From Algebra to Astrophysics

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“The miracle of the appropriateness of the language of mathematics to the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve”.

Eugene P. Wigner (1902 - 1995), 1963 Physics Nobel Prize Laureate.

Solving Algebraic Equations

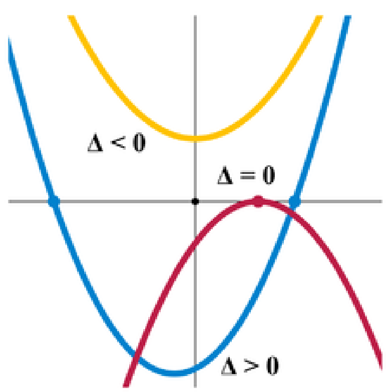
Recall quadratic equations:

$$ax^2 + bx + c = 0,$$

where a, b, c are given real numbers (coefficients). Completing the square we can easily derive

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{\Delta}}{2a},$$

where $\Delta = b^2 - 4ac$, the discriminant. Then, the number of **real** solutions of the equation depends on Δ. Yet, the number of **complex** solutions counting multiplicities is always 2!



A Brief History

Quadratic Equations - 1800-1600 B.C., Babilonians

Cubic Equations - 16th century A. D. (S. del Ferro, N. Tartaglia, G. Cardano)

Quartic Equations - 16th century A.D., (L. Ferrari)

All the above solutions are expressed as explicit formulas.

19th century - N.-H. Abel, E. Galois proved that general equations of of degree 5 and higher **CANNOT** be solved by explicit formulas.

Question: How many many complex solutions does an equation of degree $n \geq 1$ have?

Fundamental Theorem of Algebra

Theorem 1. *Every complex polynomial $p(z) := a_n z^n + \dots + a_0$, $a_n \neq 0$ of degree n has precisely n complex roots (counted with multiplicities).*

First proved in 1799 by C. F. Gauss (1777-1855).

In the 1990s T. Sheil-Small, A. Wilmshurst proposed to extend FTA to a larger class of polynomials, harmonic polynomials.

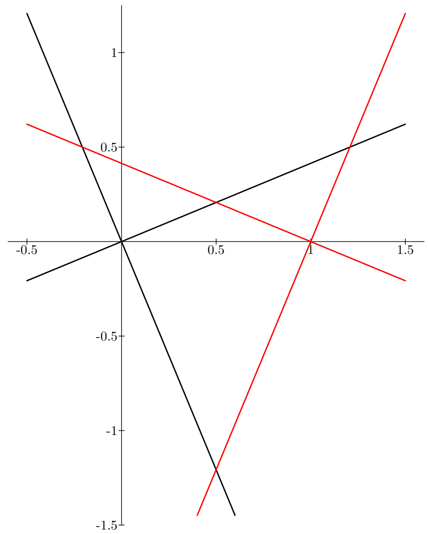
$$h(z) := p(z) - \overline{q(z)}, n := deg\, p > m := deg\, q.$$

(For a complex number $a+ib$, $a, b \in \mathbf{R}$, $\overline{a+ib} = a - ib$.)

Theorem 2. *(A. Wilmshurst, '92)*

$$\sharp\{z : h(z) = 0\} \leq n^2.$$

Moreover, there exist $p, q : deg\, q = n - 1$ *such that the upper bound* n^2 *is attained.*



Wilmshurst's example for $n = 2$.

$$h(z) := Im(e^{-\frac{i\pi}{4}} z^n) + iIm(e^{\frac{i\pi}{4}}(z-1)^n).$$

A little bit of algebra gives a more elegant example:

$$h(z) := z^n + (z-1)^n + i\bar{z}^n - i(\bar{z}-1)^n.$$

Note: m=n-1.

Question: If $m << n$, what is the precise upper bound for the number of zeros of the polynomial $p(z) - \overline{q(z)}$?

Conjecture 1. *(A. Wilmshurst, '92)*

$$\sharp\{z : p(z) - \overline{q(z)} = 0\} \leq m(m-1) + 3n - 2.$$

For $m = n - 1$, the above example shows that Conjecture 1 holds and is sharp. For $m = 1$, it becomes

Conjecture 2. *(T. Sheil-Small - A. Wilmshurst, '92)*

$$\sharp\{z : p(z) - \bar{z} = 0, n > 1\} \leq 3n - 2.$$

History of Conjecture 2

• In the 1990s D. Sarason and B. Crofoot and, independently, D. Bshouty, A. Lyzzaik and W. Hengartner verified it for $n = 2, 3$.

• In 2001, using elementary complex dynamics and the argument principle for harmonic mappings, G. Swiatek and DK proved Conjecture 2 for all $n > 1$.

• In 2003-2005 L. Geyer showed, using dynamics, that $3n - 2$ bound is sharp for all n.

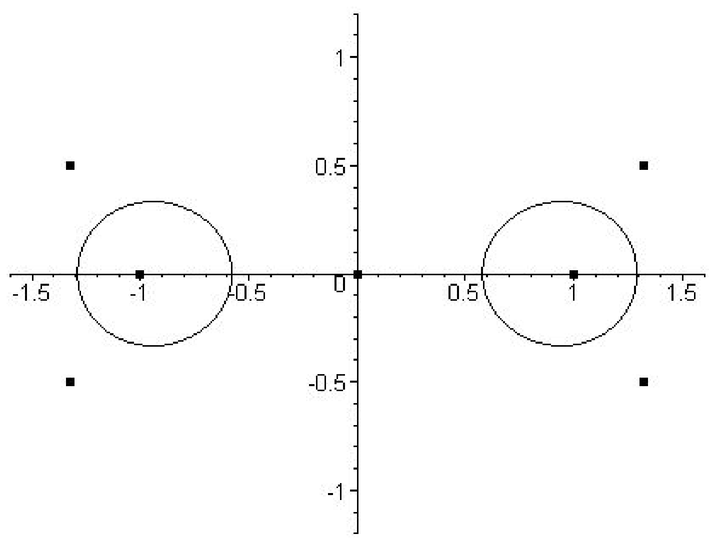
Theorem 3. *(G. Swiatek -DK, '01)*

$$\sharp\{z : p(z) - \bar{z} = 0, n > 1\} \leq 3n - 2.$$

The bound $3n-2$ is sharp for all n (L. Geyer, '03 -'05).

Example. Consider

$h(z) = z - \overline{\frac{1}{2}(3z - z^3)}$, $n = 3$. It has $3\times3-2 = 7$ zeros $0, \pm1, \frac{1}{2}(\pm\sqrt{7} \pm i)$.



Let $r(z) := \frac{p(z)}{q(z)}$ be a rational function, $p(z), q(z)$ are polynomials. $deg\, r := max\{deg\, p, deg\, q\}$. For example,

$$r(z) = \sum_{j=1}^{n} \frac{a_j}{z - z_j}.$$

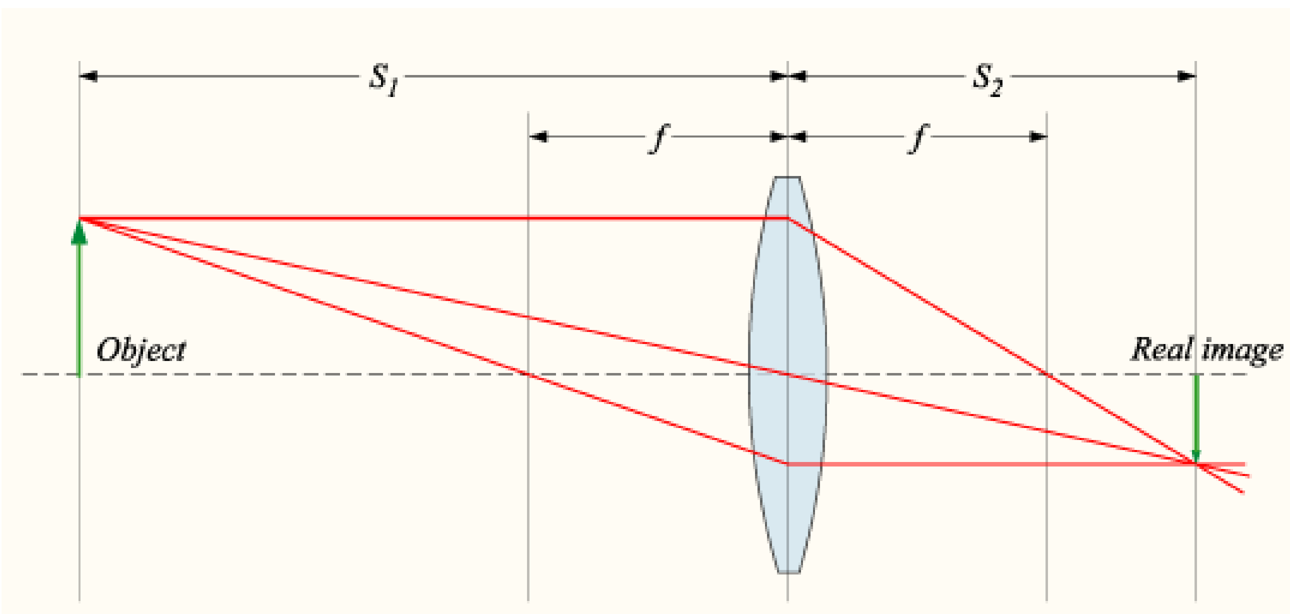
Theorem 4. *(G. Neumann -DK, '05)*

$$\sharp\{z : r(z) - \bar{z} = 0, n := deg\, r > 1\} \leq 5n - 5.$$

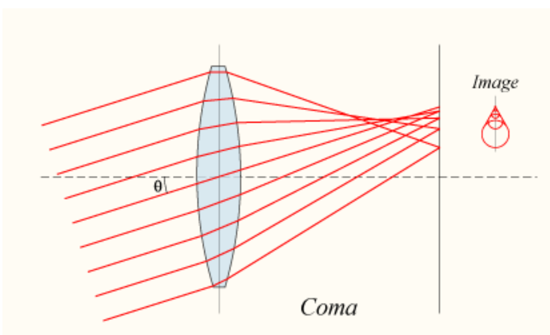
The bound $5n-5$ is sharp for all n (S. Rhie, '03).

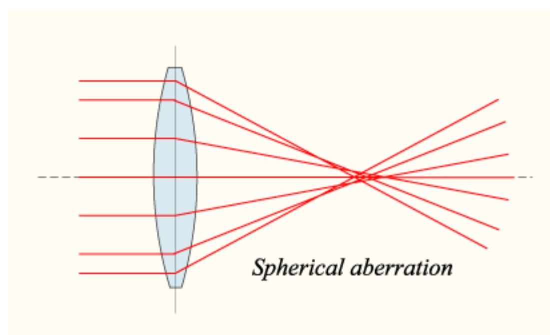
It turns out that this result opens a door to another world.

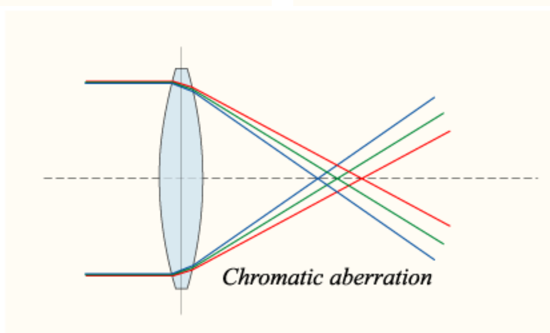
Geometric Optics in the Perfect World



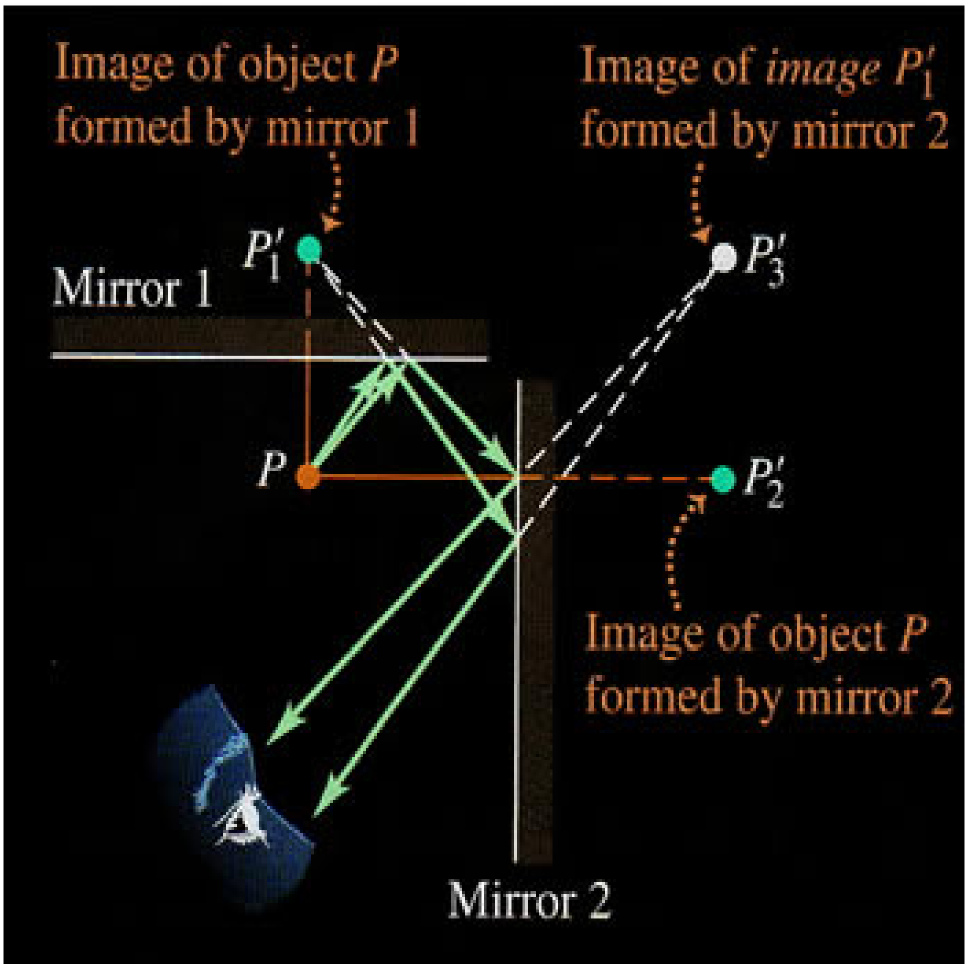
Optics in Less Than Perfect World







Multiple Images by a System of Mirrors

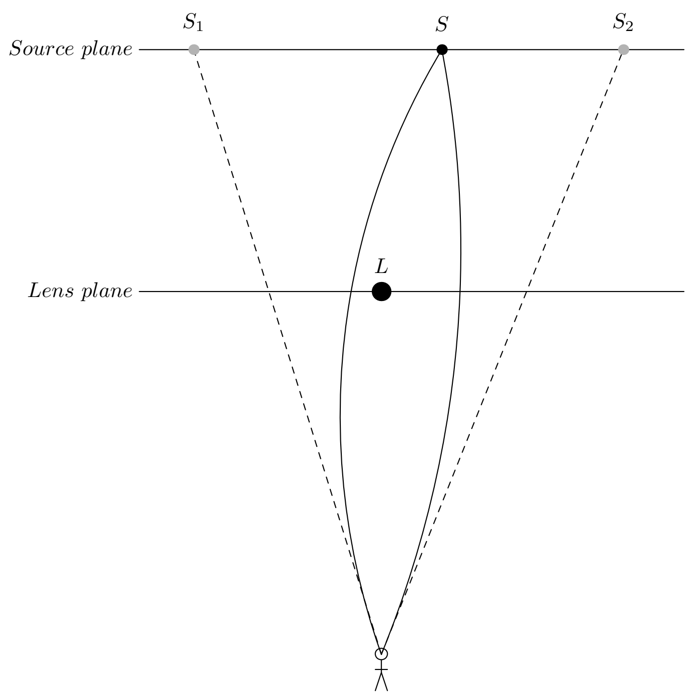


Gravitational Microlensing

• n co-planar point-masses (e.g. condensed galaxies, black holes, etc.) in *lens plane* or *deflector plane*.

• Consider a light source in the plane parallel to the lens plane (*source plane*) and perpendicular to the line of sight from the observer.

• Due to deflection of light by masses multiple images of the source are formed. This phenomenon is known as *gravitational microlensing*.

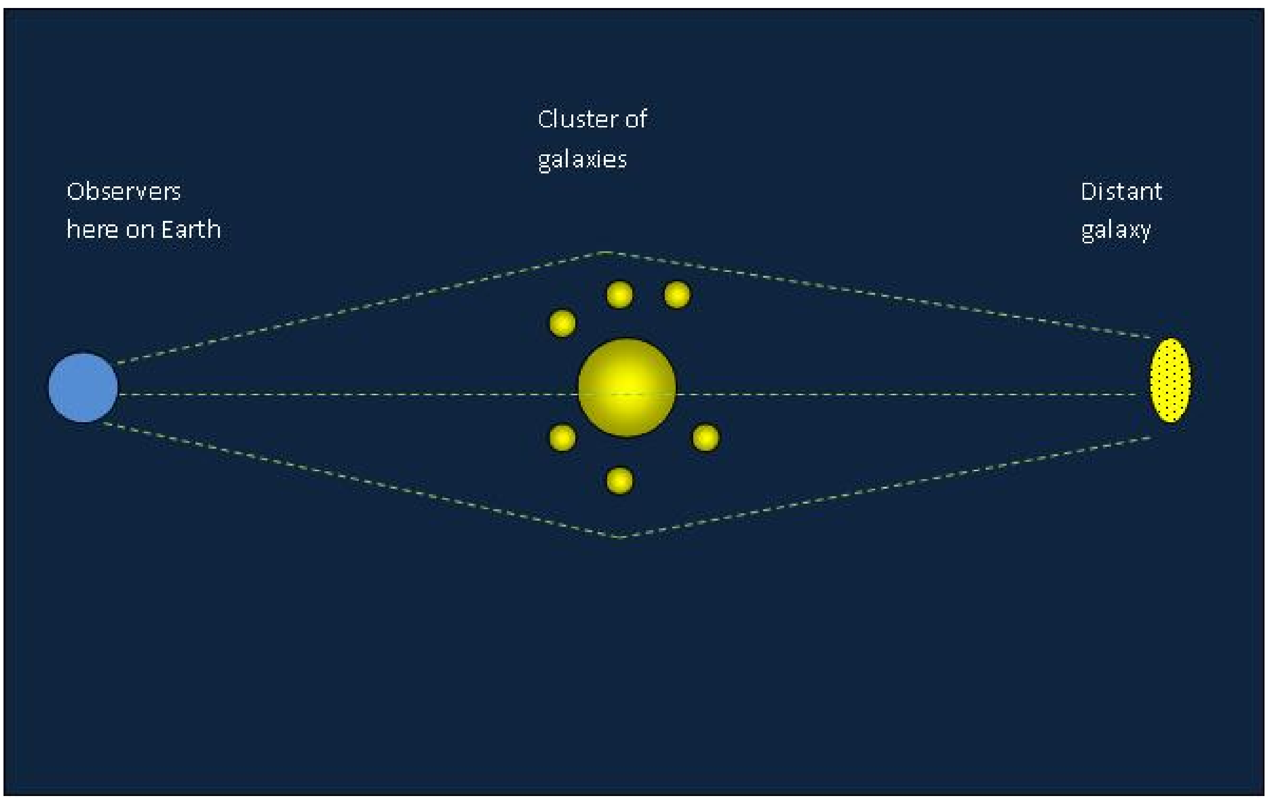


Basics of gravitational lensing.

Gravitational lensing: the gravitational field of a massive object(s) acts as a lens for background sources


Exciting fact: the map from the distorted picture to the original is a planar harmonic map.

Lensing by Multiple Massive Objects



Lens Equation

Light source is located in the position w in the *source plane*. The lensed image is located at the position z in the *lens plane* while the masses are located at the positions z_j in the *lens plane*.

$$w = z - \sum_{1}^{n} \sigma_j / (\overline{z} - \overline{z_j}),$$

where $\sigma_j \neq 0$ are real constants.

Letting $r(z) = \sum_{1}^{n} \sigma_j/(z - z_j) + \overline{w}$, the lens equation becomes

$$z - \overline{r(z)} = 0,\ deg\, r = n.$$

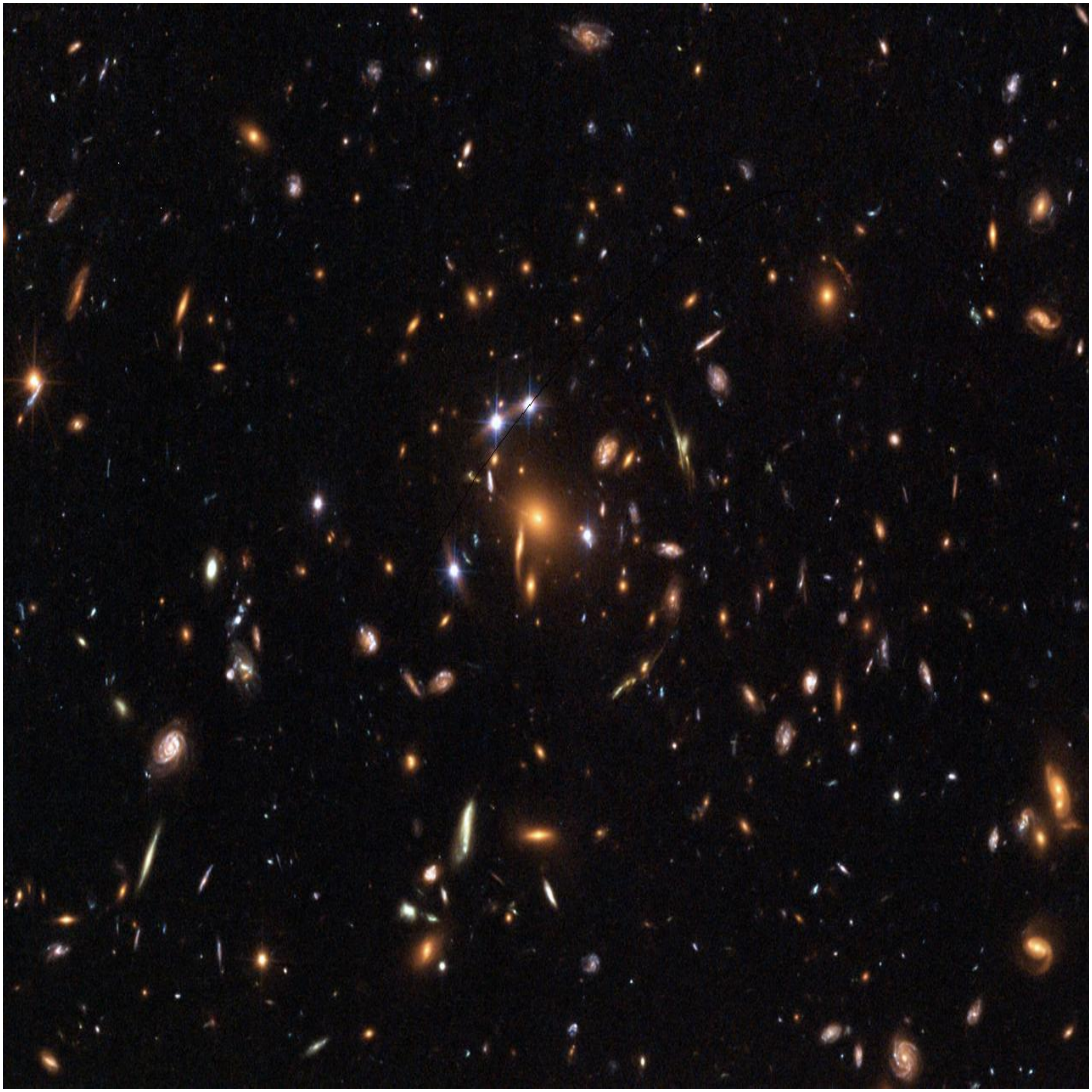
The number of solutions = the number of "lensed" images.

Gravitational Lens
Galaxy Cluster 0024+1654

HST · WFPC2

PRC96-10 · ST ScI OPO · April 24, 1996

W.N. Colley (Princeton University), E. Turner (Princeton University),
J.A. Tyson (AT&T Bell Labs) and NASA



5 images of a quasar=quasi-stellar-radio object

History

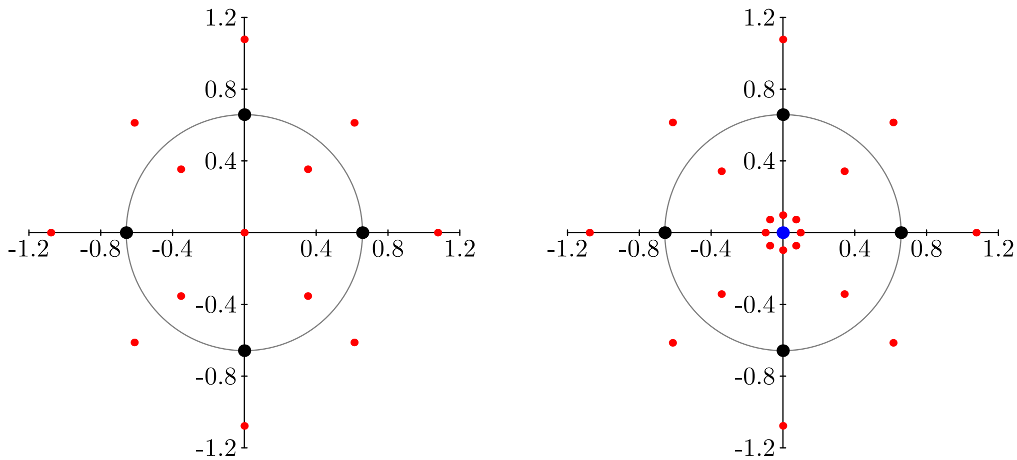
• $n = 1$ (one mass) A. Einstein (1912 - 1933), either two images or the whole circle (“Einstein ring”).

• H. Witt ('90) For $n > 1$ the maximum number of observed images is $\leq n^2 + 1$. S. Mao, A. Petters and H. Witt ('97) showed that the maximum is $\geq 3n + 1$.

• S.H. Rhie ('01) conjectured the upper bound for the number of lensed images for an n-lens is $5n - 5$.

Corollary 1. *(G. Neumann-DK, '05). The number of lensed images by an n-mass lens cannot exceed $5n - 5$ and this bound is sharp (Rhie, '03). Moreover, it follows from the proof that the number of images is even when n is odd and vice versa.*

Rhie's Construction



13 **images for the non-perturbed lens and** 20 **images after adding a small mass at the origin.**

Questions

1. How many zeros can a polynomial

$$h := \bar{z}^m - p(z), deg\, p = n > m$$

have?

Wilmshurst's conjecture for $m = 2$ suggests the upper bound $3n$. Is it true?

2. *Lensing*. G. Neumann-DK's theorem applies to n "spherically symmetric" mass distributions in the lens plane and gives at most $5n-5$-lensed images outside the support of the mass distribution.

Question. How many lensed images can a uniform elliptic mass distribution produce?

Theorem 5. *(C. Fassnacht - C. Keeton - DK, '07.)*

An elliptic galaxy Ω *with a uniform mass density may produce at most* 4 *"bright" lensing images of a point light source outside* Ω*, and at most one "dim" image inside* Ω*, i.e., at most* 5 *lensing images altogether.*

Moreover, an elliptic galaxy Ω *with mass density that is constant on ellipses confocal with* Ω*, may produce at most* 4 *"bright" lensing images of a point light source outside* Ω*.*

An “Astronomical” Proof

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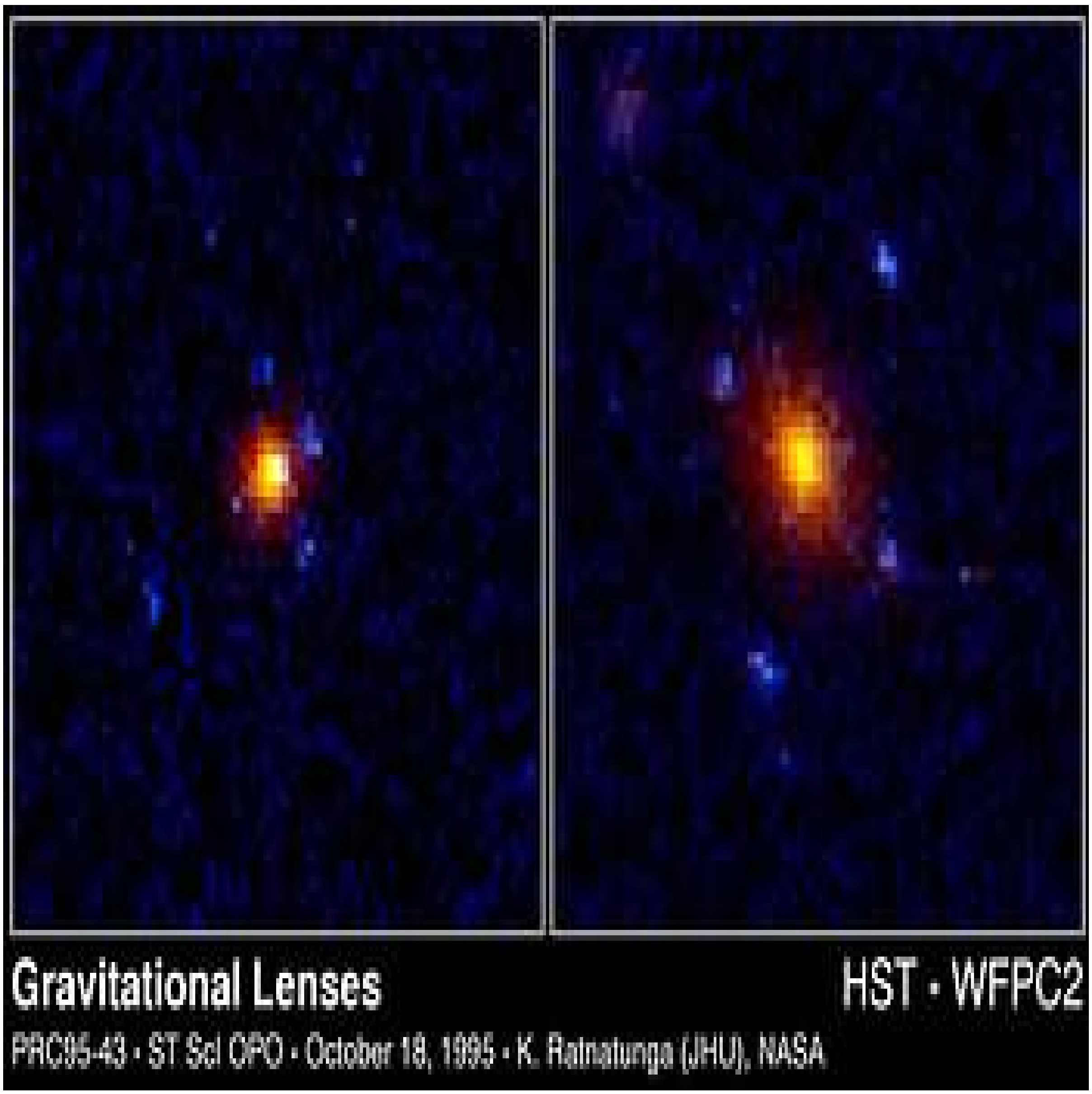
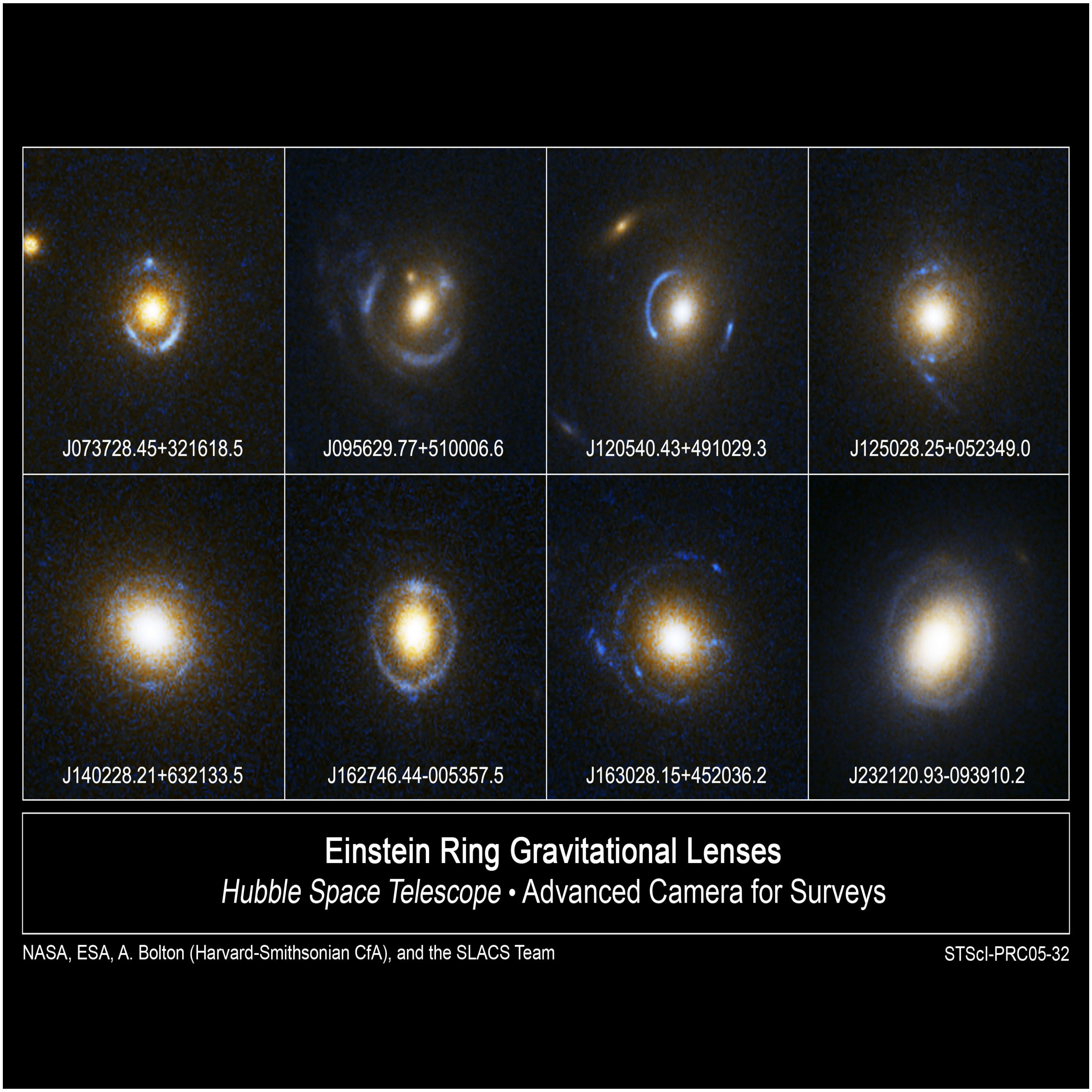


Einstein Rings are Ellipses

Theorem 6. *(Fassnacht - Keeton - DK,'07.) For any lens μ, if the lensing produces an image "curve" surrounding the lens, it is either a circle in the case when the shear, i.e., a gravitational "pull" by a galaxy "far, far away",* $= 0$*, or an ellipse.*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J095629.77+510006.6
J120540.43+491029.3
J125028.25+052349.0
J140228.21+632133.5
J162746.44-005357.5
J163028.15+452036.2
J232120.93-093910.2
Einstein Ring Gravitational Lenses
Hubble Space Telescope • Advanced Camera for Surveys
NASA, ESA, A. Bolton (Harvard-Smithsonian CfA), and the SLACS Team
STScI-PRC05-32

“Isothermal” Elliptical Lenses

- The density, important from the physical viewpoint, is a so-called “isothermal density” obtained by projecting onto the lens plane the “realistic” three-dimensional density $\sim 1/\rho^2$, where ρ is the (three-dimensional) distance from the origin. It could be included into the whole class of densities that are constant on all ellipses *homothetic* rather than confocal with the given one.

- Lens equation becomes transcendental.

$$z - const \int_0^1 \frac{dt}{\sqrt{\bar{z}^2 - c^2t^2}} - \gamma\bar{z} = w.$$

Final Remarks

- An isothermal sphere with a shear is covered by '06 DK -G. Neumann theorem and may produce at most 4 images (observed).

- DK and E. Lundberg ('09) have proved that an isothermal elliptical lens without a shear may produce up to 8 bright images. Instantly, Bergweiler and Eremenko improved the estimate to 6 images, and showed that 6 is sharp. No more than 5 images (4 bright $+1$ dim) have been observed up to now.

- In 2000 Ch. Keeton, S. Mao and H. J. Witt constructed models with a tidal gravitational perturbation (shear) having
9, (8 bright $+$ 1 dim), images.

Three-Dimensional Lensing

- The 3-dimensional lens equation with mass-distribution $dm(y)$ with source at $\vec{w}$ becomes

$$\vec{x} - \nabla_x \left(\int \frac{dm(y)}{|x-y|} \right) = \vec{w}.$$

- If the mass-distribution $dm(y)$ consists of n point-masses, there are some estimates for the maximal number of images (A. Petters, '90s) based on geometric topology (Morse theory). No sharp estimates are known.

- A difficult Maxwell's problem concerns a number of stationary points of the Newtonian potential of n point-masses (conjectured $\leq (n-1)^2$). Most recent progress due to Eremenko, Gabrielov, D. Novikov, B. Shapiro. But this is the beginning of a new tale.

THANK YOU!