

ON THE NUMBER OF ZEROS OF CERTAIN HARMONIC POLYNOMIALS

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ABSTRACT. Using techniques of complex dynamics we prove the conjecture of Sheil-Small and Wilmshurst that the harmonic polynomial $z - \overline{p(z)}$, $\deg p = n > 1$, has at most $3n - 2$ complex zeros.

1. INTRODUCTION

Let $h(z) := p(z) - \overline{q(z)}$ be a harmonic polynomial of degree $n > 1$ where p, q are analytic polynomials of degree n and m , $m < n$. The following question was raised by T. Sheil-Small [6]: what is the upper bound on the number of zeros of h ? He conjectured that the sharp upper bound was n^2 . His former student A. Wilmshurst has proved this in his thesis [7] by demonstrating the upper bound using Bézout's theorem and also showing by examples that for $m = n, n - 1$ that bound was sharp. Some of Wilmshurst's results were independently discovered by Bshouty et al. [1]. However, for $m < n - 1$ it was suggested in [7] that the upper bound should be much lower, in particular Wilmshurst conjectured that for $m = 1$ the number of zeros of $h(z)$ does not exceed $3n - 2$. The purpose of this note is to prove this result by using certain well-known techniques from complex dynamics. This is not at all surprising since in that case the zeros of h could be thought of as finite fixed points of the mapping $z \rightarrow \overline{p(z)}$ of the Riemann sphere. It must be mentioned that the first author's attention to the problem was drawn by a question posed by D. Sarason, who jointly with B. Crofoot proved in [5] the $3n - 2$ conjecture for $n = 3$ (it is trivial for $n = 2$). Also, Crofoot and Sarason obtained in [5] several intriguing reformulations of the problem in terms of coercive estimates of some linear operators on finite-dimensional spaces.

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2. PRELIMINARIES

As was mentioned before, Wilmshurst's conjecture for $m = 1$ can be reformulated in terms of the fixed points. Namely,

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Theorem 1. *Let $p(z)$, $\deg p = n > 1$, be an analytic polynomial. Then*

$$\#\{z \in \mathbf{C} : \overline{p(z)} = z\} \leq 3n - 2.$$

It is easy to see that the fixed points of $\overline{p(z)}$ are finitely many, since the latter are also fixed points of the function $Q(z) = p(\overline{p(z)})$ which is an analytic polynomial of degree n^2 . In fact, one can also observe (see [7] and [4]) that for any harmonic function $h(z) = p(z) - \overline{q(z)}$, $0 < \deg q < \deg p$, all the zeros are isolated. On the other hand, examples of quadratic polynomials show that the estimate of Theorem 1 is the best possible.

2.1. Facts from complex dynamics. If $q(z)$ is a polynomial, a fixed point $z_0 \in \mathbf{C}$ is attractive, repelling or neutral if, respectively, $|q'(z_0)| < 1$, $|q'(z_0)| > 1$ or $|q'(z_0)| = 1$. A neutral fixed point is *rationally neutral* if $q'(z_0)$ is a root of unity. We shall say that a fixed point z_0 *attracts* some point $w \in \mathbf{C}$ provided that the sequence $q^k(w) = \underbrace{q \circ \dots \circ q}_k(w)$ converges to z_0 . A point ζ is called a *critical point* of q if $q'(\zeta) = 0$.

Fact 1. *If $\deg q > 1$, z_0 is an attracting or rationally neutral fixed point, then z_0 attracts some critical point of q .*

For the proof, see [2], Ch. III, Thms. 2.2 and 2.3.

2.2. The argument principle. A harmonic function $h = f + \overline{g}$, where f and g are analytic functions, is called *sense-preserving* at z_0 if the Jacobian $J_h(z) = |f'(z)|^2 - |g'(z)|^2 > 0$ for every z in some punctured neighborhood of z_0 . We also say that h is *sense-reversing* if \overline{h} is sense-preserving at z_0 . If h is neither sense-preserving nor sense-reversing at z_0 , then z_0 is called *singular* and necessarily (but not sufficiently) $J_h(z_0) = 0$.

Note that for harmonic functions $z - \overline{p(z)}$, $\deg p > 1$, a point z_0 is sense-preserving, reversing or singular if and only if $|p'(z_0)|$ is less than 1, greater than 1 or equal to 1, respectively.

If Γ is an oriented closed curve and F does not vanish on Γ , then the notation $\Delta_\Gamma \arg F(z)$ means the increment of the argument of $F(z)$ along Γ . We will use the following argument principle which is taken from [4]. The referee pointed out that there is a newer and stronger formulation of the principle found in [3].

Fact 2. *Let h be a harmonic function in a finitely-connected domain Ω with a piecewise smooth boundary Γ . Assume that h is continuous in $\overline{\Omega}$ and $h \neq 0$ on Γ . Suppose also that h has no singular zeros in Ω and let N be the number of zeros of h inside Ω , counted with their orders and the positive sign for sense-preserving zeros and negative for the sense-reversing ones. Then,*

$$\frac{1}{2\pi} \Delta_\Gamma \arg h(z) = N.$$

We will apply Fact 2 with Ω equal to the set of sense-preserving points of $h(z) = z - \overline{p(z)}$, and then to the set of sense-reversing points of h intersected with a sufficiently large disk $D(0, R)$, chosen so that $\Gamma \subset D(0, R)$, all zeros of $z - \overline{p(z)}$ are in $D(0, R)$ and the argument change of $z - \overline{p(z)}$ along the circle $C(0, R)$ is $-n$. Then $\frac{1}{2\pi} \Delta_{C(0, R)} \arg h(z) = -n$, where $C(z_0, R)$ denotes the positively oriented circle with the given center and radius.

3. NON-REPELLING FIXED POINTS

We will now prove Theorem 1. Let us start with the following proposition which is of independent interest.

Proposition 1. *If p is a polynomial of degree $n > 1$, then the set of points for which $z = \overline{p(z)}$ and $|p'(z)| \leq 1$ has cardinality at most $n - 1$.*

We consider the function $Q(z) := \overline{p(\overline{p(z)})}$ which is an analytic polynomial of degree n^2 . Notice first that if $|p'(z_0)| = 1$ and $\overline{p(z_0)} = z_0$, then $Q'(z_0) = 1$. This follows by writing $\overline{p(z_0 + z)} = z_0 + e^{i\theta}\overline{z} + O(|z|^2)$ with $\theta \in \mathbf{R}$ and iterating. Thus, all points mentioned in Proposition 1 are fixed points of Q which are either attracting or rationally neutral. So, each of them attracts a critical point of Q by Fact 1.

Lemma 1. *If $\overline{p(z_0)} = z_0$ and $z \in \mathbf{C}$, then $(\overline{p})^k(z) \rightarrow z_0$ iff $Q^k(z) \rightarrow z_0$.*

Proof. The \Rightarrow implication is obvious. For the opposite one, we observe that $Q^k = (\overline{p})^{2k}$ by definition, and $(\overline{p})^{2k}(z) \rightarrow z_0$ implies $(\overline{p})^{2k+1}(z) \rightarrow z_0$ since z_0 is a fixed point. □

Recall that a *grand orbit* under a transformation F is an equivalence class of the relation $x \sim y$ iff $F^p(x) = F^q(y)$ for some $p, q > 0$.

Lemma 2. *If $Q'(c) = 0$, then there are at least $n + 1$ critical points of Q , counted with multiplicities, which all belong to the same grand orbit under \overline{p} .*

Proof. Note that if $p'(\zeta) = 0$, then ζ and all its preimages by \overline{p} are critical points of Q . If $\overline{\zeta}$ is not a critical value, that gives $n + 1$ *distinct* critical points of Q counted with multiplicities. If $p(\zeta_1) = \overline{\zeta}$ and ζ_1 is a critical point of p with multiplicity k , then ζ_1 is a critical point of Q with multiplicity at least $2k + 1$. In any case the sum of multiplicities of critical points of Q over the set $\{\zeta\} \cup \overline{p}^{-1}(\{\zeta\})$ is at least $n + 1$.

The condition $Q'(c) = 0$ implies that either $p'(c)$ or $p'(\overline{p}(c))$ is 0 and Lemma 2 follows from the remark of the previous paragraph applied to either c or $\overline{p}(c)$, respectively. □

Lemma 3. *If $Q'(c) = 0$, $p(z_0) = \overline{z_0}$ and $Q^k(c) \rightarrow z_0$, then there are $n + 1$ critical points of Q , counted with multiplicities, all attracted to z_0 under the iteration of Q .*

Proof. These critical points are obtained from Lemma 2. By Lemma 1, $(\overline{p})^k(c) \rightarrow z_0$, and then the same must occur for every point in its grand orbit. □

As already observed, each point z_0 which satisfies the conditions of Proposition 1 attracts a critical point of Q , but then it attracts $n + 1$ of them. Clearly, different fixed points attract disjoint sets of critical points. Since the degree of Q is n^2 , the total number of its critical points counted with multiplicities is $n^2 - 1 = (n + 1)(n - 1)$ which proves the claim of Proposition 1.

4. PROOF OF THE MAIN THEOREM

For the purpose of this section, we call the polynomial p *regular* provided that the conditions $|p'(z_0)| = 1$ and $\overline{p}(z_0) = z_0$ are not satisfied simultaneously for any $z_0 \in \mathbf{C}$.

Lemma 4. *If p is regular of degree $n > 1$, then there are at most $2n - 1$ points z in the complex plane for which both $\overline{p}(z) = z$ and $|p'(z)| > 1$ are satisfied.*

Proof. Consider the regions Ω_+ where $z - \overline{p}(z)$ is sense-preserving and Ω_- where it is sense-reversing. They are separated by a piecewise oriented analytic curve (a lemniscate) Γ which is the boundary of Ω_+ . In addition, make Ω_-^0 compact by intersecting Ω_- with a large disk $D(0, R)$ chosen so that $\Gamma \subset D(0, R)$, all zeros of $z - \overline{p}(z)$ are in $D(0, R)$ and the argument change of $z - \overline{p}(z)$ along the circle $C(0, R)$ is $-n$. By Fact 2 and Proposition 1, $\Delta_\Gamma(z - \overline{p}(z)) \leq 2\pi(n - 1)$. Hence,

$$\Delta_{C(0,R)} - \Delta_\Gamma \geq -2\pi(2n - 1).$$

Since $C(0, R) - \Gamma$ is the oriented boundary of the region Ω_-^0 , Fact 2 means that $-\frac{1}{2\pi}(\Delta_{C(0,R)} - \Delta_\Gamma)$ is the number of zeros of $z - \overline{p}(z)$ in Ω_- , which is what the lemma claims. \square

From Proposition 1 and Lemma 4 we see that Theorem 1 holds for regular p . Moreover, it also holds on the closure of the set of regular polynomials (with the topology of uniform convergence in the spherical metric). Indeed, a sufficiently small perturbation will not decrease the number of zeros of $z - \overline{p}(z)$ in Ω_- , hence Lemma 4 still holds for p in the closure of the set of regular polynomials.

It remains to see that the set of regular polynomials is dense and we show even more:

Lemma 5. *If $p(z)$ is a polynomial of degree greater than 1, then the set of complex numbers c for which $p(z) - c$ is regular is open and dense in \mathbf{C} .*

Proof. This lemma may be derived from general considerations about algebraic sets. Here we give a simple proof due to D. Sarason.

For a given p , consider the set S which is the image under the transformation $z \rightarrow p(z) - \overline{z}$ of the set $\{z \in \mathbf{C} : |p'(z)| = 1\}$. If $c \notin S$, then $p(z) - c - \overline{z} \neq 0$ whenever $|p'(z)| = 1$, in other words $p(z) - c$ is a regular polynomial. But S is compact with empty interior and hence Lemma 5 follows. \square

Theorem 1 is now proved.

5. FINAL REMARKS

Sharpness of the result. As was mentioned before, easy examples of polynomials of degree 2 and 3 show that when considered for *all* $n > 1$, the estimate of Theorem 1 is sharp. For example, the equation

$$\frac{1}{2}(z^3 - 3z) + \overline{z} = 0$$

has seven roots: $0, \pm 1, \frac{1}{2}(\pm\sqrt{7} \pm i)$, with any combinations of the signs in the last pattern allowed. This realizes the bound $3n - 2$. Moreover, there are two roots ± 1 at which the function is sense-preserving, and five sense-reversing roots realizing the estimates of Proposition 1 and Lemma 4.

However, if Theorem 1 is sharp for every $n \geq 2$ remains to be seen. Along those lines B. Crofoot and D. Sarason [5] raised the following important question.

Conjecture 1. *For $n > 1$ there exist $n - 1$ points z_1, \dots, z_{n-1} and a polynomial p of degree n such that $p(z_j) = \overline{z_j}$ and $p'(z_j) = 0$ for all j .*

If true, this implies that the bound of Proposition 1 is sharp for each n , and then so are the bounds of Lemma 4 and Theorem 1.

On Proposition 1. An analogue of that proposition with $p(z)$ replacing $\overline{p(z)}$ has a cute elementary proof which does not use Fact 1 and is sketched below.

Proposition 2. *Let $p(z)$ be a polynomial of degree $n > 1$. Then, the number of its fixed points with derivative in the set $\overline{D(0,1)} \setminus \{1\}$ is at most $n - 1$.*

Let a_1, \dots, a_n be the fixed points of p . Then

$$p(z) = z + C(z - a_1) \cdots (z - a_n) =: z + q(z)$$

with $C \in \mathbf{C}$, $C \neq 0$. If $p'(a_j) = 1$ for some j , then the claim is obvious, so without loss of generality all a_j are simple fixed points. To see that $|p'(a_j)| > 1$ for some j , it suffices to show that the points $q'(a_j)$ cannot all belong to the closed unit disk centered at -1 with 0 excepted. To this end, we demonstrate that 0 belongs to the convex hull of points $q'(a_j)$, $j = 1, \dots, n$. This follows at once since

$$(1) \quad \sum_{j=1}^n \frac{1}{q'(a_j)} = C_1 \sum_{j=1}^n \operatorname{res} \frac{1}{q(z)} = 0$$

where

$$\frac{1}{q'(a_j)} = \frac{\overline{q'(a_j)}}{|q'(a_j)|^2}$$

and so equality (1) indeed means that 0 can be realized as a convex combination of $q'(a_j)$.

Examples of the form

$$p(z) = (1 + \epsilon)z + z^n, \quad n \geq 2, \quad 0 < \epsilon < \frac{2}{n-1},$$

show that the bound of Proposition 2 is the best possible for any n .

Possible extensions. Wilmschurst has conjectured (see [7]) that for a general $h = p(z) - \overline{q(z)}$, $\deg p = n > m = \deg q > 0$, the maximal number of zeros is $m(m-1) + 3n - 2$. It is not clear whether the ideas of this paper can be extended sufficiently to treat his conjecture. Even in the simplest case $q(z) = z^m$, $1 \leq m \leq n-1$, it immediately requires a profound study of the dynamics of the map $\sqrt[m]{\overline{p(z)}}$ on the Riemann surface. Perhaps, such an investigation will lead to a beginning of a new tale.

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