SMOOTH FUNCTIONS IN STAR-INVARIANT SUBSPACES

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Dedicated to Joseph Cima on the occasion of his 70th birthday

Abstract. In this note we summarize some necessary and sufficient conditions for subspaces invariant with respect to the backward shift to contain smooth functions. We also discuss smoothness of moduli of functions in such subspaces.

1. Introduction

For $0 < p \le \infty$, let H^p denote the classical Hardy space of analytic functions on the disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. As usual, we also treat H^p as a subspace of $L^p(\mathbb{T}, m)$, where $\mathbb{T} := \partial\mathbb{D}$ and m is the normalized arc length measure on $\mathbb{T}$.

Now suppose θ is an inner function on $\mathbb{D}$, that is, $\theta \in H^\infty$ and $|\theta(\zeta)| = 1$ for m-almost all $\zeta \in \mathbb{T}$. Factoring θ canonically, we get $\theta = BS$, where B is a Blaschke product and S is a singular inner function (see [12], Chapter II). The latter is thus of the form

$$S(z) = S_\mu(z) := \exp\left\{-\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta)\right\},$$

where μ is a (positive) singular measure on $\mathbb{T}$, and we shall write $\mu = \mu_\theta (= \mu_S)$ to indicate that μ is associated with θ (or S) in this way.

We shall be concerned with the *star-invariant subspace*

$$K_\theta := H^2 \ominus \theta H^2 \tag{1.1}$$

that θ generates in H^2. Here, the term *star-invariant* stands for *invariant under the backward shift operator*

$$f \mapsto (f - f(0))/z, \qquad f \in H^2,$$

and it is well known that the general form of a (closed and proper) star-invariant subspace in H^2 is actually given by (1.1), with θ inner; see, e. g., [5] or [14].

This paper treats two questions related to the (boundary) smoothness of functions in K_θ. The first of these concerns the very existence of nontrivial functions in $K_\theta \cap X$, where X is a given smoothness class. The answer should of course depend on X, but for a wide range of X's it turns out to be the same. Before we can state it, let

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us recall that a closed subset E of $\mathbb{T}$ is said to be a *Carleson* (or *Beurling–Carleson*) *set* if

$$\int_{\mathbb{T}} \log \operatorname{dist}(\zeta, E)\, dm(\zeta) > -\infty.$$

Originally, Carleson sets arose in [4] and the earlier work of Beurling as boundary zero-sets of analytic functions on $\mathbb{D}$ that are smooth, say of class C^1 or C^n, up to $\mathbb{T}$. Later on, they emerged in Korenblum's and Roberts' description of cyclic inner functions in Bergman spaces; see [13], [15] and [6], Chapter 8.

Our first result, Theorem 2.1, basically says that for many – or "most" – natural smoothness spaces X, one has $K_\theta \cap X = \{0\}$ if and only if θ is a singular inner function with the property that

$$\mu_\theta(E) = 0 \quad \text{for every Carleson set } E \subset \mathbb{T}. \tag{1.2}$$

This contrasts with the fact that the intersection $K_\theta \cap C(\mathbb{T})$ is always dense in K_θ (and hence always nontrivial), a result due to A. B. Aleksandrov; cf. [1], Theorem 6.

We admit that our Theorem 2.1 is not completely original, and the appearance of Carleson sets in this context should not be surprising. For instance, it was proved by H. S. Shapiro in [16] that if θ is a singular inner function for which (1.2) fails, then K_θ contains nonzero functions of class $C^n(\mathbb{T})$, for any fixed n. The new feature is, however, that our theorem applies to a larger scale of smoothness classes X. These range from the nicest possible space $C^\infty(\mathbb{T})$ to certain Bergman–Sobolev (or Besov) spaces that contain unbounded functions and enjoy very little smoothness indeed. In fact, those Bergman–Sobolev spaces are "almost the largest ones" for which the Korenblum–Roberts condition (1.2) is still relevant; we shall explain this in more detail below.

Our second theme is related to moduli of K_θ-functions. Roughly speaking, the question is how various smoothness properties of $f \in K_\theta$ are affected by those of $|f|$. More precisely, we seek to determine the nonnegative functions φ on $\mathbb{T}$ for which the set $\{f \in K_\theta : |f| = \varphi\}$ is nonempty and lies in a given smoothness class. This time we restrict our attention to the Lipschitz spaces Λ_ω defined in terms of a majorant ω; the solution is then given by Theorem 3.1.

2. Smooth functions in K_θ: existence

First let us fix some additional notations. We write σ for the normalized area measure on $\mathbb{D}$, and A^p for the Bergman p-space defined as the set of analytic functions in $L^p(\mathbb{D}, \sigma)$; we also need the Bergman–Sobolev spaces $A^{p,1} := \{f \in A^p : f' \in A^p\}$.

Further, we recall the definition of the Lipschitz–Zygmund spaces $\Lambda^\alpha = \Lambda^\alpha(\mathbb{T})$ with $\alpha > 0$. Given $\alpha = k + \beta$, where $k \ge 0$ is an integer and $0 < \beta \le 1$, the space Λ^α consists of those functions $f \in C^k(\mathbb{T})$ which satisfy

$$f^{(k)}(e^{ih}\zeta) - f^{(k)}(\zeta) = O(|h|^\beta), \quad \text{if} \quad 0 < \beta < 1,$$

and

$$f^{(k)}(e^{ih}\zeta) - 2f^{(k)}(\zeta) + f^{(k)}(e^{-ih}\zeta) = O(|h|), \quad \text{if} \quad \beta = 1,$$

uniformly in $\zeta \in \mathbb{T}$ and $h \in \mathbb{R}$. Finally, we put $\Lambda^\alpha_A := H^\infty \cap \Lambda^\alpha$ and $\mathcal{A}^\infty := H^\infty \cap C^\infty(\mathbb{T})$.

Theorem 2.1. *Let θ be an inner function on $\mathbb{D}$. The following statements are equivalent.*

(i.1) *K_θ contains a nontrivial function of class $\mathcal{A}^\infty$.*

(ii.1) *K_θ contains a nontrivial function of class $\bigcup_{p>1} A^{p,1}$.*

(iii.1) *Either θ has a zero in $\mathbb{D}$, or there is a Carleson set $E \subset \mathbb{T}$ with $\mu_\theta(E) > 0$.*

Proof. The implication (i.1) $\Longrightarrow$ (ii.1) is obvious.

(ii.1) $\Longrightarrow$ (iii.1). Suppose (iii.1) fails, so that θ is a purely singular inner function, whose associated measure μ_θ vanishes on every Carleson set. By the Korenblum–Roberts theorem (see [6], p. 249), it follows that θ is a *cyclic* vector in each Bergman space A^q with $q \ge 1$.

Now if (ii.1) holds, then we can find a nontrivial function $F \in K_\theta \cap A^{p,1}$, with some $p > 1$. Being orthogonal to θH^2 (in H^2), this F satisfies

$$\int_{\mathbb{T}} \overline{zF(z)}\, \theta(z) z^n \, dz = 0 \qquad (n = 0, 1, \dots). \tag{2.1}$$

Using Green's formula, we rewrite (2.1) as

$$\int_{\mathbb{D}} \overline{f(z)}\, \theta(z) z^n \, d\sigma(z) = 0 \qquad (n = 0, 1, \dots), \tag{2.2}$$

where $f := (zF)'$. Letting $q = p/(p-1)$, we further rephrase (2.2) by saying that the family $\{\theta z^n : n \ge 0\}$ (and hence the subspace it spans in A^q) is annihilated by a nonzero functional in $(A^q)^* = A^p$, namely by the functional $g \mapsto \int_{\mathbb{D}} \overline{f} g \, d\sigma$. Indeed, we have $f \in A^p$ because $F \in A^{p,1}$, and $f \not\equiv 0$ because $F \not\equiv 0$. Thus θ generates a *proper* shift-invariant subspace in A^q and is, therefore, a *noncyclic* vector therein. This contradiction implies that (iii.1) holds as soon as (ii.1) does.

(iii.1) $\Longrightarrow$ (i.1). If θ has a zero z_0 in $\mathbb{D}$, then $z \mapsto (1 - \bar{z}_0 z)^{-1}$ is a nontrivial function in $K_\theta \cap \mathcal{A}^\infty$.

Now assume that θ is a singular inner function and that $E \subset \mathbb{T}$ is a Carleson set with $\mu_\theta(E) > 0$. Put $\nu := \mu_\theta \big| E$, and let $S = S_\nu$ be the corresponding singular inner function. Since S divides θ, and hence $K_S \subset K_\theta$, it suffices to find a nontrivial $\mathcal{A}^\infty$-function in K_S. First we observe that, since ν lives on a Carleson set, S must divide the inner part of some nontrivial $\mathcal{A}^\infty$-function (see [19], Corollary 4.8). Thus, $GS \in \mathcal{A}^\infty$ for some $G \in H^\infty$, $G \not\equiv 0$, whence it actually follows (see, e. g., [19], Theorem 4.1) that $G \in \mathcal{A}^\infty$. In fact, there is no loss of generality in assuming that G is outer (once again, because division by inner factors preserves membership in $\mathcal{A}^\infty$). Next, we claim that

$$G\bar{S} \in C^\infty(\mathbb{T}), \tag{2.3}$$

a fact we shall soon verify.

Postponing this verification for a moment, let us now use (2.3) to complete the proof. Put $\Phi := \bar{z}\bar{G}S$ and $f := P_+\Phi$, where P_+ stands for the orthogonal projection from $L^2(\mathbb{T})$ onto H^2. Our plan is to show that f is a nontrivial function in $\mathcal{A}^\infty \cap K_S$.

First of all, $f \not\equiv 0$, because otherwise we would have $\Phi \in \bar{z}\overline{H^2}$, or equivalently $G\bar{S} \in H^2$, which is impossible since G is outer and S is inner. To see that f is in $C^\infty = C^\infty(\mathbb{T})$, and hence in $\mathcal{A}^\infty$, we note that $\Phi \in C^\infty$ by virtue of (2.3) and then recall that P_+ maps C^∞ into itself. Finally, since $f \in H^2$, the inclusion $f \in K_S$ will be established as soon as we check that f is orthogonal to the subspace SH^2. This we now do: if $h \in H^2$, then

$$\int_{\mathbb{T}} f\bar{S}\bar{h}\,dm = \int_{\mathbb{T}} \Phi\bar{S}\bar{h}\,dm = \int_{\mathbb{T}} \bar{z}\bar{G}\bar{h}\,dm = 0,$$

where the first equality holds because the antianalytic function $(I - P_+)\Phi$ is automatically orthogonal to Sh.

It remains to prove (2.3). Fix $\alpha > 0$ and an integer n with $n > \alpha$. This done, we invoke Proposition 1.5 of [8] which says, in particular, that given a function $F \in \Lambda^\alpha_A$ and an inner function I, the inclusions $FI^n \in \Lambda^\alpha_A$ and $F/I^n \in \Lambda^\alpha$ are equivalent. Applying this to $F = G$ and $I = S^{1/n}$, while recalling that $GS \in \mathcal{A}^\infty \subset \Lambda^\alpha_A$, we deduce that $G/S(= G\bar{S})$ is in Λ^α. And since this happens for *each* $\alpha > 0$, we finally conclude that

$$G\bar{S} \in \bigcap_{\alpha>0} \Lambda^\alpha = C^\infty(\mathbb{T}),$$

as desired. □

Remarks. (1) In connection with condition (ii.1) above, we observe that K_θ always contains nontrivial H^∞-functions; one example is $1 - \overline{\theta(0)}\theta$. Now if $\mathcal{B}$ stands for the *Bloch space* (i.e., the set of analytic functions f on $\mathbb{D}$ with $\sup_{z\in\mathbb{D}}(1-|z|)|f'(z)| < \infty$), then we have

$$H^\infty \subset \mathcal{B} \subset \bigcap_{0<p<1} A^{p,1},$$

so the intersection $K_\theta \cap \bigcap_{0<p<1} A^{p,1}$ is always nontrivial. Thus, the smoothness class $\bigcup_{p>1} A^{p,1}$ in Theorem 2.1 cannot be made "much larger" (say, by extending the union to $p > 1 - \varepsilon$) if the result is to remain true.

(2) We do not know, however, if the latter class can be replaced by $A^{1,1}$. The dual of A^1 being the Bloch space $\mathcal{B}$ (see [6], p. 48), the question can be rephrased in terms of weak* cyclicity of an inner function in $\mathcal{B}$. While the Korenblum–Roberts condition (1.2) on a singular inner function θ is necessary for θ to be weak* cyclic in $\mathcal{B}$, the sufficiency of that condition seems to present an open problem. This was mentioned in [3], and we are unaware of any further progress on that matter.

On the other hand, some sufficient conditions for weak* cyclicity in $\mathcal{B}$ – and a construction of an inner function satisfying them – can be found in [2]. In particular, there do exist inner functions θ with the property that $K_\theta \cap A^{1,1} = \{0\}$.

3. Smooth functions in K_θ and their moduli

This section deals with the following problem. Suppose φ is a nonnegative function on $\mathbb{T}$ that coincides a. e. with the modulus of some K_θ-function (this will be written as $\varphi \in |K_\theta|$). *When does it happen that all functions $f \in K_\theta$ with $|f| = \varphi$ are smooth, in some sense or other?*

We shall address this question when smoothness is understood as membership in $\Lambda_\omega = \Lambda_\omega(\mathbb{T})$, the Lipschitz space associated with a majorant (alias, modulus of continuity) ω. It will be assumed that $\omega : [0,2] \to \mathbb{R}$ is a continuous increasing function with $\omega(0) = 0$ and that $\omega(t)/t$ is non-increasing. The space Λ_ω is then formed by those functions $f \in C(\mathbb{T})$ which satisfy

$$f(z_1) - f(z_2) = O\left(\omega(|z_1 - z_2|)\right), \qquad z_1, z_2 \in \mathbb{T}.$$

Thus, we want the set

$$M(\theta, \varphi) := \{f \in K_\theta : |f| = \varphi\}$$

to be contained in Λ_ω, and we shall soon characterize the pairs (θ, φ) for which this happens.

Before going any further, we recall that there is a simple characterization of the set $|K_\theta|$, as well as a parametrization of $M(\theta, \varphi)$ for $\varphi \in |K_\theta|$. These results are contained in [7] (see also Lemma 5 in [10]) and can be summarized as follows. In order that $\varphi \in |K_\theta|$, it is necessary and sufficient that $\bar{z}\varphi^2\theta \in H^1$. If that is so, we can factor the latter function as

$$\bar{z}\varphi^2\theta = \mathcal{O}_\varphi^2 I, \tag{3.1}$$

where $\mathcal{O}_\varphi := \exp\left(\log\varphi + i\,\widetilde{\log\varphi}\right)$ is the outer function with modulus φ and I is an inner function. This done, it is easy to see that the functions $\mathcal{O}_\varphi$ and $\mathcal{O}_\varphi I$ are both in K_θ, and hence in $M(\theta, \varphi)$, while any other function in $M(\theta, \varphi)$ lies "in between". Precisely speaking, we have

$$M(\theta, \varphi) = \{\mathcal{O}_\varphi J : J \in \mathcal{D}(I)\}, \tag{3.2}$$

where $\mathcal{D}(I)$ stands for the set of all inner divisors of I. We also point out, for future reference, that (3.1) implies

$$\bar{z}\bar{\mathcal{O}}_\varphi\theta = \mathcal{O}_\varphi I \tag{3.3}$$

(to see why, write $\varphi^2 = \mathcal{O}_\varphi\bar{\mathcal{O}}_\varphi$ and substitute this in (3.1)).

Finally, with an inner function θ we associate the sets

$$\Omega(\theta, \varepsilon) := \{z \in \mathbb{D} : |\theta(z)| < \varepsilon\}, \qquad 0 < \varepsilon < 1,$$

and

$$\rho(\theta) := \{z \in \mathbb{D} \cup \mathbb{T} : \liminf_{\mathbb{D}\ni w \to z} |\theta(w)| = 0\}.$$

Of course, $\rho(\theta) \cap \mathbb{D}$ is just the zero-set of θ, while $\rho(\theta) \cap \mathbb{T}$ consists of its boundary singularities.

Theorem 3.1. *Let θ be an inner function, and let $\varphi \in |K_\theta|$. The following are equivalent:*

(i.2) $M(\theta, \varphi) \subset \Lambda_\omega$.

(ii.2) $\mathcal{O}_\varphi\theta \in \Lambda_\omega$.

(iii.2) $\mathcal{O}_\varphi \in \Lambda_\omega$ *and* $\varphi\theta \in \Lambda_\omega$.

(iv.2) $\mathcal{O}_\varphi \in \Lambda_\omega$, *and for some (or any)* $\varepsilon \in (0,1)$ *one has*

$$\mathcal{O}_\varphi(z) = O\left(\omega(1 - |z|)\right), \qquad z \in \Omega(\theta, \varepsilon). \tag{3.4}$$

Proof. Put $F := \mathcal{O}_\varphi$, and let I be as in (3.1). Taking (3.2) into account and recalling that division by inner factors preserves membership in $\Lambda_\omega \cap H^\infty$ (see [17]), we deduce that (i.2) holds iff the "extremal" function FI is in Λ_ω. Using the identity $FI = \bar{z}\bar{F}\theta$ (this is precisely (3.3)), we restate the condition $FI \in \Lambda_\omega$ as $F\bar{\theta} \in \Lambda_\omega$. The latter can be further rephrased by saying that the quantity

$$\begin{aligned} Q(z_1, z_2) : &= (F\bar{\theta})(z_1) - (F\bar{\theta})(z_2) \\ &= [F(z_1) - F(z_2)]\,\bar{\theta}(z_1) + F(z_2)\left[\bar{\theta}(z_1) - \bar{\theta}(z_2)\right] \end{aligned} \tag{3.5}$$

is $O\left(\omega(|z_1 - z_2|)\right)$ whenever z_1, z_2 are in $\mathbb{T} \setminus \rho(\theta)$.

It should be noted that if $F\bar{\theta}$ satisfies a Λ_ω-condition over $\mathbb{T} \setminus \rho(\theta)$, then $F\bar{\theta} \in \Lambda_\omega$ (the converse being trivially true). Indeed, since $F\bar{\theta}$ is at least continuous on $\mathbb{T}$, it follows that $F = 0$ on $\rho(\theta) \cap \mathbb{T}$. And if $F \not\equiv 0$, which we may safely assume, then we conclude that $m(\rho(\theta) \cap \mathbb{T}) = 0$ and so $\mathbb{T} \setminus \rho(\theta)$ is dense in $\mathbb{T}$.

Going back to (3.5), we observe that the first of the two terms on the right will be $O\left(\omega(|z_1 - z_2|)\right)$ as soon as

$$F \in \Lambda_\omega \tag{3.6}$$

(and this happens if any of the conditions (i.2)–(iv.2) holds). Therefore, the estimate

$$Q(z_1, z_2) = O\left(\omega(|z_1 - z_2|)\right), \qquad z_1, z_2 \in \mathbb{T} \setminus \rho(\theta),$$

reduces to

$$\varphi(z_2)|\theta(z_1) - \theta(z_2)| = O\left(\omega(|z_1 - z_2|)\right), \qquad z_1, z_2 \in \mathbb{T} \setminus \rho(\theta), \tag{3.7}$$

where we have also used the fact that $|F| = \varphi$ on $\mathbb{T}$. Thus, (i.2) is equivalent to (3.6)&(3.7) (that is, to (3.6) and (3.7) taken together).

A similar argument now enables us to rewrite the condition (3.6)&(3.7) as (ii.2). Indeed, (ii.2) says that $F\theta \in \Lambda_\omega$, which in turn means that

$$\widetilde{Q}(z_1, z_2) := (F\theta)(z_1) - (F\theta)(z_2)$$

is $O\left(\omega(|z_1 - z_2|)\right)$ for z_1, $z_2 \in \mathbb{T} \setminus \rho(\theta)$. A formula similar to (3.5), but with $\widetilde{Q}$ in place of Q and with no bar over θ, convinces us that the required estimate on $\widetilde{Q}(z_1, z_2)$ reduces to (3.6)&(3.7), exactly as before.

We now know that (i.2) $\iff$ (ii.2). That (iii.2) is also equivalent to (3.6)&(3.7), and hence to (i.2) and (ii.2), is verified in very much the same way. Indeed, (iii.2) obviously implies (3.6) and, *a fortiori*, the weaker condition that $\varphi \in \Lambda_\omega$, while the rest follows from the formula

$$(\varphi\theta)(z_1) - (\varphi\theta)(z_2) = [\varphi(z_1) - \varphi(z_2)]\,\theta(z_1) + \varphi(z_2)\,[\theta(z_1) - \theta(z_2)]\,.$$

Finally, the equivalence between (ii.2) and (iv.2) is contained in Theorem 5 of [11]. ◻

Remarks. (1) It was proved by Shirokov that an inner function θ divides (the inner part of) some nontrivial function in $\Lambda_\omega \cap H^\infty$ if and only if

$$\int_{\mathbb{T}} \log \omega\left(\operatorname{dist}(\zeta, \rho(\theta))\right)\, dm(\zeta) > -\infty; \tag{3.8}$$

see [18], Chapter III. Thus, a nontrivial function φ satisfying (i.2)–(iv.2) can only exist if the Carleson-type condition (3.8) is fulfilled.

(2) Under the additional assumption

$$\int_0^\delta \frac{\omega(t)}{t}dt \le \mathrm{const}\cdot\omega(\delta), \qquad 0<\delta<1,$$

the nonnegative functions $\varphi\in\Lambda_\omega$ with the property $\mathcal{O}_\varphi\in\Lambda_\omega$ can be characterized as those satisfying $\log\varphi\in L^1(m)$ and

$$\varphi(z/|z|)-|\mathcal{O}_\varphi(z)|=O\left(\omega(1-|z|)\right);$$

see [9] and [11] for a proof. Consequently, condition (3.4) in (iv.2) is then equivalent to saying that

$$\varphi\left(\frac{z}{|z|}\right)=O(\omega(1-|z|))$$

as $|z|\to1$, $z\in\Omega(\theta,\varepsilon)$.

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