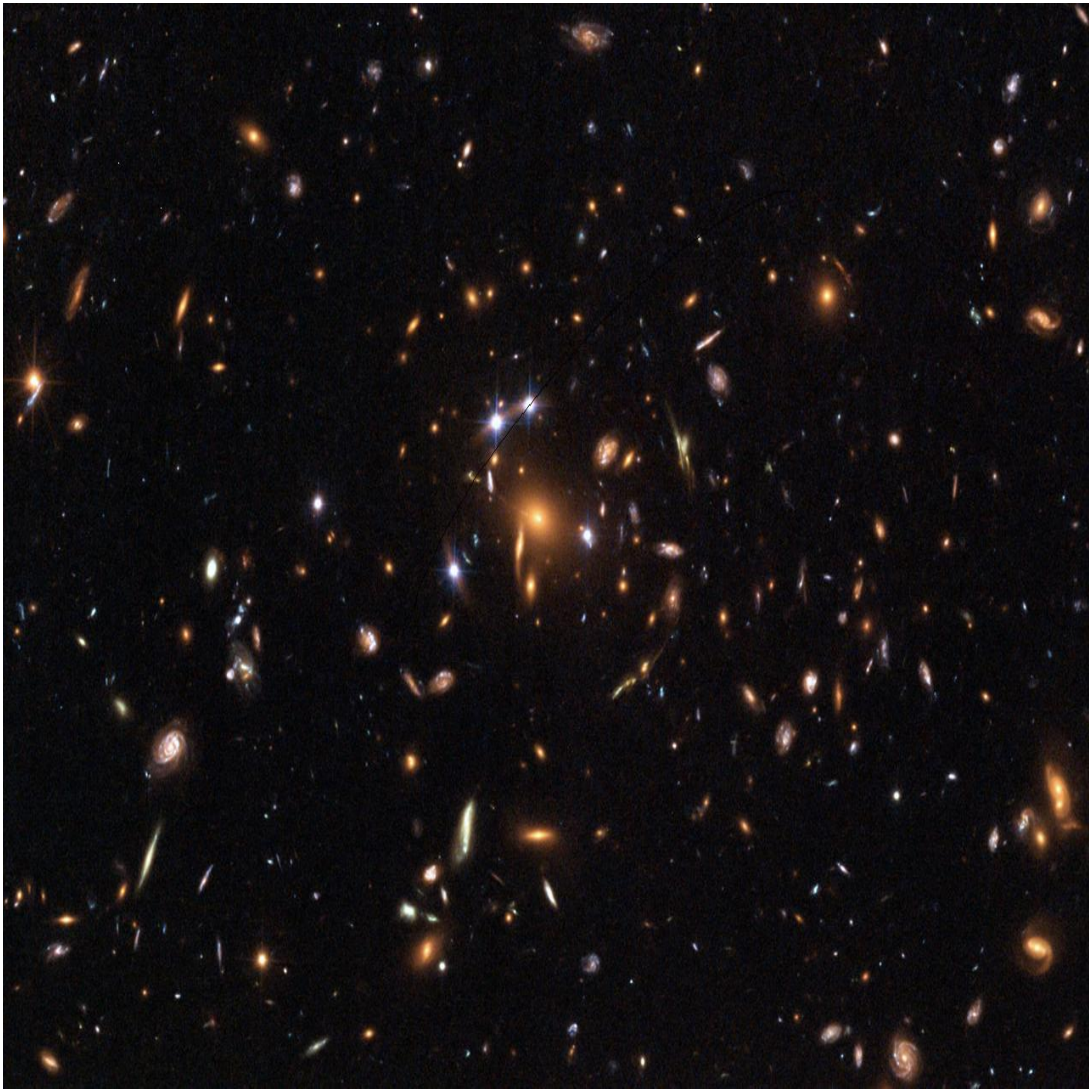
The number of solutions = the number of “lensed” images.

Gravitational Lens HST · WFPC2

Galaxy Cluster 0024+1654

PRC96-10 · ST ScI OPO · April 24, 1996

W.N. Colley (Princeton University), E. Turner (Princeton University),
J.A. Tyson (AT&T Bell Labs) and NASA



5 images of a quasar=quasi-stellar-radio object

History

• $n = 1$ (one mass) A. Einstein (1912 - 1933), either two images or the whole circle ("Einstein ring").

• H. Witt ('90) For $n > 1$ the maximum number of observed images is $\leq n^2 + 1$. S. Mao, A. Petters and H. Witt ('97) showed that the maximum is $\geq 3n + 1$.

• S.H. Rhie ('01) conjectured the upper bound for the number of lensed images for an n-lens is $5n - 5$.

Corollary 1. *(G. Neumann-DK, '05). The number of lensed images by an n-mass lens cannot exceed $5n - 5$ and this bound is sharp (Rhie, '03). Moreover, it follows from the proof that the number of images is even when n is odd and vice versa.*

The Ideas Involved

$h := z - \overline{p(z)}, deg\, p = n > 1$

Critical set of h

$\{z : Jacobian(h) = 1 - |p'|^2 = 0\} =: L,$

a lemniscate with at most $n-1$ connected components.

• Inside each of these components, h is sense-preserving. God willing, h **would be univalent inside L**, so there will be at most $n-1$ zeros of h where $|p'| \leq 1$.

• Outside L, h is sense reversing and all of its n_- sense-reversing zeros are finite.

$$-(n-1) \leq -\Delta_L arg\, h = n - n_-$$

i.e., $n_- \leq 2n-1$.

• The total number of zeros of h is

$\leq n-1+2n-1 = 3n-2$, done.

Example Consider

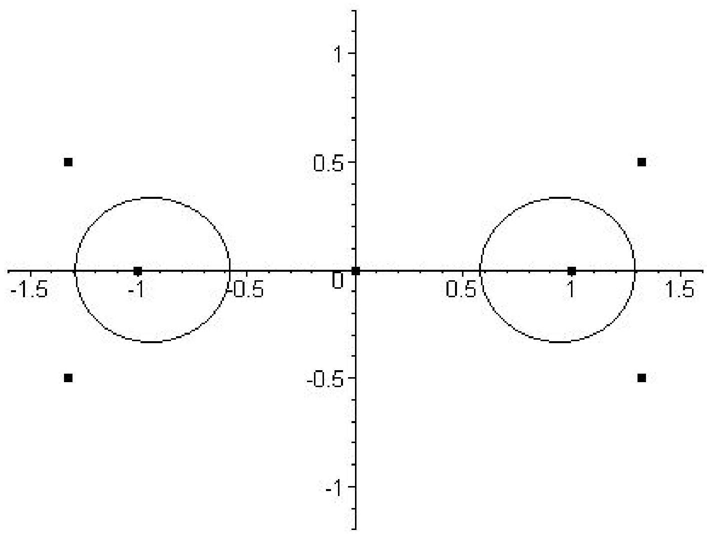
$$h(z) = z - \overline{\frac{1}{2}(3z - z^3)},$$

$n = 3$. It has $3 \times 3 - 2 = 7$ zeros

$$0, \pm 1, \frac{1}{2}(\pm\sqrt{7} \pm i).$$

$2 \times 3 - 1 = 5$ sense-reversing roots $0, \frac{1}{2}(\pm\sqrt{7} \pm i)$

and $3 - 1 = 2$ sense preserving roots ± 1, where $p' = 0$. Great!



2 **sense preserving zeros at** ± 1 **and** 5 **sense reversing zeros at** 0 **and** $\frac{1}{2}(\pm\sqrt{7} \pm i)$**.**

But God is NOT willing and h need not be univalent inside L. Thus the above argument fails.

Help From Dynamics

Proposition 1. *Let $deg\, p = n$. Then,*

$$\sharp\{\text{attracting fixed points of } \overline{p(z)}\} =$$

$$\sharp\{z : z - \overline{p(z)} = 0, |p'(z)| < 1\} \leq n - 1.$$

• $Q(z) := \overline{p(\overline{p(z)})}$ is an ANALYTIC polynomial of degree n^2.

• Every attracting fixed point of $\overline{p(z)}$ is an attracting fixed point of Q and by Fatou's theorem it "attracts" at least one CRITICAL (i.e., $z : Q'(z) = 0$) point of Q.

Lemma 1. *Each attracting fixed point of $\overline{p(z)}$ attracts at least a group of $n+1$ critical points of Q.*

• Q has $n^2 - 1$ critical points. Divided into groups of at least $n + 1$ points they "run" to at most $\frac{n^2-1}{n+1} = n - 1$ fixed points of $\overline{p(z)}$ and the proposition and then the theorem follow.

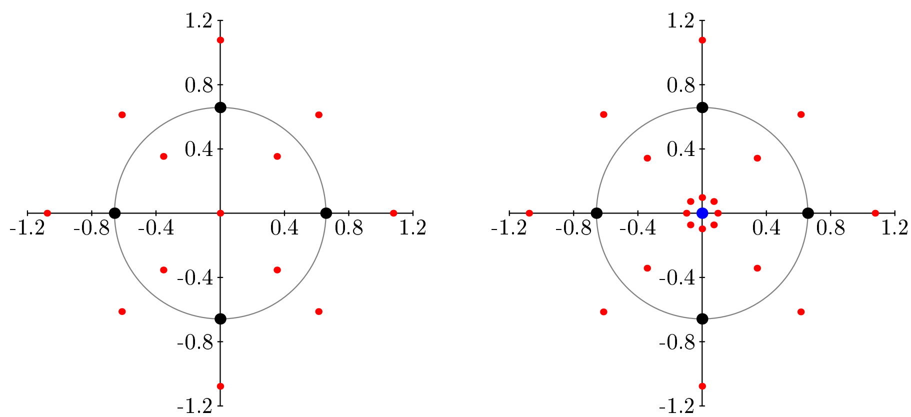
Sharpness Results

Theorem 5. *(L. Geyer, 2003-05) For every $n > 1$ there exists a complex analytic polynomial p of degree n and mutually distinct points $z_1, ..., z_{n-1}$ with $p'(z_j) = 0$ and $p(z_j) = \overline{z_j}$.*

Theorem 6. *(S. H. Rhie, 2001) For every $n > 1$ there exists a gravitational lens with n masses that produces precisely $5n - 5$ images of a point source.*

Corollary 2. *For every $n > 1$ there exists a complex analytic polynomial p of degree n such that $\overline{p(z)} - z$ has precisely $3n - 2$ zeros. Similarly, there exists a rational function $r(z)$ with (finite) poles $z_1, ..., z_n$ such that the $\overline{r(z)} - z$ has precisely $5n - 5$ zeros.*

Rhie's Construction



13 **images for the non-perturbed lens and** 20 **images after adding a small mass at the origin.**

Topological Dynamics Tools

Two polynomials p and q are called *conjugate* if there is an affine linear map T such that $p = T^{-1} \circ q \circ T$.

They are *equivalent* if there are affine linear maps S and T such that $p = S \circ q \circ T$.

Theorem 7. *(L. Geyer, 2005) The number E_n of equivalence classes of real polynomials p of degree n having $n-1$ distinct critical points $c_1, ..., c_{n-1}$ and satisfying $p(c_j) = \overline{c_j}$ for $j = 1, ..., n-1$ is $E_n = C_{[\frac{n-1}{2}]}$, where C_m is the m-th Catalan number. The number of conjugacy classes is $Q_n = E_n + C_k$ if $n = 4k+3$ and $Q_n = E_n$ if $n \not\equiv 3 (mod\, 4)$.*

Questions

1. How many zeros can a polynomial

$$h := \bar{z}^m - p(z), deg\, p = n > m$$

have?

Wilmshurst's conjecture for $m = 2$ suggests the upper bound $3n$. Is it true?

2. *Lensing*. G. Neumann-DK's theorem applies to n "spherically symmetric" mass distributions in the lens plane and gives at most $5n-5$-lensed images outside the support of the mass distribution.

Question. How many lensed images can a uniform elliptic mass distribution produce?

Theorem 8. *(C. Fassnacht - C. Keeton - DK, '07.)*

An elliptic galaxy Ω *with a uniform mass density may produce at most* 4 *"bright" lensing images of a point light source outside* Ω*, and at most one "dim" image inside* Ω*, i.e., at most* 5 *lensing images altogether.*

Moreover, an elliptic galaxy Ω *with mass density that is constant on ellipses confocal with* Ω*, may produce at most* 4 *"bright" lensing images of a point light source outside* Ω*.*

An “Astronomical” Proof

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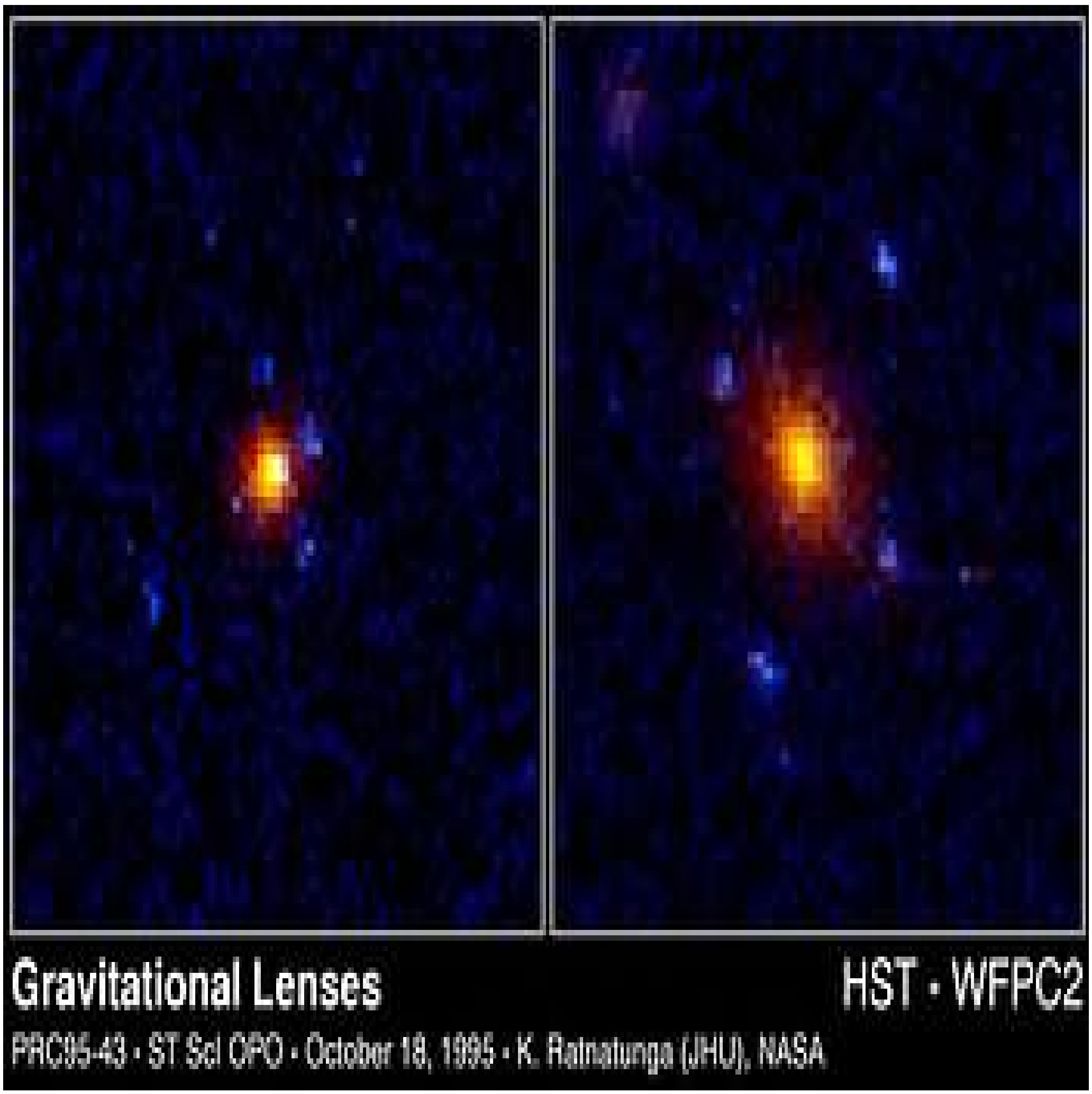
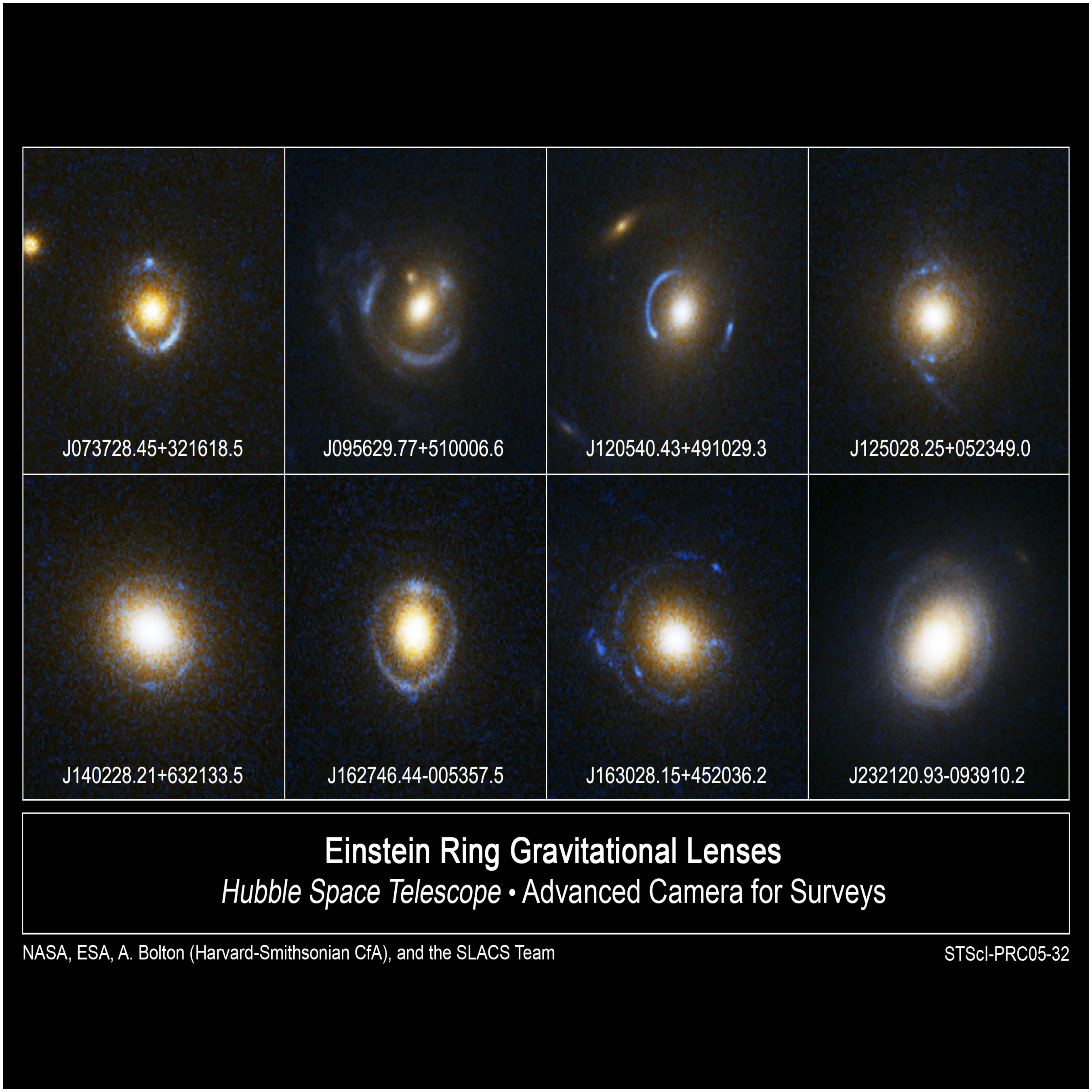


Einstein Rings are Ellipses

Theorem 9. *(Fassnacht - Keeton - DK,'07.) For any lens μ, if the lensing produces an image "curve" surrounding the lens, it is either a circle in the case when the shear, i.e., a gravitational "pull" by a galaxy "far, far away", $= 0$, or an ellipse.*


J073728.45+321618.5
J095629.77+510006.6
J120540.43+491029.3
J125028.25+052349.0
J140228.21+632133.5
J162746.44-005357.5
J163028.15+452036.2
J232120.93-093910.2
Einstein Ring Gravitational Lenses
Hubble Space Telescope • Advanced Camera for Surveys
NASA, ESA, A. Bolton (Harvard-Smithsonian CfA), and the SLACS Team
STScI-PRC05-32

“Isothermal” Elliptical Lenses

- The density, important from the physical viewpoint, is a so-called “isothermal density” obtained by projecting onto the lens plane the “realistic” three-dimensional density $\sim 1/\rho^2$, where ρ is the (three-dimensional) distance from the origin. It could be included into the whole class of densities that are constant on all ellipses *homothetic* rather than confocal with the given one.

- Lens equation becomes transcendental.

- There have been no more than 5 images $(4 + 1)$ observed as of today.

Final Remarks

- An isothermal sphere with a shear is covered by ’06 DK -G. Neumann theorem and may produce at most 4 images (observed).

- A rigorous proof that an isothermal elliptical lens may only produce finitely many images is still missing. Up to today, no more than 5 images (4 bright $+1$ dim) have been observed.

- In 2000 Ch. Keeton, S. Mao and H. J. Witt constructed models with a tidal gravitational perturbation (shear) having 9, (8 bright $+$ 1 dim), images.

THANK YOU,

SAFE SPACE TRAVEL!