ON A GEOMETRIC APPROACH TO PROBLEMS CONCERNING CAUCHY INTEGRALS AND RATIONAL APPROXIMATION

by

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Thesis

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VITA

Dmitry Khavinson was born on January 8, 1956 in Moscow, USSR. He received his M.S. in Mathematics from Moscow State Pedagogical Institute in June, 1978. At Brown University, he has been the recipient of a University Fellowship and has served as both a Teaching Assistant and Research Assistant.

To my parents and
my friends

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PREFACE

The problem of approximating continuous functions defined on a compact subset of the complex plane by rational functions has been studied since the classical work of K. Weierstrass and C. Runge.

In this thesis we are approaching questions concerning rational approximation by means of the following geometric idea. As it is well known each compact set can be thought of as an intersection of "nicely" bounded finitely connected compact sets shrinking to the given set. The problems discussed in this thesis are centered on the following general question: how completely can we describe the properties of a given set with regard to rational approximation in terms of a sequence of "nice" sets approximating the given set?

This work consists of 3 parts. In the first chapter we investigate the problem posed in [25] of geometric characterization of the sets whose characteristic functions are Cauchy transforms of a certain measure a.e. with respect to the area measure. Results in Geometric Measure Theory by E. De Giorgi, H. Federer and W. Fleming provide the answer for this problem. We show that such sets are "limits" in a certain sense of a sequence of finitely connected sets with smooth boundaries whose perimeters are uniformly bounded from above. Then, we discuss various questions concerning the integral representation of certain classes of functions on such sets. In particular, we establish the answer of the Cauchy formula for the algebra H^∞ on such sets, where H^∞ is defined as the weak (*)-closure of rational functions in L^∞ .

In the second part we consider general compact sets $X \subset C$. We are interested in describing the measures supported on the boundary of X and annihilating rational functions, (generalization of the F. and M. Riesz Theorem). We introduce the concept of "analytic measures" defined as limits of analytic differentials supported on the boundaries of "nice" sets approximating X . We prove that non-trivial analytic measures always exists as long as there exist continuous functions on X which can not be approximated uniformly by rational functions. We also state and prove the theorem giving a sufficient condition when the linear span of analytic measures is weak (*) dense in the whole annihilator of rational functions. We regard this result as a generalization of F. and M. Riesz theorem.

Finally, in the third chapter we discuss a "localization formula" for the Cauchy transform of an arbitrary measure. This formula allows us to write down the restriction of the Cauchy transform onto any disk as the Cauchy transform of a certain new measure associated with a given measure. We give an explicit formula for this measure and discuss some applications of the localization formula.

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CHAPTER 1

SETS OF FINITE PERIMETER IN THE COMPLEX PLANE

§1. Introductory remarks.

Let μ be a complex Baire measure in $\mathbb{C}$. The Cauchy transform $\hat{\mu}$ of μ is defined by

$$\hat{\mu}(z) = \int_{\mathbb{C}} \frac{d\mu(\zeta)}{\zeta - z} .$$

Clearly, $\hat{\mu}(z)$ is analytic outside of $\operatorname{supp}\mu$. It is well known that the integral defining $\hat{\mu}$ converges absolutely, a.e. and is locally integrable with respect to area measure.

Consider the following well-known example. One defines a Swiss cheese to be a compact set obtained by deleting from the closed unit disk Δ_0 a sequence Δ_j of pairwise disjoint open disks whose radii have a finite sum and whose union is dense in Δ_0. We denote such a set by K.

Let $R(K)$ be a uniform closure of the space of rational functions with poles outside of K.

Let μ be the measure on K defined by

$$\mu|_{\partial\Delta_0} = \frac{1}{2\pi i}dz \quad , \quad \mu|_{\partial\Delta_j} = -\frac{1}{2\pi i}dz \quad , \quad j=1,2, \ldots$$

Then, μ is a finite measure on K and $\mu \perp R(K)$. This is equivalent to the following

$$\hat{\mu}(z) \overset{\text{def}}{=} \int_K \frac{d\mu(\zeta)}{\zeta - z} \equiv 0 \quad , \qquad z \in C\backslash K .$$

(see [24]). Furthermore,

$$\hat{\mu}(z) = \int_K \frac{d\mu(\zeta)}{\zeta - z} \equiv 1 \quad \text{a.e. on } K .$$

In reality, fix any $z \in K$, such that

$$\int_K \frac{d|\mu(\zeta)|}{|\zeta - z|} < +\infty .$$

Then, by monotone convergence theorem we obtain for such z

$$\hat{\mu}(z) = \int_K \frac{d\mu(\zeta)}{\zeta - z} = \lim_{n\to\infty} \sum_{j=0}^{n} \int_{\partial\Delta_j} \frac{d\mu(\zeta)}{\zeta - z} = 1$$

and the statement follows.

This example motivated J. Garnett to pose the following problem in [25].
(A) Describe the sets in $\mathbb{C}$ whose characteristic functions are Cauchy transforms of complex measures almost everywhere with respect to area measure.

It turns out that results of geometric measure theory described below give a complete solution of Garnett's problem. This was first noted by K. Pietz in his thesis [42].

In this section we describe the application of geometric measure theory to Garnett's problem.

One reason for the interest of Garnett's problem is the following observation, made by Garnett in [25].

As usual, two points $z,w \in K$ are said to be in the same Gleason part of the algebra $R(K)$ if

$$\sup\{|f(w)| : f \in R(K) \ , \ \|f\|_\infty \le 1 \ , \ f(z) = 0\} < 1 .$$

It is known, (see [6],[24]), that K can be decomposed as follows: $K = \bigcup_{n\ge 0} P_n$, where P_0 is the set of one-point parts (peak points), and each P_n $n \ge 1$ is a non-trivial part. Moreover, each P_n , $n \ge 1$ has a positive area (see [24]). Let K be a Swiss cheese. As we have shown above $\hat{\mu}(z) = \chi_K$ a.e., where χ_K means the characteristic function of K . By the abstract version of F. and M. Riesz Theorem due to I. Glicksberg and others (see [6], [24]), $\mu = \sum_{n=1}^{\infty} \mu_n$, where $\mu_n \perp R(K)$ for all n , $\mu_n \perp \mu_m$ as $n \neq m$ and $\hat{\mu}_n = \chi_{P_n}$ a.e. So an answer to the problem (A) should yield some information on the structure of the parts P_n .

An unpublished result of J. Wermer gives a precise answer, when the set is the interior of a Jordan curve.

<u>Theorem</u> (J. Wermer). Let U be the region bounded by a Jordan curve Γ . Assume that there exists a measure μ on Γ such that $\hat{\mu}(z) \equiv 1$ for $z \in U$ and $\hat{\mu}(z) \equiv 0$ for $z \notin \Gamma \cup U$. Then, Γ is rectifiable and $\mu = \frac{1}{2\pi i}dz$ on Γ .

For the proof see [25].

At the same time since 1945 H. Federer and later E. De Giorgi and then W. Fleming have been studying the following problem.

(B) Describe sets $E \subset \mathbb{R}^n$ such that there exists a vector-valued measure $(\mu_1, \ldots, \mu_n)$ such that

$$\operatorname{grad} \chi_E = (\mu_1, \ldots, \mu_n)$$

in the distribution sense. In the series of papers [10], [11], [17], [18], [19], [20] they have obtained the complete solution of the problem (B).

Assuming that $n = 2$, let us make the following observation.

Let $E \subset \mathbb{C}$ and $\operatorname{grad} \chi_E = (\mu_1, \mu_2)$. Then,

$$\frac{\partial}{\partial \bar{z}} \chi_E = \frac{1}{2}\left(\frac{\partial}{\partial x}\chi_E + i \frac{\partial}{\partial y}\chi_E\right) = \frac{1}{2}(\mu_1 + i\mu_2) = \mu ,$$

where μ is a complex measure. But, as it is known

$$\frac{\partial}{\partial \bar{z}} f = \mu \iff f = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{d\mu(\zeta)}{\zeta - z}$$

a.e. (see [24]) for any compactly supported $L^1(dxdy)$ function f. So, $\chi_E = -\frac{1}{\pi}\hat{\mu}(z)$. The converse is also true, i.e. if

$$\chi_E = \hat{\mu}(z) \quad \text{a.e.}$$

then

$$\frac{\partial}{\partial \bar{z}}\chi_E = -\pi(\mathrm{Re}\mu + i\ \mathrm{Im}\mu)\ .$$

Hence,

$$\mathrm{grad}\ \chi_E = (-2\pi\mathrm{Re}\mu, -2\pi\ \mathrm{Im}\mu)\ .$$

Thus, the problem (A) is completely equivalent to the problem (B) for $n = 2$. Therefore, translating the corresponding results from the geometric measure theory into the language of complex analysis we obtain the complete answer to the question (A).

To state the theorem combining the results of H. Federer and E. De Giorgi and describing the geometrical structure of the sets satisfying (A) (or (B)) in "complex" terms we have to introduce the concept of exterior normal due to H. Federer (see [19], [21], [23]).

<u>Definition</u>. Let $E \subset \mathbb{C}$ be a measurable bounded set. We say, that E has an <u>exterior normal</u> $n(x)$ at the point x if $|n(x)| = 1$, and letting

$$S(x,r) = \{y \in C : |y-x| < r\}$$
$$S_+(x,r) = \{y \in S(x,r) : \overrightarrow{y-x} \cdot \overrightarrow{n(x)} \geq 0\}$$
$$S_-(x,r) = \{y \in S(x,r) : \overrightarrow{y-x} \cdot \overrightarrow{n(x)} \leq 0\}\ ,$$

we have

$$\lim_{r\to 0^+} \frac{m_2\{S_-(x,r) \cap E\}}{m_2\{S_-(x,r)\}} = 1\ ;$$

$$\lim_{r\to 0^+} \frac{m_2\{S_+(x,r) \cap E\}}{m_2\{S_+(x,r)\}} = 0 .$$

Here, m_2 denotes Lebesgue area measure $dxdy$.

Let $B_E = B(E) = \{x \in \mathbb{C} : E$ has an exterior normal $n(x)$ at $x\}$. Following H. Federer we shall call B_E the reduced boundary of E. We shall view the vector $n(x)$ as an element of $\mathbb{C}$.

The following theorem puts together the general results due to H. Federer and E. De Giorgi (see [11], [19], [21], [23]) in case of 2 dimensions.

Theorem 1.1. Let E be a bounded measurable set in $\mathbb{C}$ satisfying the property (A) (or (B)!), i.e. there exists a measure μ such that $\hat{\mu}(z) = \chi_E$ a.e. Let $P(E) = 2\pi\|\mu\|$ and let m_1 be 1-dimensional Hausdorff measure in $\mathbb{C}$. Then, the following statements hold

(i) B_E is m_1-measurable, $m_1(B_E) < +\infty$ and, moreover,

$$P(E) = m_1(B_E) .$$

(ii) For any Borel set $A \subset \mathbb{C}$,

$$\mu(A) = \mu(A \cap B_E) = \frac{1}{2\pi}\int_{A\cap B_E} n(x)\, d\, m_1(x)$$

(iii) The following holds a.e. with respect to m_1 on B_E:

$$\lim_{r\to 0^+} \frac{m_1\{B_E \cap S(x,r)\}}{2r} = 1 .$$

(iv) B_E is a <u>regular set</u>, i.e. the whole B_E except, maybe, a set of m_1-measure zero is contained in a countable union of rectifiable arcs. Moreover,

$$d\mu|_{B_E} = \frac{1}{2\pi i} d\zeta|_{B_E} .$$

Theorem 1.1 gives a complete answer to the question (A).

<u>Terminology</u>. If E satisfies (A), i.e. $\chi_E = \hat{\mu}(z)$ for a certain measure μ , we shall call E a set of <u>finite perimeter</u>. We shall also call μ a <u>perimeter measure</u>. P(E) is called the <u>perimeter</u> of E .

<u>Note</u>. When this paper was being typed I became aware of the work done by K. Pietz in his thesis [42]. Also, see his papers [43], [44]. He was the first to have observed the equivalence of the problems (A) and (B), and answered J. Garnett's question in the form analogous to Theorem 1.1 using the results in the geometric measure theory. Also, he has obtained many interesting results generalizing J. Wermer's theorem in different directions.

This chapter consists of 4 sections. In §2 we follow the real-variable technique of E. De Giorgi to study the Cauchy transforms (see [10], [11]). In §3 we investigate the generalization of the problem (A). Namely, we consider X such that there exists a <u>real-valued</u> function f supported on X and such that f is a Cauchy transform of a certain measure. In particular, we obtain that under this hypothesis there exists a sequence of finitely connected smoothly bounded sets $\{\Pi_n\}$ "converging inside" X and such that $P(\Pi_n) = \|d\zeta\|_{\partial\Pi_n} \le M < +\infty$ and the Cauchy transform of the measure $\mu =$ weak (*) $\lim_{n\to\infty}(\frac{1}{2\pi i} d\zeta|_{\partial\Pi_n})$ is a real-valued <u>bounded</u> function in $\mathbb{C}$. (Theorem 1.2, Corollary 1.2). As a corollary from Theorem 1.2 we also obtain E. De Giorgi's theorem describing the sets of finite perimeter as "geometric limits" of finitely connected smoothly bounded regions. In §4 we apply the obtained results to certain problems concerning algebras $R(K)$ and $H^\infty(K)$, when K is a compact set of finite perimeter (e.g. K is a Swiss cheese). In particular, we establish the analog of the Cauchy representation formula for $H^\infty(K)$ for any such K.

We also study the question under what hypothesis the function satisfying the Lipschitz condition belongs to the algebra $R(K)$. The answer we obtain is the following (Proposition 1.9). If the function f satisfies the Lipschitz condition, then $f \in R(K)$ if and only if $\frac{\partial f}{\partial \bar z} \equiv 0$ a.e. on K. At the same time we obtain the analog of F. and M. Riesz theorem for the nowhere dense sets of finite perimeter (Proposition 1.10). We want to mention here the papers of A. O'Farrell [14]-[16] containing many interesting results concerning rational approximation of Lipschitz functions in the Lipschitz norm.

Notation. For $p \geq 1$ $L^p = \{f : \mathbb{C} \to \mathbb{C}$, such that $\|f\|_p = (\int_{\mathbb{C}} |f|^p d\, m_2)^{1/p} < \infty\}$.

L^p_{loc}, $p \geq 1$ denotes the space of all complex-valued functions f such that

$$\int_K |f|^p dxdy < \infty$$

for any compact set K in $\mathbb{C}$.

$L^p_0 = \{f : f \in L^p_{loc}$ and $f \equiv 0$ outside of a certain compact set $K\}$.
$C^\infty_0 = \{f : f \in C^\infty \cap L^1_0\}$.

$$\mathrm{Lip}(\alpha) = \{\phi \in C(\mathbb{C}) \; ; \; |f(z)-f(w)| \leq \mathrm{const}|z-w|^\alpha\} .$$

§2. Preliminaries.

Let $\lambda > 0$. Define the operator:

$$W_\lambda : L^1(\mathbb{C}) \to L^1(\mathbb{C}) \cap C^\infty(\mathbb{C})$$

(the convolution with the approximate identity) as follows:

$$W_\lambda f(z) = \frac{1}{\pi\lambda}\int_{\mathbb{C}} e^{\frac{-|z-\zeta|^2}{\lambda}} f(\zeta)\,d\xi d\eta \ , \quad \text{where} \quad \zeta = \xi + i\eta \ .$$

It is known that $\{W_\lambda\}_{\lambda>0}$ forms a contraction semigroup (see [12]).

Proposition 1.1. For any $f \in L_0^1$

$$\lim_{\lambda\to 0}\int_{\mathbb{C}} |W_\lambda f - f|\,dxdy = 0 \ .$$

Proof. Fix $\varepsilon > 0$. Since $f \in L_0^1$, there exists a compact set K such that $f \equiv 0$ on $\mathbb{C}\setminus K$. Put $C(K) = \max|\zeta|$. Let T be a certain rectangle in $\mathbb{C}$ containing K . Then,

$$\int_{\mathbb{C}} |W_\lambda f-f|\,dxdy \le \int_T |W_\lambda f-f|\,dxdy + \int_{\mathbb{C}\setminus T} |W_\lambda f|\,dxdy \ .$$

Furthermore, applying Fubini's Theorem we obtain

$$\int_{\mathbb{C}\setminus T} |W_\lambda f|\,dxdy = \frac{1}{\pi\lambda} \int_{\mathbb{C}\setminus T} dxdy \left| \int_K e^{\frac{-|z-\zeta|^2}{\lambda}} f(\zeta)\,d\xi d\eta \right|$$

$$\leq \frac{1}{\pi\lambda} \int_{\mathbb{C}\setminus T} dxdy \int_K e^{\frac{-|z-\zeta|^2}{\lambda}} |f(\zeta)|\,d\xi d\eta \tag{1.1}$$

$$\leq \frac{1}{\pi\lambda} \int_K |f(\zeta)| \{ \int_{\mathbb{C}\setminus T} e^{\frac{-|z-\zeta|^2}{\lambda}} dxdy \} d\xi d\eta \ .$$

$|z-\zeta|^2 \geq |z|^2 - 2C(K)|z| - C(K)^2$. Hence, choosing T such that $r_0 = \mathrm{dist}(0,\partial T) \geq 4C(K)$ from (1.1) we obtain the following.

$$\int_{\mathbb{C}\setminus T} |W_\lambda f|\,dxdy \leq \frac{e^{C(K)}}{\pi\lambda} \int_{\mathbb{C}\setminus T} e^{-\frac{1}{2}\frac{|z|^2}{\lambda}} dxdy \ . \tag{1.2}$$

Since,

$$\frac{1}{\pi\lambda} \int_{\mathbb{C}\setminus T} e^{-\frac{1}{2}\frac{|z|^2}{\lambda}} dxdy \leq \frac{1}{4} \int_{r_0^2/2\lambda}^{\infty} e^{-1/2t}\,dt = 1/2e^{-r_0^2/\lambda} \to 0$$

as $\lambda \to 0$, we can choose T such that the left hand side in (1.2) is smaller than ε . At the same time, the standard argument with the approximate identity shows that $\exists\ \lambda_0$: for $\lambda < \lambda_0$

$$\int_T |W_\lambda f - f|\,dxdy < \varepsilon \ .$$

From this our proposition follows.

Proposition 1.2. Let $f \in L^1$. Then, $\frac{\partial}{\partial\bar{z}}W_\lambda f \in L^1$. Moreover, $\|\frac{\partial}{\partial\bar{z}}W_\lambda f\|_{L^1}$ increases as $\lambda \downarrow 0$.

Proof. Applying Fubini's theorem we obtain

$$\|\frac{\partial}{\partial\bar{z}}W_\lambda f\|_{L^1} = \int_{\mathbb{C}} |\frac{\partial}{\partial\bar{z}}W_\lambda f|\,dxdy$$

$$= \int_{\mathbb{C}} |\frac{\partial}{\partial\bar{z}} \frac{1}{\pi\lambda}\int_{\mathbb{C}} e^{\frac{-|z-\zeta|^2}{\lambda}} f(\zeta)\,d\xi d\eta dxdy$$

$$\le \frac{1}{\pi\lambda^2}\|f\|_{L^1}\int_{\mathbb{C}} |z-\zeta|\, e^{\frac{-|z-\zeta|^2}{\lambda}}\,d\xi d\eta < +\infty .$$

Furthermore,

$$|\frac{\partial}{\partial\bar{z}}W_{\lambda+\mu}f(z)| = |\frac{\partial}{\partial\bar{z}}W_\lambda W_\mu f(z)| = |W_\lambda \frac{\partial}{\partial\bar{z}}W_\mu f(z)| , \text{ for } \lambda,\mu > 0 .$$

Hence,

$$\|\frac{\partial}{\partial\bar{z}}W_{\lambda+\mu}f(z)\|_{L^1} = \int_{\mathbb{C}} |W_\lambda \frac{\partial}{\partial\bar{z}}W_\mu f(z)| \le \int_{\mathbb{C}} |\frac{\partial}{\partial\bar{z}}W_\mu f| .$$

Proposition is proved.

Definition. For any $f \in L^1$ we define

$$J[f] = \lim_{\lambda\to 0} \int_{\mathbb{C}} \left|\frac{\partial}{\partial \bar z} W_\lambda f\right| dxdy \tag{1.3}$$

According to Proposition 1.2 the limit (maybe, infinite) in (1.3) always exists.

Remark. This is the "complexification" of the similar definition in the E. De Giorgi paper [10], where he deals with grad f instead of $\partial/\partial z$. We also point out H. Federer's book [21].

Proposition 1.3. Let $f \in L^1_0$. Then, the following statements are equivalent.

(i) $f(z) = \int_C \frac{d\mu}{\zeta - z}$, for a certain Borel measure μ with a compact support.

(ii) $\frac{\partial}{\partial \bar z} f = -\pi\mu$ in the distribution sense.

(iii) $J[f] < +\infty$

(iv) For any $\phi \in \mathrm{Lip}(1,\mathbb{C}) \cap L^1_0$, we have

$$\left|\int_{\mathbb{C}} \frac{\partial}{\partial \bar z} \phi\; f\; dxdy\right| \le \mathrm{const}(f) \|\phi\|_{L^\infty} .$$

Further, for each $\phi \in \mathrm{Lip}(1,\mathbb{C}) \cap L^1_0$, the following holds

$$-\int_{\mathbb{C}} \frac{\partial}{\partial \bar z}\phi f\; dxdy = \int_{\mathbb{C}} \phi\; d\mu ,$$

where μ is a measure. Moreover, if one of these conditions holds, then

(v) $J(f) = \pi\|\mu\|$.

(vi) $\exists\{\lambda_n\}_1^\infty : \lambda_n \to 0$ and if we define the measures $\{\mu_n\}_1^\infty$ by

$$\mu_n(B) = -\int_B \frac{\partial}{\partial \bar{z}} W_{\lambda_n} f \, dxdy \ , \text{ for any Borel set } B \ ,$$

then μ_n weak (*) converge to $\pi\mu$.

Proof. (i)<=>(ii) is well known (see [24], [25]). (ii)=>(iii). Applying Fubini's theorem we obtain

$$\int_{\mathbb{C}} \left|\frac{\partial}{\partial \bar{z}} W_\lambda f\right| dxdy = \pi \int_{\mathbb{C}} \left|\frac{1}{\pi\lambda}\int_{\mathbb{C}} e^{\frac{-|z-\zeta|^2}{\lambda}} d\mu(\zeta)\right| dxdy \le \pi\|\mu\| \ .$$

(iii)<=>(iv). Fix $\phi \in L_0^1 \cap \mathrm{Lip}(1,\mathbb{C})$. Then, $\frac{\partial}{\partial \bar{z}}\phi \in L^\infty$. Since $f \in L_0^1$, according to Proposition 1.1 we have

$$\left|\int_{\mathbb{C}} \frac{\partial}{\partial \bar{z}}\phi \ f \ dxdy\right| = \lim_{\lambda\to 0} \left|\int_{\mathbb{C}} \frac{\partial}{\partial \bar{z}}\phi W_\lambda f \ dxdy\right|$$

$$= \lim_{\lambda\to 0} \left| -\int_{\mathbb{C}} \phi \frac{\partial}{\partial \bar{z}} W_\lambda f \ dxdy\right|$$

$$\le \|\phi\|_{L^\infty} \lim_{\lambda\to 0} \int_{\mathbb{C}} \left|\frac{\partial}{\partial \bar{z}} W_\lambda f\right| dxdy \le J[f]\|\phi\|_{L^\infty} \ .$$

This implies that the functional $\frac{\partial}{\partial \bar{z}} f : \mathrm{Lip}(1,\mathbb{C}) \cap L_0^1 \to \mathbb{C}$ defined by

$$\left\langle \frac{\partial}{\partial \bar{z}} f, \phi \right\rangle = -\int_{\mathbb{C}} \frac{\partial}{\partial \bar{z}}\phi \ f \ dxdy \ , \text{ for } \forall\phi \in \mathrm{Lip}(1,\mathbb{C}) \cap L_0^1$$

is bounded with respect to the uniform norm. Therefore, according to the Hahn-Banach theorem it can be extended to a continuous linear functional on $C(K)$, where K is a compact set containing $\operatorname{supp} f$. Then, by F. Riesz's theorem there exists a Borel measure μ : $\operatorname{supp}\mu \subset K$ such that

$$\langle \frac{\partial}{\partial \bar{z}} f, \phi \rangle = \int_K \phi \, d\mu \quad \text{for} \quad \forall \phi \in C(K) .$$

This proves (iv). (iv)<=>(ii). Applying (iv) to an arbitrary function $\phi \in C_0^\infty$, we obtain

$$\int_K \phi \, d\mu = \int_{\mathbb{C}} \phi \, d\mu = - \int_{\mathbb{C}} \frac{\partial}{\partial \bar{z}} \phi \, f \, dm_2 .$$

Hence, $\mu \equiv \frac{\partial}{\partial \bar{z}} f$ in the distribution sense.

Let us now assume that one of the conditions (i)-(iv) holds. For each $\lambda > 0$ and for every Borel set $B \subset \mathbb{C}$ we put

$$\mu^\lambda(B) = \int_B \frac{\partial}{\partial \bar{z}} W_\lambda f \, dxdy .$$

Then, in accordance with (iii), we have

$$\|\mu^\lambda\| \le \|\frac{\partial}{\partial \bar{z}} W_\lambda f(z)\|_{L^1} \le J[f] < +\infty .$$

Hence, $\exists \{\lambda_n\}_1^\infty : \mu^{\lambda_n}$ converge to μ' in the weak (*) topology as

$\lambda_n \downarrow 0$. Since $\|\mu^{\lambda_n}\| \le J[f]$, $\|\mu'\| \le J[f]$. Moreover, for an arbitrary $\phi \in C_0^\infty(\mathbb{C})$ by Proposition 1.1 we have

$$\int_{\mathbb{C}} \phi \, d\mu' = \lim_{n\to\infty} \int_{\mathbb{C}} \phi \, d\mu^{\lambda_n} = \lim_{n\to\infty} \int_{\mathbb{C}} \phi \frac{\partial}{\partial \bar z} W_{\lambda_n} f(z)\, dxdy$$

$$= - \lim_{n\to\infty} \int_{\mathbb{C}} \frac{\partial}{\partial \bar z}\phi \, W_{\lambda_n} f \, dxdy = - \int_{\mathbb{C}} \frac{\partial}{\partial \bar z}\phi f \, dxdy$$

$$= \langle \frac{\partial}{\partial \bar z} f, \phi \rangle .$$

By (ii) , $\frac{\partial f}{\partial \bar z} = - \pi\mu$. So, $\mu \equiv \pi^{-1}\mu'$. Furthermore, in the proof of the implication (ii) => (iii) we actually showed that $\pi\|\mu\| \geq J[f]$. At the same time, $\pi\|\mu\| \le \varliminf_{n\to\infty} \|\mu^{\lambda_n}\| \le J[f]$. Therefore, $\pi\|\mu\| = J[f]$ and the proof is complete.

Proposition 1.4. Let $f \in L_0^1$. If one of the contitions (i)-(iv) in Proposition 1.3 holds, then the measure μ defined as $-\frac{1}{\pi}\frac{\partial}{\partial \bar z} f$ is unique.

Proof. Assuming that there exist two measures μ_1, μ_2 defined by (ii). Using (i), we obtain

$$\int_{\mathbb{C}} \frac{d(\mu_1 - \mu_2)}{\zeta - z} \equiv 0 \quad \text{a.e. in } \mathbb{C} .$$

This implies that $\mu_1 - \mu_2 \equiv 0$ (see [24], [25]).

In the next section we will be using the following simple fact.

Lemma 1.1. Let K be a compact set. Let F be a subset of $\mathbb{C}\backslash K$ such that $d = \text{dist}(K,F) > 0$. If $f \in L^1_0$ and $\text{supp}\, f \subset K$ then $W_\lambda f(z) \to 0$ uniformly on F as $\lambda \downarrow 0$.

Proof. For any $z \in F$, we have

$$|W_\lambda f(z)| \le (\pi\lambda)^{-1} \int_K e^{\frac{-|z-\zeta|^2}{\lambda}} |f(\zeta)|\, d\zeta d\eta \le (\pi\lambda)^{-1} e^{-\frac{d}{\lambda}} \|f\|_{L^1} \to 0$$

uniformly as $\lambda \downarrow 0$.

3. Geometric Approximation.

Proposition 1.5. Let $\{\mu_n\}_1^\infty$ be a sequence of finite Borel measures which converges in the weak (*) topology to the measure μ . For the sake of simplicity we assume that $\hat{\mu}_n \equiv 0$ for all n outside of a fixed compact set K. Then, $\hat{\mu}_n \to \hat{\mu}$ in the weak topology of L^1 .

Proof. Fix $\phi \in L^\infty$. Then

$$\int_K \mu_n \phi = \int_K \phi \int_{\mathbb{C}} \frac{d\mu_n(\zeta)}{\zeta - z} dxdy$$

$$= [\tfrac{1}{z} * \mu_n) * \phi](0) = [\mu_n * (\tfrac{1}{z} * \phi)](0)$$

Since $\frac{1}{z} \in L^1_{loc}$ and $\phi \in L^\infty$, $\frac{1}{z} * \phi$ is continuous in $\overline{C}$. Therefore,

$$\lim_{n\to\infty} \int_K \hat{\mu}_n \phi \, dxdy = \lim_{n\to\infty} \int_K (\tfrac{1}{z} * \phi) d\mu_n = \int_K (\tfrac{1}{z} * \phi) d\mu = \int_K \hat{\mu} \phi \, dxdy$$

and the Proposition is proved.

The following theorem generalizes the result of E. De Giorgi (Corollary 1.1). However, the crucial point of the proof, i.e. the ingenuous geometric construction of the approximating sequence of smoothly bounded finitely connected sets is due to E. De Giorgi (cf. to [10]).

Theorem 1.2. Let X be a bounded measurable set in $\mathbb{C}$. Let us assume that there exists $f \in L_0^1$, $\overline{\text{supp } f} \subset X$, $f \not\equiv 0$ such that $f = \hat{\Psi}(z)$ for a certain measure Ψ and f is real valued. Then, there exists a sequence of finitely connected compact sets $\{\Pi_n\}$ bounded by finitely many smooth Jordan curves such that

(i) $m_2(\Pi_n) \geq \text{const} > 0$ for all n.

(ii) $m_2(\Pi_n \backslash X) \to 0$ as $n \to \infty$.

(iii) If E_δ denotes the set $\{z \in X : |f(z)| > \delta\}$, then $m_2(E_\delta \backslash \Pi_n) \to 0$ as $n \to \infty$.

(iv) $m_1(\partial\Pi_n) \leq \text{const} < +\infty$.

(v) There exists a measure $\mu \not\equiv 0$, $d\mu \equiv \text{weak}(*) \lim_{n\to\infty} \frac{1}{2\pi i} d\zeta|_{\partial\Pi_n}$. Moreover, $\hat{\mu} \equiv 0$ outside of $\overline{\text{supp } f}$ and $0 \leq \hat{\mu} \leq 1$ a.e.

Proof. Without loss of generality we can assume that $\overline{X} \subset \{z : |z| < 1/2\} \overset{\text{def}}{=} D_0$, and $\|f\|_{L^1} = 1$. Let $\mathbb{C}\backslash\overline{\text{supp } f} = \bigcup_{i=1}^{\infty} U_i$, where U_i, $i=1, \ldots$ are connected open sets such that $U_i \cap U_j = \emptyset$ as $i \neq j$. Let $U_i = \bigcup_{j=1}^{\infty} K_i^j$, where $K_i^1 \subset K_i^2 \subset \ldots$ are compact subsets of U_i, $i=1,2, \ldots$. Let us choose two sequences of positive numbers $\{\varepsilon_n\}_1^\infty$, $\{\eta_n\}_1^\infty$ such that $\varepsilon_n \downarrow 0$, $\eta_n \downarrow 0$, $\varepsilon_n > \eta_n$ for all n, $\varepsilon_n < 1/8$ and $\lim_{n\to\infty} \frac{\varepsilon_n}{\eta_n} = 0$.

Construction of Π_n. Fix n. In accordance with Propisition 1.1 and Lemma 1.1 we find $\lambda_n > 0$ such that

$$\int_{\mathbb{C}} |W_{\lambda_n} f - f| \, dxdy < \varepsilon_n \; ;$$

$$\max_{1 \le j \le n} \sup_{z \in K_j^n} |W_{\lambda_n} f(z)| < \frac{\eta_n}{2} \; ; \qquad (1.4)$$

$$|W_{\lambda_n} f(z)| < \frac{\eta_n}{2} \quad \text{for all} \quad z : |z| > \tfrac{1}{2}(z \in D_0) \; .$$

Note, that since f is real-valued, $W_\lambda f$ is also a real-valued function. According to this let us define the C^∞ surface Γ_{λ_n} in $\mathbb{R}^3$ as follows:

$$\Gamma_{\lambda_n} = \{(x,y,\mathcal{Z}) : \mathcal{Z} = W_{\lambda_n} f(z) \quad , \quad \text{where} \quad z = x+iy\} \; .$$

Put

$$D_n = \{(x,y,\mathcal{Z}) \subset \Gamma_{\lambda_n} : x+iy \in D_0\} \; .$$

According to (1.4), on ∂D_0 we have the following

$$|W_{\lambda_n} f(z)| \Big|_{\partial D_0} \le \frac{\eta_n}{2} \; ;$$

Consider the sections of Γ_{λ_n} by the pairs of parallel planes $\{(x,y,\mathcal{Z}) : \mathcal{Z} = \pm\, \theta \; , \; \theta \ge \eta_n\}$. Since Γ_{λ_n} is a C^∞-surface, then applying the Sard theorem we can assume that $D_n \cap \{\;\; = \pm\, \theta\}$ consists of finitely many smooth Jordan curves. Let $\rho_n(\theta)$ denote the total length (i.e. the perimeter) of $D_n \cap \{\;\; = \pm\, \theta\}$.

Put

$$\Gamma^*_{\lambda_n} = \{(x,y,\mathscr{Z}) : |\mathscr{Z}| \geq \eta_n\} \cap \Gamma_{\lambda_n} .$$

Let $d\sigma$ denote the element of a surface area on Γ_{λ_n}. According to a well known relation between the surface area and the lengths of the sections of the surface by planes parallel to the (x,y)-plane, we have

$$\int_{|\theta|>\eta_n} \rho_n(\theta)d\theta = \int_{\Gamma^*_{\lambda_n}} \nu_{\mathscr{Z}}\, d\sigma ,$$

where $\nu_{\mathscr{Z}}$ denotes the projection of a unit normal vector ν to Γ_{λ_n} onto the plane $\{\mathscr{Z} = \theta\}$ (or $\mathscr{Z} = -\theta$). Using an elementary result from the calculus of several variables we obtain

$$\int_{\Gamma^*_{\lambda_n}} \nu_{\mathscr{Z}}\, d\sigma \leq \int_{\Gamma_{\lambda_n}} \nu_{\mathscr{Z}}\, d\sigma = \int_{\mathbb{C}} |\text{grad } W_{\lambda_n} f|\, dxdy \tag{1.5}$$

But

$$|\text{grad } W_{\lambda_n} f| = |(\frac{\partial}{\partial x} W_{\lambda_n} f, \frac{\partial}{\partial y} W_{\lambda_n} f)| = |2\frac{\partial}{\partial \bar{z}} W_{\lambda_n} f| ,$$

since $W_{\lambda_n} f$ is real-valued. Therefore, from (1.5) and Proposition 1.3 we obtain

$$\int_{|\theta|>\eta_n} \rho_n(\theta)d\theta = \int_{\Gamma^*_{\lambda_n}} \nu_{\mathscr{Z}}\, d\sigma \le 2\int_{\mathbb{C}} \left|\frac{\partial}{\partial \bar{z}} W_{\lambda_n} f\right| dxdy \le 2J[f] < \infty .$$

Hence,

$$\int_{|\theta|>\eta_n}^{1/2} \rho_n(\theta)d\theta \le 2J[f] \tag{1.6}$$

In view of (1.6), there exists $\bar{\theta}_n$; $\eta_n < \bar{\theta}_n < 1/2$ such that

$$\rho_n(\bar{\theta}_n) < 2\frac{J[f]}{2(\frac{1}{2} - \eta_n)} = 2\frac{J[f]}{1 - 2\eta_n} ;$$

Let $\Pi(\bar{\theta}_n) = \Pi_n$ be a projection of $\{(x,y\mathscr{Z}) : |\mathscr{Z}| \ge \bar{\theta}_n\} \cap \Gamma_{\lambda_n}$ onto $\mathbb{C}$-plane. Then, according to the definition of Γ_{λ_n} we have

$$|W_{\lambda_n} f(z)| \ge \bar{\theta}_n > \eta_n , \text{ for all } z \in \Pi_n .$$

Moreover,

$$\|d\zeta\|_{\partial\Pi_n} = P(\Pi_n) = \rho(\bar{\theta}_n) \le 2\frac{J[f]}{1 - 2\eta_n} \le \text{const} < +\infty \tag{1.7}$$

for all n . According to (1.4), we have

$$\int_{\Pi_n \setminus X} |W_{\lambda_n} f - f|\,dxdy \le \|W_{\lambda_n} f - f\|_{L^1} < \varepsilon_n .$$

At the same time, by the definition of Π_n we have

$$|W_{\lambda_n} f(z) - f(z)| = |W_{\lambda_n} f| > \eta_n$$

for all $z \in \Pi_n \setminus X$. From the last two inequalities we obtain

$$m_2(\Pi_n \setminus X) \cdot \eta_n < \varepsilon_n , \text{ or } m_2(\Pi_n \setminus X) < \frac{\varepsilon_n}{\eta_n} \to 0$$

$$\text{as } n \to \infty .$$

This proves (ii). Furthermore, according to (1.4) we have

$$\int_{D_0} |W_{\lambda_n} f|\,dxdy \ge \|f\|_{L^1} - \varepsilon_n = 1 - \varepsilon_n \tag{1.8}$$

Since $|W_{\lambda_n} f(z)| < \overline{\theta}_n < \frac{1}{2}$ on $D_0 \setminus \Pi_n$, we have

$$\int_{D_0} |W_{\lambda_n} f|\,dxdy = \int_{\Pi_n} |W_{\lambda_n} f| + \int_{D_0 \setminus \Pi_n} |W_{\lambda_n} f|$$

$$\le \int_{\Pi_n} |W_{\lambda_n} f| + \frac{1}{2} m_2(D_0)$$

$$\le \int_{\Pi_n} |W_{\lambda_n} f| + \frac{1}{2} .$$

Hence, from (1.8) we obtain

$$\int_{\Pi_n} |W_{\lambda_n} f|\,dxdy \geq \frac{1}{2} - \varepsilon_n \geq \text{const} > 0 \tag{1.9}$$

for all n. At the same time, by (1.4) again, we have

$$\int_{\Pi_n} |W_{\lambda_n} f|\,dxdy \leq \int_{\Pi_n} |f|\,dxdy + \varepsilon_n .$$

From this and (1.9) it follows that

$$\lim_{n\to\infty} \int_{\Pi_n} |f|\,dxdy \geq \text{const} > 0 .$$

Therefore, according to absolute continuity of integrals we obtain that $\underline{\lim}_{n\to\infty}\ m_2(\Pi_n) \geq \text{const} > 0$, which proves (i).

Fix $\delta > 0$. For $n : \eta_n < \delta$ we have in accordance with (1.4)

$$\varepsilon_n > \int_{E_\delta \setminus \Pi_n} \left| |W_{\lambda_n} f| - |f| \right| dxdy \geq m_2(E_\delta \setminus \Pi_n)(\delta - \eta_n)$$

(We recall that $|W_{\lambda_n} f| \leq \eta_n$ outside of Π_n). Hence,

$$m_2(E_\delta \setminus \Pi_n) < \frac{\varepsilon_n}{\delta - \eta_n} \to 0 \quad \text{as} \quad n \to \infty .$$

So, we have (iii).

(1.7) verifies (iv). Moreover, since $\frac{1}{2\pi}\|d\zeta\|_{\partial\Pi_n} \le \text{const} < +\infty$, we can choose a subsequence which we also denote by $\{\Pi_n\}$ and such that

$$\frac{1}{2\pi i} d\zeta|_{\partial\Pi_n} \xrightarrow{\text{weak}(*)} \mu .$$

In view of Green's formula applied to Π_n , we have

$$\int_{\mathbb{C}} \overline{\zeta} d\mu = \lim_{n\to\infty} \frac{1}{2\pi i} \int_{\partial\Pi_n} \overline{\zeta} d\zeta = \lim_{n\to\infty} \frac{1}{\pi} \iint_{\Pi_n} dxdy$$

$$= \frac{1}{\pi} \lim_{n\to\infty} m_2(\Pi_n) \ge \text{const} > 0 .$$

So, $\mu \not\equiv 0$. Take any component U_{n_0} of $\mathbb{C}\setminus\overline{\text{supp } f}$. Fix $z_0 \in U_{n_0}$. Then, $\exists N_0$: such that for all $n > N_0$ $z_0 \in K^n_{n_0}$. Therefore, for $n > \max(N_0, n_0)$ in accordance with (1.4) and our construction of $\{\Pi_n\}$ we obtain $K^n_{n_0} \cap \Pi_n = \emptyset$. Hence, we have

$$\hat{\mu}(z_0) = \int_{\mathbb{C}} \frac{d\mu(\zeta)}{\zeta - z_0} = \lim_{n\to\infty} \frac{1}{2\pi i} \int_{\partial\Pi_n} \frac{d\zeta}{\zeta - z} = 0 .$$

So, $\hat{\mu}(z) \equiv 0$ on U_{n_0} . Moreover,

$$\chi_{\Pi_n} \equiv \frac{1}{2\pi i} \int_{\partial \Pi_n} \frac{d\zeta}{\zeta - z} ,$$

By Proposition 1.5 we obtain that $\chi_{\Pi_n} \to \hat{\mu}$ in the weak topology of L^1. Then, there exists a sequence of the convex combinations of χ_{Π_n} converging to $\hat{\mu}$ in the strong (normed) topology of L^1. Taking a subsequence if necessary, we can assume that the convex combinations of $\{\chi_{\Pi_n}\}_1^\infty$ converge to $\hat{\mu}$ a.e. This implies that $0 \le \hat{\mu} \le 1$ a.e. in $\mathbb{C}$ and (v) has been verified. Thus, the proof of Theorem 1.2 is complete.

Corollary 1.1. (E. de Giorgi, see [10]). Let X be a set of finite perimeter, i.e. $\chi_X = \hat{\mu}(z)$ a.e. where μ is a certain measure. Then, there exists a sequence $\{\Pi_n\}_1^\infty$ of finitely connected smoothly bounded compact sets such that

(i) $m_2(\Pi_n \Delta X) = m_2(\Pi_n \backslash X) + m_2(X \backslash \Pi_n) \to 0$ as $n \to \infty$.

(ii) $\| d\zeta|_{\partial \Pi_n} \| \le \text{const} < +\infty$ for all n.

(iii) $\chi_{\Pi_n} \to \chi_X$ a.e.

(iv) $\frac{1}{2\pi i} d\zeta|_{\partial \Pi_n} \to d\mu$ in the weak (*) topology. Moreover,

$$\| d\zeta|_{\partial \Pi_n} \| \to 2\pi \|\mu\| \overset{\text{def}}{=} P(X) = 2J[\chi_X] .$$

Remark. In [17], [18] W. Fleming has proved much stronger result. Namely, he showed that if X is a set of finite perimeter then there exists a sequence $\{\Pi'_n\}_1^\infty$ as above satisfying the properties (i)-(iv) of Corollary 1.1 and, moreover, such that

$$\|d\zeta|_{\partial\Pi'_n} - 2\pi d\mu\| \to 0 \quad \text{as} \quad n \to \infty .$$

In other words the sequence $\{\Pi'_n\}$ stabilizes in a much stronger sense than the sequence $\{\Pi_n\}$ in Corollary 1.1.

Proof of Corollary 1.1. Let us apply Theorem 1.2 to the function χ_X. Let $\{\Pi_n\}_1^\infty$ be a sequence of finitely connected smoothly bounded compact sets guaranteed by Theorem 1.2. According to pp. (i) and (iii) of Theorem 1.2, we have

$$m_2(\Pi_n \Delta X) = m_2(\Pi_n \setminus X) + m_2(X \setminus \Pi_n) \to 0$$

as $n \to \infty$, since $X = \{z : \chi_X(z) = 1\}$. This means that $\chi_{\Pi_n} \to \chi_X$ in the strong topology of L^1. Taking a subsequence if necessary, we can assume that $\chi_{\Pi_n} \to \Pi_X$ a.e. Let μ' be a weak (*) limit of the sequence of measures $\frac{1}{2\pi i} d\zeta|_{\partial\Pi_n}$. According to Proposition 1.5 we obtain that $\widehat{\frac{1}{2\pi i} d\zeta|_{\partial\Pi_n}}(z) \to \hat{\mu}'(z)$ in the weak topology of L^1. But $\widehat{\frac{1}{2\pi i} d\zeta|_{\partial\Pi_n}}(z) \equiv \chi_{\Pi_n} \to \chi_X$ a.e. Hence,

$$\chi_X = \hat{\mu}(z) = \hat{\mu}'(z) \quad \text{a.e.}$$

So, by Proposition 1.4, we obtain that $\mu \equiv \mu'$. In view of the construction of Π_n in the proof of Theorem 1.2 (see the formula (1.7)), we have

$$\|d\zeta|_{\partial\Pi_n}\| \le 2 \cdot \frac{J[\chi_X]}{1 - 2\eta_n}$$

Therefore, $\lim_{n\to\infty} \|d\zeta|_{\partial\Pi_n}\| \le 2 \cdot J[\chi_X]$. By p. (v) in Proposition 1.3 $J[\chi_X] = \pi\|\mu\|$. So, we obtain $\|d\zeta|_{\partial\Pi_n}\| \le 2\pi\|\mu\|$. On the other hand, since $\chi_{\Pi_n} \to \chi_X$ a.e. we have for each $\lambda > 0$

$$\lim_{n\to\infty} \frac{\partial}{\partial \bar{z}} W_\lambda \chi_{\Pi_n} = \frac{\partial}{\partial \bar{z}} W_\lambda \chi_X \quad \text{for all} \quad z .$$

From Propositions 1.2 and 1.3 we conclude

$$\|d\zeta|_{\partial\Pi_n}\| \ge 2 \int_{\mathbb{C}} \left|\frac{\partial}{\partial \bar{z}} W_\lambda \chi_{\Pi_n}\right| dxdy .$$

From this, using P. Fatou's lemma we obtain:

$$\begin{aligned}\lim_{n\to\infty} \|d\zeta|_{\partial\Pi_n}\| &\ge 2 \lim_{n\to\infty} \int_{\mathbb{C}} \left|\frac{\partial}{\partial \bar{z}} W_\lambda \chi_{\Pi_n}\right| dxdy \\ &\ge \int_{\mathbb{C}} \lim_{n\to\infty} \left|\frac{\partial}{\partial \bar{z}} W_\lambda \chi_{\Pi_n}\right| dxdy \\ &= 2\left\|\frac{\partial}{\partial \bar{z}} W_\lambda \chi_X\right\|_{L^1} .\end{aligned}$$

This inequality holds for each $\lambda > 0$. Therefore, letting $\lambda \downarrow 0$, we conclude that

$$\lim_{n\to\infty} \|d\zeta|_{\partial\Pi_n}\| \ge 2J[\chi_X] = 2\pi\|\mu\| .$$

Corollary 1.1 is proved.

Corollary 1.2. Let f satisfy the hypothesis of Theorem 1.2 and let $X = \overline{\operatorname{supp} f}$. Then, there exists a measure $\mu = \text{weak } (*) \lim_{n\to\infty}(\frac{1}{2\pi i}d\zeta|_{\partial\Pi_n})$, where $\{\Pi_n\}_1^\infty$ are the same as in Theorem 1.2, such that $\mu \perp R(X)$ and $0 \le \hat{\mu}(x) \le 1$ a.e. on X.

Proof. Since $\hat{\mu}(z) \equiv 0$ on $\mathbb{C}\backslash X$, $\mu \perp R(X)$ (see [24], [25]). By (v) of Theorem 1.2, $0 \le \hat{\mu} \le 1$ a.e.

§4. Applications to the algebra $R(K)$.

We start out with a simple proposition which we will make use of later in this section.

Proposition 1.6. Let E be a set of finite perimeter, i.e. $\chi_E = \hat{\mu}(z)$, where μ is a measure. Let $z_0 \in \partial E$. Assume that for any disk $S(z_0,r)$ $m_2\{S(z_0,r) \cap (\mathbb{C}\backslash E)\} > 0$. If there exists a sequence $\{z_n\}_1^\infty$ such that $z_n \to z_0$ and $\hat{\mu}(z_n) = 1$, then $z_0 \in \overline{B(E)}$.

Proof. Suppose $z_0 \notin \overline{B(E)}$. Then, we can find a disk $S(z_0,r_0)$ such that $S(z_0,r_0) \cap \overline{B(E)} = \emptyset$. Therefore the function

$$\chi_E(z) = \int_{B(E)} \frac{d\mu(\zeta)}{\zeta - z}$$

is continuous in $S(z_0,r_0)$. According to our assumption $\exists\{z_n'\} : z_n' \in S(z_0,r_0)$, $z_n' \to z_0$ and $\hat{\mu}(z_n') = 0$. Hence, $\hat{\mu}(z_0) = \lim_{n\to\infty} \hat{\mu}(z_n') = 0$. On the other hand, $\hat{\mu}(z_0) = \lim_{n\to\infty} \hat{\mu}(z_n) = 1$. This contradiction proves our proposition.

Let us recall the following definition (see [6]).

Definition. A compact set K is called an essential set for the algebra $R(K)$ if K coincides with the closure of the union of the supports of all annihilating measures for $R(K)$.

Theorem 1.3. Let K be a compact set of finite perimeter, i.e. $\chi_K = \hat{\mu}(z)$ where μ is a Borel measure. Then, the following statements hold

(i) Each non-trivial Gleason part P_n , $n \geq 1$ of $R(K)$ is a set of finite perimeter, i.e. $\chi_{P_n} \equiv \hat{\mu}_n$ a.e. where $d\mu_n = \frac{1}{2\pi i} d\zeta|_{B(P_n)}$.

(ii) $m_1(B(P_n) \cap B(P_m)) = 0$ as $n \neq m$. Moreover, $B(K) = \bigcup_{n=1}^{\infty} B(P_n) \pm A$, where $m_1(A) = 0$, $\mu = \sum_{n=1}^{\infty} \mu_n$, and the series converges in norm.

(iii) All statements of Theorem 1.1 and Corollary 1.1 hold for each non-trivial Gleason part P_n , $n \geq 1$.

(iv) $m_2(P_0) = 0$, where P_0 is a set of all peak points for $R(K)$.

(v) Each point $z_0 \in B(K)$ is a peak point. If K is an essential set for $R(K)$, then $P_0 \subset \overline{B(K)}$.

Remark. It can occur that $P_0 \neq \overline{B(K)}$. For an example, let K be a Swiss cheese. Then $B(K) = \bigcup_{j=0}^{\infty} \partial\Delta_j \pm A$, where $m_1(A) = 0$ and $\overline{B(K)} = K \neq P_0$.

Proof. (i)-(iv). Since $\hat{\mu} \equiv 0$ on $\mathbb{C}\backslash K$, $\mu \perp R(K)$. Then, according to the abstract version of F. and M. Riesz theorem due to I. Glicksberg (see [6], [24]), $\mu = \sum_{n=1}^{\infty} \mu_n$, where $\mu_n \perp \mu_m$, $\mu_n \perp R(K)$, $|\mu_n|(P_m) = 0$ as $n \neq m$ and the series converges in norm. Moreover, $\hat{\mu}_n|_{P_m} = 0$ a.e. if $m \neq n$. In fact, assume there exists $E \subset P_m$, $m_2(E) > 0$ and $\hat{\mu}_n|_E \neq 0$. Then, for almost all $z \in P_m$, there exists a representing measure m which is absolutely continuous with respect to μ_n (see [24], Ch. II, VI). At the same time, according to I. Glicksberg theorem (see [24], Ch. VI Theorem 5.4), there exist Borel sets $\{E_n\}_1^\infty$ such that $E_n \cap E_m = \emptyset$, $m \neq n$ $E_n \supset P_n$, $\mu_n = \mu_{E_n}$ for all $n \geq 1$ and each representing measure for the points in P_n is concentrated on E_n . Thus, m is concentrated on E_m . Hence, μ_n is also concentrated on E_m . But μ_n is concentrated on E_n . We arrived at the contradiction. So, $\hat{\mu}_n|_{P_m} \equiv 0$ a.e. for each $m \neq n$. Therefore, $\hat{\mu}_n \equiv 0$ on $(C\backslash K) \cup (\bigcup_{\substack{m=1 \\ m\neq n}}^{\infty} P_m)$ and $\hat{\mu}_n \equiv 1$ on P_n a.e. with respect to Lebesgue measure. Let $z_0 \in K$ be any point such that $\chi_K(z_0) = \hat{\mu}(z_0)$ and $\int \frac{d|\mu|}{|\zeta - z_0|} < \infty$. Then, according to the Wilken theorem (see [24]), $\frac{1}{\zeta - z_0} d\mu(\zeta)$ is a complex representing measure for $R(K)$ at z_0 . Since $\frac{1}{\zeta - z_0} d\mu(\zeta)$ is not a point-mas measure, $z_0 \notin P_0$. (see [24], E. Bishop's criterion for the peak points.) Therefore, $m_2(P_0) = 0$. This proves (iv). Moreover, $\hat{\mu}_n \equiv \chi_{P_n}$ a.e., $n \geq 1$. Thus, for each $n \geq 1$ P_n is a set of finite perimeter. Applying Theorem 1.1 we immediately obtain (i)-(ii). (iii) follows from (i). According to the definition of $B(K)$ we have for each $z_0 \in B(K)$

$$\overline{\lim_{r\to 0^+}} \frac{m_2(S(z_0,r)\backslash K)}{\pi r^2} = 1/2 > 0 .$$

Then, applying P. Curtis's criterion for the peak points (see [24]), we obtain that $z_0 \in P_0$. Finally, let us assume that K is an essential set for $R(K)$. Take $z_0 \notin B(K)$. The same argument as in the proof of Proposition 1.6 shows that there exists a neighborhood U of z_0 such that $\hat{\mu} \equiv 1$ or $\hat{\mu} \equiv 0$ on U . If $\hat{\mu}|_U \equiv 1$, then $z_0 \in \mathring{K}$ and, therefore, $z_0 \notin P_0$. Assume $\hat{\mu}|_U \equiv 0$. Then, $m_2(K \cap \overline{U}) = 0$ $(\hat{\mu} = \chi_K !)$. By a standard argument based on Bishop's localization theorem (Theorem 10.3, Ch. II in [24]) it follows that

$K \cap U = \emptyset$. Thus, $z_0 \in U \subset \mathbb{C}\backslash K$. This proves (v). Theorem 1.3 is now proved.

Corollary 1.3. Let K be a Swiss cheese. Then $B(K) = \bigcup_{j=0}^{\infty}(\partial\Delta_j) \pm A$, where $m_1(A) = 0$ and all statements of Theorem 1.3 hold.

The following proposition gives an estimate on the distribution of the deleted disks for the Swiss cheeses.

At first, let us introduce some more notation. Let K be a Swiss cheese. Take $z_0 \in \partial\Delta_{j_0}$. Let $\rho > 0$. We denote by $\Delta^{(j)}_{z_0,\rho}$, j=1, ... all Δ_j such that $S(z_0,\rho) \cap \Delta_j \neq \emptyset$. If z_{j_0} is a center of Δ_{j_0} and r_{j_0} is its radius, then we denote by $\Delta^{(j)}_{j_0,\rho}$, j=1, ... all Δ_j such that $\Delta_j \cap S(z_{j_0},r_{j_0}+\rho) \neq \emptyset$.

Proposition 1.7. Let K be a Swiss cheese. Then the following statements hold.

(i) For all j_0 and for m_1- almost all $z_0 \in \partial\Delta_{j_0}$

$$\lim_{\rho\to 0^+} \frac{m_2\left\{\left(\bigcup_{j=1}^{\infty}\Delta^{(j)}_{z_{j_0},\rho}\right) \cap S(z_0,\rho)\right\}}{\rho^2} = 0 . \tag{1.10}$$

Moreover, for all j_0

$$\lim_{\rho\to 0^+} \frac{m_2\left\{\left(\bigcup_{j=1}^{\infty}\Delta^{(j)}_{j_0,\rho}\right) \cap S(z_{j_0}, r_{j_0}+\rho)\right\}}{\rho} = 0 . \tag{1.11}$$

(ii) Let $z_0 \in \Delta_{j_0}$ belong to $B(P_n)$, where P_n , $n \geq 1$ is a non-trivial Gleason part for $R(K)$. Then

$$\lim_{\rho\to 0^+} \frac{m_2\{S(z_0,\rho) \cap P_n\}}{1/2\pi\rho^2} = 1 .$$

Proof. (1.10) follows directly from the definition of $B(K)$ and the fact that m_1-almost all $z \in \partial\Delta_j$ belong to $B(K)$. Let $A(\rho) = S(z_{j_0}, r_{j_0}+\rho)\setminus\Delta_{j_0}$. Suppose that (1.11) does not hold. Then,

$$\overline{\lim_{\rho\to 0^+}} \frac{m_2\left\{\left(\bigcup_{j=1}^{\infty}\Delta^{(j)}_{j_0,\rho}\right) \cap S(z_{j_0}, r_{j_0}+\rho)\right\}}{\pi(2r_{j_0}+\rho)\rho} \geq \varepsilon_0 > 0 .$$

Since $m_2\{A(\rho)\} = \pi(2r_{j_0}+\rho)\rho$ we obtain that

$$\underline{\lim_{\rho\to 0^+}} \frac{m_2\{K \cap A(\rho)\}}{m_2\{A(\rho)\}} \leq 1 - \varepsilon_0 .$$

For the sake of simplicity let us assume that $j_0 > 0$ and z_{j_0} = center of $\Delta_{j_0} = 0$. Define a function $\phi \in \mathrm{Lip}(1) \cap L_0^1$ such that

$$\phi|_{A(\rho)} = -\frac{1}{\rho}\,\overline{\zeta} + \frac{2r_{j_0}}{\zeta} + \frac{r_{j_0}^2}{\zeta\rho}$$

and

$$\phi|_{\Delta_0\setminus S(0,r_{j_0}+\rho)} = -\frac{\rho}{\zeta} .$$

It is possible, since

$$\phi\Big|_{|\zeta|=r_{j_0}+\rho} = -\frac{(r_{j_0}+\rho)^2}{\rho\zeta} + \frac{2r_{j_0}}{\zeta} + \frac{r_{j_0}^2}{\zeta\rho} = -\frac{\rho}{\zeta} ;$$

Note, that

$$\phi\Big|_{|\zeta|=r_{j_0}} = -\frac{r_{j_0}^2}{\rho\zeta} + \frac{2r_{j_0}}{\zeta} + \frac{r_{j_0}^2}{\rho\zeta} = \frac{2r_{j_0}}{\zeta} .$$

Then, $\frac{\partial\phi}{\partial\bar{\zeta}}\Big|_{A(\rho)} = -\frac{1}{\rho}$ a.e. and $\frac{\partial\phi}{\partial\bar{\zeta}}\Big|_{K\setminus A(\rho)} \equiv 0$ a.e. ($\phi = -\frac{\rho}{\zeta}$ is analytic on $\Delta_0 \setminus S(0,r_{j_0}+\rho)$). According to Theorem 1.1 and Proposition 1.3, we have

$$\int_K \frac{\partial\phi}{\partial\bar{\zeta}} dxdy = \frac{1}{2i}\int_{B(K)} \phi(\zeta)d\zeta = \frac{1}{2i}\int_{\partial\Delta_{j_0}} \phi(\zeta)d\zeta$$
$$+ \frac{1}{2i}\int_{B(K)\cap A(\rho)\setminus\partial\Delta_{j_0}} \phi(\zeta)d\zeta + \frac{1}{2i}\int_{B(K)\setminus(B(K)\cap A(\rho))} \phi(\zeta)d\zeta . \qquad (1.12)$$

Since $\bigcap_{\rho>0} A(\rho) = \partial\Delta_{j_0}$, we can choose ρ_0 so small that for $\rho < \rho_0$ $m_1\{B(K) \cap A(\rho)\setminus\partial\Delta_{j_0}\} < \delta_0$, where $\delta_0 > 0$ is an arbitrary fixed positive number. At the same time,

$$\Big|\frac{1}{2i}\int_{B(K)\setminus(B(K)\cap A(\rho))} \phi(\zeta)d\zeta\Big| \le \text{Const} \cdot \rho = O(\rho)$$

in view of our definition of ϕ . Moreover,

$$\frac{1}{2i}\int_{\partial\Delta_{j_0}} \phi(\zeta)d\zeta = -\frac{1}{2i}\int_D^{2\pi} \frac{2r_{j_0}}{r_{j_0}e^{i\theta}r_{j_0}} i\, e^{i\theta}d\theta = -2\pi r_{j_0} .$$

Therefore, for all $\rho < \rho_0$ the right hand side of (1.12) differs from $-2\pi r_j$ at most by $\frac{1}{2}\|\phi\|\delta_0 + O(\rho)$. But, according to our assumption $\exists\{\rho_n\} : \rho_n \downarrow 0$ such that

$$\int_K \frac{\partial\phi}{\partial\bar\zeta}dxdy = -\frac{1}{\rho_n}\int_{K\cap A(\rho_n)} dxdy \geq -(1-\varepsilon_0)\pi(2r_{j_0}+\rho_n) \to -(1-\varepsilon_0)2\pi r_{j_0}$$

Hence, choosing ρ_n and δ_0 sufficiently small we obtain a contradiction with (1.12). The proof of (i) is complete.

(ii) follows directly from the definition of $B(P_n)$, $n \geq 1$. Proposition is proved.

Before we move to the last topic of this section, namely, integral representations on the sets of finite perimeter we want to pose a problem which we have been unable to solve.

First, let us give the following definition.

Definition. Let E be a set of finite perimeter. We say that E is *decomposible* if there exist sets E_1, E_2 of finite perimeter, such that

$$m_2((E_1 \cup E_2) \,\Delta\, E) = 0 \ , \ m_2(E_1 \cap E_2) = 0 \ ,$$

$$B(E) = B(E_1) \cup B(E_2) \pm A \quad , \quad m_1(A) = 0 \ ,$$

$$\text{and} \quad m_1\{B(E_1) \cap B(E_2)\} = 0 \ .$$

Conjecture. Let X be a compact set of finite perimeter. Then, each non-trivial Gleason part P_n, $n \geq 1$ of $R(X)$ is not decomposible.

For the last part of this chapter we need the following well known result.

Lemma 1.2. Let X be a finitely connected compact set with a smooth boundary Γ consisting of finitely many Jordan curves. Let $\phi \in C^1(\mathbb{C})$. Then,

$$\frac{1}{2\pi i}\int_{\Gamma} \frac{\phi(\zeta)}{\zeta - z}d\zeta - \frac{1}{\pi}\iint_{X} \frac{1}{\zeta - z}\frac{\partial\phi}{\partial\bar{\zeta}}dxdy = \begin{cases} \phi(z) \ , \text{ if } \ z \in \overset{\circ}{X} \ , \\ 0 \ , \text{ if } \ z \notin X \ . \end{cases}$$

Proposition 1.8. Let X be a compact set of finite perimeter. Then, for each $\phi \in \mathrm{Lip}(1,\mathbb{C})$ the following holds

$$\frac{1}{2\pi i}\int_{B(X)} \frac{\phi(\zeta)}{\zeta - z}d\zeta - \frac{1}{\pi}\iint_{X} \frac{\partial\phi}{\partial\bar{\zeta}}\frac{1}{\zeta - z}dxdy = \begin{cases} \phi(z) \quad \text{a.e. on } \ X \\ 0 \ , \ z \in \mathbb{C}\backslash X \ . \end{cases} \qquad (1.13)$$

Proof. At first let us assume that $\phi \in C^1(\mathbb{C})$. Let $\{\Pi_n\}_1^\infty$ be a sequence of finitely connected compact sets with smooth Jordan boundaries converging to X and satisfying (i)-(iv) of Corollary 1.1 and in addition

$$\| d\zeta|_{\partial\Pi_n} - d\zeta|_{B(X)} \| \to 0 \qquad \text{as} \quad n \to \infty$$

The existence of such a system follows from W. Fleming's theorem (see [18] and remark after Corollary 1.1). Let $z_0 \in \mathbb{C}\setminus X$ be such that $\chi_{\Pi_n}(z_0) \to 0$ as $n \to \infty$. Hence, $\exists n_0 : \forall n > n_0$, $z_0 \notin \Pi_n$. Then according to Lemma 1.2 we have

$$\frac{1}{2\pi i}\int_{\partial\Pi_n} \frac{\phi(\zeta)d\zeta}{\zeta - z_0} - \frac{1}{\pi}\iint_{\Pi_n} \frac{\partial\phi}{\partial\bar{\zeta}} \frac{1}{\zeta - z_0} dxdy = 0 \text{ , as } n > n_0 \text{ .}$$

Since $\frac{\phi(\zeta)}{\zeta - z_0}$ is continuous near X, $\frac{1}{2\pi i}d\zeta|_{\partial\Pi_n} \to \frac{1}{2\pi i}d\zeta|_{B(X)}$ and $\chi_{\Pi_n} \to \chi_X$ a.e., we obtain

$$\frac{1}{2\pi i}\int_{B(X)} \frac{\phi(\zeta)d\zeta}{\zeta - z_0} - \frac{1}{\pi}\iint_{\mathbb{C}} \chi_X \frac{\partial\phi}{\partial\bar{\zeta}} \frac{1}{\zeta - z} dxdy$$

$$= \lim_{n\to\infty}\left\{\frac{1}{2\pi i}\int_{\partial_n} \frac{\phi(\zeta)d\zeta}{\zeta - z_0} - \frac{1}{\pi}\iint_{\mathbb{C}} \chi_{\Pi_n} \frac{\partial\phi}{\partial\bar{\zeta}} \frac{1}{\zeta - z_0} dxdy\right\} =$$

$$= 0 \text{ .}$$

So, (1.13) holds for almost all $z \in \mathbb{C}\setminus X$. Since $\mathbb{C}\setminus X$ is open and the left-hand side of (1.13) is continuous on $\mathbb{C}\setminus X$, (1.13) holds for all $z \in \mathbb{C}\setminus X$. Now fix $z_0 \in X$ satisfying the following conditions: (a) $\chi_{\Pi_n}(z_0) \to 1$ as $n \to \infty$ (b) $\int_{B(X)} \frac{|d\zeta|}{|\zeta - z_0|} < \infty$ (c) $\frac{1}{2\pi i}\int_{B(X)} \frac{d\zeta}{\zeta - z_0} = 1$. Clearly, (a), (b), (c) hold for almost all $z_0 \in X$. By (a) $\exists n_0 : \forall n > n_0$, $z_0 \in \mathring{\Pi}_n$. Then, applying Lemma 1.2 again we have

$$\phi(z_0) = \frac{1}{2\pi i}\int_{\partial\Pi_n} \frac{\phi(\zeta)}{\zeta - z_0}d\zeta - \frac{1}{\pi}\iint_{\Pi_n} \frac{\partial\phi}{\partial\bar{\zeta}}\frac{1}{\zeta - z_0}dxdy$$

as $n > n_0$. Since $\chi_{\Pi_n} \to \chi_X$ a.e. and $\frac{1}{\zeta - z_0} \in L^1_{loc}$ we obtain

$$\frac{1}{\pi}\iint_{\Pi_n} \frac{\partial\phi}{\partial\bar{\zeta}}\frac{1}{\zeta - z_0}dxdy \to \frac{1}{\pi}\iint_X \frac{\partial\phi}{\partial\zeta}\frac{1}{\zeta - z_0}dxdy \tag{1.14}$$

as $n \to \infty$. Note, that $\left|\frac{\phi(\zeta) - \phi(z_0)}{\zeta - z_0}\right| \le K$ for $\zeta \in \mathbb{C}\setminus\{z_0\}$, where K is a constant. Then, using properties (b) and (c) of z_0 we obtain for $n > n_0$

$$\left|\frac{1}{2\pi i}\int_{B(X)} \frac{\phi(\zeta)d\zeta}{\zeta - z_0} - \frac{1}{2\pi i}\int_{\partial\Pi_n} \frac{\phi(\zeta)d\zeta}{\zeta - z_0}\right|$$

$$= \left|\frac{1}{2\pi i}\int_{B(X)} \frac{\phi(\zeta) - \phi(z_0)}{\zeta - z_0}d\zeta - \frac{1}{2\pi i}\int_{\partial\Pi_n} \frac{\phi(\zeta) - \phi(z_0)}{\zeta - z_0}d\zeta\right|$$

$$\le \text{const}\left\| d\zeta|_{B(X)} - d\zeta|_{\partial\Pi_n}\right\| \to 0$$

as $n \to \infty$. Therefore, by (1.14)

$$\phi(z_0) = \lim_{n\to\infty}\left\{\frac{1}{2\pi i}\int_{\partial\Pi_n} \frac{\phi(\zeta)}{\zeta - z_0}d\zeta - \frac{1}{\pi}\iint_{\Pi_n} \frac{\partial\phi}{\partial\bar{\zeta}}\frac{1}{\zeta - z_0}dxdy\right\}$$

$$= \frac{1}{2\pi i}\int_{B(X)} \frac{\phi(\zeta)d\zeta}{\zeta - z_0} - \frac{1}{\pi}\iint_X \frac{\partial\phi}{\partial\bar{\zeta}}\frac{1}{\zeta - z_0}dxdy .$$

This proves (1.13) for $\phi \in C^1$. Assume now, that $\phi \in \mathrm{Lip}(1,C)$. As it is well-known, there exists a sequence $\{\phi_n\}_1^\infty$ $\phi_n \in C_0^1$ such that $\|\phi_n - \phi\|_{C(X)} \to 0$ and $\|\frac{\partial \phi_n}{\partial \bar z}\|_{L^\infty} \le M < +\infty$ (it suffices to take $\phi_n = \phi * \Psi_{\varepsilon_n}$ where Ψ_ε is any approximate identity, $\varepsilon_n \to 0$). Taking a subsequence if necessary we can assume that $\frac{\partial \phi_n}{\partial \bar z} \to \frac{\partial \phi}{\partial \bar z}$ in the weak (*) topology of L^∞. In reality, $\frac{\partial \phi_n}{\partial \bar z} \to \frac{\partial \phi}{\partial \bar z}$ in the distribution sense, at the same time, there exists a subsequence converging weak (*) to a function ϕ_0 in L^∞. Then, ϕ_0 must equal to $\frac{\partial \phi}{\partial \bar z}$ a.e. Take $z_0 \in X$ such that the conditions (a)-(c) are satisfied. Then, using the fact that we have proved (1.13) for C^1-functions, we have

$$\phi(z_0) = \lim_{n\to\infty} \phi_n(z_0) = \lim_{n\to\infty} \left\{ \frac{1}{2\pi i} \int_{B(X)} \frac{\phi_n(\zeta)}{\zeta - z_0} d\zeta - \frac{1}{\pi} \iint_X \frac{\partial \phi_n}{\partial \bar\zeta} \frac{1}{\zeta - z_0} dxdy \right\}.$$

Since $\frac{\partial \phi_n}{\partial \bar z} \to \frac{\partial \phi}{\partial \bar z}$ weak (*) and $\frac{1}{\zeta - z_0} \in L^1_{loc}$ we have

$$\lim_{n\to\infty} \frac{1}{\pi} \iint_X \frac{\partial \phi_n}{\partial \bar\zeta} \frac{1}{\zeta - z_0} dxdy = \frac{1}{\pi} \iint_X \frac{\partial \phi}{\partial \bar\zeta} \frac{1}{\zeta - z_0} dxdy .$$

Also, according to (b), we have

$$\left|\frac{1}{2\pi i}\int_{B(X)} \frac{\phi_n(\zeta)}{\zeta - z_0}d\zeta - \frac{1}{2\pi i}\int_{B(X)} \frac{\phi(\zeta)}{\zeta - z_0}d\zeta\right|$$

$$\leq \frac{1}{2\pi i}\|\phi_n - \phi\|_{C(X)} \int_{B(X)} \frac{d|\zeta|}{|\zeta - z_0|} \to 0 \quad \text{as} \quad n \to \infty .$$

Therefore, (1.13) also holds $B(X)$ in this case. To prove (1.13) for $z \in \mathbb{C}\backslash X$, $\phi \in Lip(1,\mathbb{C})$ we have to repeat the same argument and again use the fact that (1.13) holds for all ϕ_n. Proposition is proved.

The following proposition characterizes those of $Lip(1,\mathbb{C})$ functions which belong to $R(X)$.

Proposition 1.9. Let X be a compact set in $\mathbb{C}$. If $f \in Lip(1,\mathbb{C})$ and $\frac{\partial f}{\partial \bar{z}} \equiv 0$ a.e. on X, then $f \in R(X)$. Moreover, if X has a finite perimeter, then the converse statement is also true. Namely, if $f \in R(X) \cap Lip(1,\mathbb{C})$, then $\frac{\partial f}{\partial \bar{z}} \equiv 0$ a.e. on X.

Proof. Let X be an arbitrary compact set. Take any measure $\mu \perp R(X)$. Then, according to Proposition 1.3 we have

$$\int_X f\, d\mu = - \int_{\mathbb{C}} \frac{\partial f}{\partial \bar{z}}\hat{\mu}(z)dxdy$$

$$= - \int_X \frac{\partial f}{\partial \bar{z}}\hat{\mu}(z)dxdy - \int_{\mathbb{C}\backslash X} \frac{\partial f}{\partial \bar{z}}\hat{\mu}(z)dxdy = 0 .$$

since $\frac{\partial f}{\partial \bar{z}} \equiv 0$ on X and $\hat{\mu}(z) \equiv 0$ on $\mathbb{C}\backslash X$. Therefore, $\mu \perp f$. Applying the Hahn-Banach Theorem we obtain that $f \in R(X)$.

Now, suppose that X has a finite perimeter. At first, let us note that for each $z_0 \in X$ satisfying the conditions (b) and (c) from the proof of the previous proposition, the measure

$$\frac{1}{2\pi i}\frac{1}{\zeta - z_0}d\zeta|_{B(X)}$$

is a representing measure for $R(X)$. In reality, let $\phi \in R(X)$ be analytic in the neighborhood of X . Then, $\dfrac{\phi(\zeta) - \phi(z_0)}{\zeta - z_0} \in R(X)$. Since $\frac{1}{2\pi i}d\zeta|_{B(X)}$ is orthogonal to $R(X)$ and in accordance with our choice of z_0 we obtain

$$\frac{1}{2\pi i}\int_{B(X)} \frac{\phi(\zeta) - \phi(z_0)}{\zeta - z_0}d\zeta \equiv 0$$

and, hence,

$$\phi(z_0) = \frac{1}{2\pi i}\int_{B(X)} \frac{\phi(\zeta)}{\zeta - z_0}d\zeta \ . \tag{1.15}$$

In view of (c) we can take the uniform limits exactly as we did in the proof of Proposition 1.9. Therefore, (1.15) holds for all $\phi \in R(X)$. Let $f \in R(X) \cap \mathrm{Lip}(1,\mathbb{C})$. Then, by (1.15) we have for almost all $z_0 \in X$

$$f(z_0) = \frac{1}{2\pi i}\int_{B(X)} \frac{f(\zeta)}{\zeta - z_0}d\zeta$$

Since $f \in \mathrm{Lip}(1,\mathbb{C})$, for almost all $z_0 \in X$ according to Proposition 1.8 we also have

$$f(z_0) = \frac{1}{2\pi i}\int_{B(X)} \frac{f(\zeta)}{\zeta - z_0}d\zeta - \frac{1}{\pi}\iint_X \frac{\partial f}{\partial\bar\zeta}\,\frac{1}{\zeta - z_0}dxdy .$$

Hence,

$$\iint_X \frac{\partial f}{\partial\zeta}\,\frac{1}{\zeta - z}dxdy \equiv 0 \quad \text{a.e. on} \quad X .$$

For any $z_1 \in \mathbb{C}\setminus X$, $f \cdot \frac{1}{\zeta - z} \in R(X)$. As $\frac{1}{2\pi i}d\zeta|_{B(X)} \perp R(X)$, we get

$$\frac{1}{2\pi i}\int_{B(X)} \frac{f(\zeta)}{\zeta - z}d\zeta \equiv 0 \quad \text{on} \quad \mathbb{C}\setminus X .$$

Then, according to (1.13) we obtain that

$$\iint_X \frac{\partial f}{\partial\bar\zeta}\,\frac{1}{\zeta - z}dxdy \equiv 0 \quad \text{on} \quad C\setminus X .$$

Therefore, the Cauchy transform of the measure

$$\frac{\partial f}{\partial\bar\zeta}\chi_X dxdy$$

is equal to zero a.e. in $\mathbb{C}$. So, $\frac{\partial f}{\partial\bar\zeta}|_X \equiv 0$ a.e. Proposition is proved.

Corollary 1.4. Let K be a Swiss cheese. A function $f \in Lip(1,\mathbb{C})$ belongs to $R(K)$ if and only if $\frac{\partial f}{\partial \bar{\zeta}} \equiv 0$ a.e. on K.

Proposition 1.10. Let X be a nowhere dense set of finite perimeter. Let $f \in Lip(1,\mathbb{C})$. Then, $f \in R(X)$ if and only if

$$\int_{B(X)} f(\zeta)r(\zeta)d\zeta = 0 \tag{1.16}$$

for all $r(\zeta) \in R(X)$.

Remark. This Proposition can be considered as a version of F. and M. Riesz theorem for such sets (cf. to [27], [29], [45]).

Proof. Let $f \in R(X)$. Then, $f \cdot r \in R(X)$ for each $r(\zeta) \in R(X)$. Therefore,

$$\int_{B(X)} f(\zeta)r(\zeta)d\zeta = 0$$

($d\zeta|_{B(X)}$ is orthogonal to $R(X)$). Assume that (1.16) holds. It is equivalent to the following.

$$\int_{B(X)} f(\zeta) \cdot \frac{1}{\zeta - z}d\zeta = 0 \quad \text{for all} \quad z \in \mathbb{C}\setminus X .$$

Therefore, in accordance with (1.13) we obtain

$$\iint_X \frac{\partial f}{\partial \bar{\zeta}} \frac{1}{\zeta - z} dxdy \equiv 0 \quad \text{for all} \quad z \in \mathbb{C} \setminus X . \tag{1.17}$$

But $\frac{\partial f}{\partial \zeta} \in L^{\infty}$, $\frac{1}{z} \in L^1_{loc}$. Then, the convolution $\frac{\partial f}{\partial} * \frac{1}{z}$ is a continuous function in $\mathbb{C}$ (see [24]). Since X is nowhere dense, from (1.17) we obtain the following:

$$\iint_X \frac{\partial f}{\partial \bar{\zeta}} \frac{1}{\zeta - z} dxdy \equiv 0 \quad \text{for all} \quad z \in \mathbb{C} .$$

Therefore, $\frac{\partial f}{\partial \zeta}|_X \equiv 0$ a.e. Applying Proposition 1.9 we complete the proof.

The following Corollary has been first observed by D. Luecking (see [38]).

Corollary 1.5. (F. and M. Riesz Theorem for the Swiss cheeses). Let K be a Swiss cheese. Let $f \in Lip(1,\mathbb{C})$. Then $f \in R(K)$ if and only if

$$\int_{\partial\Delta_0} f(\zeta)r(\zeta)d\zeta - \sum_{j=1}^{\infty} \int_{\partial\Delta_j} f(\zeta)r(\zeta)d\zeta = 0$$

for all $r(\zeta) \in R(K)$.

Let us recall the definition of the algebra $H^{\infty}(X)$ on an arbitrary compact set X (see [26]).

Definition. Let Q be the set of all non-peak points in X of $R(X)$. Then $H^{\infty}(X)$ is defined as the weak (*) closure of $R(X)$ in $L^{\infty}(Q,m^2)$.

Remark. If X has a finite perimeter, then according to (iv) of Theorem

1.3 $m_2(Q) = m_2(X)$. So we can take X for Q in this case.

Theorem 1.4. Let X be an essential compact set with finite perimeter. Let f be an arbitrary function in $H^\infty(X)$. Then, there exists a unique function $\tilde{f} \in L^\infty(B(X),d\zeta)$ such that

$$f(z) = \frac{1}{2\pi i}\int_{B(X)} \frac{\tilde{f}(\zeta)}{\zeta - z}d\zeta \quad \text{for almost all} \quad z \in X$$

and

$$\int_{B(X)} \frac{\tilde{f}(\zeta)}{\zeta - z}d\zeta \equiv 0 \quad , \quad \text{for all} \quad z \in C\backslash X \ . \tag{1.18}$$

Moreover, $\|f\|_{L^\infty(B(X),d\zeta)} = \|f\|_{L^\infty(X)}$.

Proof. Since $f \in H^\infty(X)$, there exists a sequence $\{f_n\}_1^\infty$, $f_n \in R(X)$ such that $f_n \to f$ weak (*) in L^∞ . Then, obviously, $f_n \to f$ in the weak topology of L^1 $(X,dxdy)$. Further, since $f_n \to f$ weak (*) in $L^\infty(X)$, then $\|f_n\|_{L^\infty(X)} = \|f_n\|_{C(X)} \le M < +\infty$, where M is a certain constant. As $f_n \to f$ weakly in L^1 , then there exists a sequence of their convex combinations

$$g_n = \sum_{i=1}^{j_n} \alpha_i^{(n)} f_i \quad , \quad \alpha_i^{(n)} \ge 0 \quad , \quad \sum_i^{j_n}\alpha_i^{(n)} = 1 \ .$$

such that $g_n \to f$ in the normed topology of L^1 on X (see [47]). Furthermore, $g_n \in R(X)$ and $\|g_n\|_{C(X)} \le \sum_{i=1}^{j_n} \alpha_i^{(n)}\|f_i\|_{C(X)} \le M$ for all n . Taking a subsequence if it is necessary we can assume that

$$g_n \to f \quad \text{a.e. on } X . \tag{1.19}$$

Put $\tilde{g}_n = g_n|_{B(X)}$. Then, $\|\tilde{g}_n\|_{L^\infty(B(X),d\zeta)} \le \sup_{z\in B(X)} |g_n(z)| \le \|g_n\|_{C(X)} \le M$ for all n . Then, $\{\tilde{g}_n\}_1^\infty$ contains a subsequence which converges in the weak (*) topology of $L^\infty(B(X),d\zeta)$ to a certain function $\tilde{f}$. We also denote this subsequence by $\tilde{g}_n$. We have

$$\|\tilde{f}\|_{L^\infty(B(X),d\zeta)} \le \underline{\lim}_{n\to\infty} \|\tilde{g}_n\| \le M .$$

Fix $z_0 \in C\backslash X$. Then $\frac{1}{\zeta - z_0}$ is continuous, and therefore integrable in a neighborhood of X . Moreover, $g_n \cdot \frac{1}{\zeta - z_0} \in R(X)$ for all n . Therefore, we have

$$\frac{1}{2\pi i}\int_{B(X)} \frac{\tilde{f}(\zeta)d\zeta}{\zeta - z_0} = \frac{1}{2\pi i} \lim_{n\to\infty} \int_{B(X)} \frac{\tilde{g}_n(\zeta)d\zeta}{\zeta - z_0} = 0 ,$$

as $d\zeta|_{B(X)}$ is orthogonal to $R(X)$. So, we have proved the second equality in (1.18). To prove the remaining part, find $z_0 \in X$ such that the following conditions hold.

(a) $\frac{1}{2\pi i}\int_{B(X)} \frac{d\zeta}{\zeta - z_0} = 1$ (b) $\int_{B(X)} \frac{d|\zeta|}{|\zeta - z_0|} < \infty$ (c) $\lim_{n\to\infty} g_n(z_0) = f(z_0)$.

As we have seen in the proof of Proposition 1.9, the measure $\frac{1}{2\pi i}\frac{1}{\zeta - z_0}d\zeta|_{B(X)}$ is a representing measure for $R(X)$ at z_0 . Then, according to (1.15), we have

$$g_n(z_0) = \frac{1}{2\pi i}\int_{B(X)} \frac{\tilde{g}_n(\zeta)}{\zeta - z_0}d\zeta$$

for all n. In view of (b), $\frac{1}{\zeta - z_0} \in L^1(B(X),d\zeta)$. So, by (c) we obtain

$$f(z_0) = \lim_{n\to\infty} g_n(z_0) = \lim_{n\to\infty} \frac{1}{2\pi i}\int_{B(X)} \frac{\tilde{g}_n(\zeta)}{\zeta - z_0}d\zeta = \frac{1}{2\pi i}\int_{B(X)} \frac{\tilde{f}(\zeta)}{\zeta - z_0}d\zeta .$$

According to (1.19), (a), (b) and (c) hold for almost all $z \in X$. Thus, (1.18) holds for almost all $z \in X$. To obtain the uniqueness of $\tilde{f}$, suppose there exists another function $\tilde{f}_1 \in L^\infty(B(X),d\zeta)$ such that (1.18) holds, Then,

$$\int_{B(X)} \frac{\tilde{f}(\zeta) - \tilde{f}_1(\zeta)}{\zeta - z}d\zeta \equiv 0 \quad \text{a.e. in } \mathbb{C} .$$

Hence, $\tilde{f} = \tilde{f}_1$ a.e. with respect to $d\zeta$ on $B(X)$. Since H^∞ is an algebra then (1.18) also holds for all powers f^n. So, for almost all $z \in X$ (a), (b), (c) hold and (1.18) is valid for all f^n, $n=1, \ldots$. Take such $z_0 \in X$. Since $\|\tilde{f}^n\| \le \|\tilde{f}\|^n$ we have

$$|f(z_0)|^n = |f^n(z_0)| \le \frac{1}{2\pi}\int_{B(X)} \frac{\|\tilde{f}\|^n}{|\zeta - z_0|}d|\zeta|$$

Hence

$$|f(z_0)| \le \|\tilde{f}\|_{L^\infty(B(X),d\zeta)} \cdot \sqrt[n]{\frac{1}{2\pi}\int_{B(X)} \frac{d|\zeta|}{|\zeta - z_0|}}$$

$$\le \|\tilde{f}\|_{L^\infty(B(X),d\zeta)} \cdot \sqrt[n]{\text{const}}$$

As $n \to \infty$ we obtain

$$|f(z_0)| \le \|\tilde{f}\|_{L^\infty(B(X),d\zeta)} .$$

Therefore, $|f(z)| \le \|\tilde{f}\|$ a.e. on X. So, $\|\tilde{f}\|_{L^\infty(X)} \le \|\tilde{f}\|_{L^\infty(B(X),d\zeta)}$. To prove the inverse inequality, i.e. $\|\tilde{f}\|_{L^\infty(B(X),d\zeta)} \le \|f\|_{L^\infty(X)}$ we quote A. Davie's result according to which there exists a sequence $f_n' \in R(X)$ such that $\|f_n'\|_{C(X)} \le \|f\|_{L^\infty(X)}$ and $f_n' \to f$ weak (*) in L^∞ (see [9], [26]). Then, starting the construction with the sequence $\{f_n'\}$ we immediately obtain our inequality.

Remark. The trick applied to prove the inequality $\|f\| \le \|\tilde{f}\|$ is due to E. Landau. We are grateful to Professor A. Browder for pointing out to us that it is applicable in this situation.

Corollary 1.6. Let X be a set of finite perimeter. Let $f_1, f_2 \in H^\infty(X)$ and $\tilde{f}_1 = \tilde{f}_2$ a.e. with respect to $d\zeta$ on $B(X)$. Then $f_1 \equiv f_2$ (a.e.).

Note. This corollary shows that although in general a function f of $H^\infty(X)$ is defined almost everywhere with respect to m_2, there exists a universal set $B(X)$ of m_2-measure zero, (even more, $m_1(B(X)) < +\infty$), and a function

$\tilde{f}$ on $B(X)$ such that $\tilde{f}$ defines f uniquely. It seems natural to call $\tilde{f}$ boundary values of f on X.

Corollary 1.7. Let K be a Swiss cheese. Then, (1.18) and Corollary 1.6 hold for each $f \in H^{\infty}(K)$ with

$$B(K) = \bigcup_{j=0}^{\infty} (\dot{\partial} \Delta_j) .$$

Chapter II

ANNIHILATING MEASURES OF THE ALGEBRA $R(X)$

§1. Introductory remarks.

In this Chapter we study the general compact set X on the complex plane and the algebra $R(X)$ defined as the uniform closure on X of all rational funtions with poles outside of X. One of the major problems of the subject is to describe the space of annihilating measures of this algebra supported on the boundary of X. In case when X is the unit disk the answer is given by the celebrated theorem of F. and M. Riesz, which has numerous applications (see [13], [27], [29]).

In the series of papers (see [2], [4], [5]) E. Bishop has investigated possible generalizations of the F. and M. Riesz theorem for the compact sets K with a connected complement. He introduced the following concept of "analytic differentials".

Let U denote the interior of K Let $\{\Gamma_i\}$ be a system of curves such that (i) each Γ_i is a finite union of rectifiable closed Jordan curves in U and every connected component of U contains only one of these curves; (ii) every compact set $S \subset U$ belongs to the union of the components bounded by Γ_i for all sufficiently large i. Let $g(z)$ be an analytic function in U. If the total variations of the measures $g(z)dz|_{\Gamma_n}$ are uniformly bounded we say that these measures define an analytic differential in K. In that case, there exists a subsequence of these measures which converges weak (*) to the measure μ supported on the boundary of K. It is clear that μ is orthogonal to all polynomials. Then, Bishop has proved that the measures defined by such "analytic differentials" coincide with the whole space of measures $(R(K)|_{\partial K})^{\perp}$. (We recall here, that if E_0 is a linear subspace of a topological vector space E, then $E_0^{\perp} = \{f \in E^* : f|_{E_0} = 0\}$.

Using this description of $(R(K)|_{\partial K})^{\perp}$ Bishop has obtained a proof of a famous theorem of S.N. Mergelyan (see [39]).

However, his construction of an "analytic differential" can not be extended directly to the general sets. Moreover, in case of the sets like a Swiss cheese without interior the above definition even does not make any sense.

In this chapter we study objects which are related to Bishop's idea of an "analytic differential" but which are applicable to an arbitrary compact set on the plane.

Before giving a precise definition we have to recall the definition of <u>Smirnov's class</u> $\underline{E_1}$ in a multiply connected region.

Let G be a finitely connected region in $\mathbb{C}$ with a boundary Γ. The function $f(z)$ is said to belong to the class $E_1(G)$ if $f(z)$ is analytic in the interior of G and there exists a sequence of domains $\{G_n\}$ bounded by a finite number of rectifiable curves such that $G_n \subset G_{n+1}\ \forall n$, $\bigcup_{n=1}^{\infty} G_n = \overset{\circ}{G}$ and

$$\|f\|_{E_1(G)} = \limsup_{n \to \infty} \int_{\partial G_n} |f||dz| < +\infty$$

Note. If the boundary Γ of ∂G is rectifiable then

$$\|f\|_{E_1(G)} = \int_{\Gamma} |f(\zeta)| \ |d\zeta| \ ,$$

where $f(\zeta)$ means angular boundary values of $f(z)$ on Γ. It is known that $f(\zeta)$ exist a.e. on Γ. It is also known, that $\|f\|_{E_1(G)}$ defines a norm on $E_1(G)$. More details on E_p - classes can be found in [13], [21], [45].

Definition. Let X be a compact set in C. Fix a sequence of finitely connected compact set $\{X_n\}$ bounded by analytic Jordan curves and such that $X_1 \supset X_2 \supset \ldots, \bigcap_{n=1}^{\infty} X_n = X$.

Let μ be a complex measure supported on ∂X. We say that μ is an analytic measure relative to the given sequence $\{X_i\}$ if there exists a sequence of functions $f_n(z)$ such that $f_n \in E_1(X_n)$, $\|f_n\|_{E_1(X_n)} \leq M < +\infty$ and a subsequence of the sequence of the measures $\mu_n = f_n(z)dz|_{\partial X_n}$ converges to μ in the weak (*) topology of the space of measures on X_1. We want to point out that this definition can be also considered as a generalization of Di Georgi's ideas discussed in Chapter I. There we had a sequence of "nice" sets Π_n and the "perimeter" measures $d\zeta|_{\partial \Pi_n}$ converging weak (*) to the measure on X. Here, we consider the "weighted" perimeter measures on ∂X_n.

The first question arising here is the existence of nontrivial analytic measures on arbitrary compact set. We investigate this problem using the duality method for the extremal problems of analytic functions. This method has been developed in the series of papers by S. Ya Khavinson (see [31], [32], [33]). In the case of the unit disk the duality approach has been independently discovered by W. Rogosinski and H. Shapiro (see [13], [46]). To state our main result we have to give one more definition

Definition. Let $f \in C(X)$. We define

$$\lambda_f(X) \overset{\text{def}}{=} \inf_{\phi \in R(X)} \| f - \phi \|_{C(X)}$$

For $f(\zeta) = \bar{\zeta}|_X$ we shall call $\lambda_{\bar{\zeta}}(X)$ the rational capacity of X and denote it by $\lambda(X)$.

Then, the main result of this chapter is given by Theorem 2.1. Namely, let h be a function harmonic in the neighborhood of X.

Then the following equality holds

$$\sup_{\substack{\|\mu\| \le 1 \\ \mu \in R(X)^{\perp} \\ \operatorname{supp} \mu \subset \partial X}} \left| \int h \, d\mu \right| = \lambda_h(X)$$

and there exists an analytic measure μ^* for which the supremum in the right-hand side is attained.

From this we derive (Corollary 2.1) that on any compact set X such that $R(X) \neq C(X)$ there exist non-trivial analytic measures.

Let us give a brief description of the contents of this chapter.

In §2 we list out S. Ya. Khavinson's results on extremal problems for analytic functions in finitely connected domains. These results are used in the proof of Theorem 2.1. In our presentation we follow S. Ya. Khavinson's lecture notes [34]. For the sake of completeness we indicate the proofs of the most of his results.

In §3 we prove a few auxiliary results on analytic measures and λ_f .

In §4 we give a proof of Theorem 2.1 and Corollary 2.1.

§5 contains further properties of analytic measures and $\lambda(X)$.

In §6 we consider the problem of characterization of annihilating measures of $R(X)$ on an arbitrary compact set. The main result obtained here is given by Theorem 2.3. Namely, let $H(X)$ denote the uniform closure of all functions harmonic in a neighborhood of X . We prove that if $H(X)|_{\partial X} = C(\partial X)$ then the weak (*) closure of the linear span of all analytic measures coincides with $(R(X)|_{\partial(X)})^{\perp}$. At the end of this section we formulate a few problems which we have been unable to solve.

§2. S. Ya. Khavinson's theory of extremal problems for analytic functions in finitely connected domains.

Note. If G is a domain in C, then $H_p(G)$, $0 \le p \le \infty$ denotes the Hardy classes of analytic functions in G. Details concerning the definitions and properties of Hardy classes in multiply-connected domains can he found in [13], [33], [35], [36], [45]. Also, we mention [8], [30].

Proposition 2.1. Let E be a Banach space and let $E_0 \subset E$ be a subspace. Let $\ell_0 \in E^*$. Then,

$$\sup_{\substack{f \in E_0^* \\ \|f\| \le 1}} |\ell_0(f)| = \inf_{\ell \in E_0^\perp} \|\ell_0 - \ell\| \tag{2.1}$$

and there exists $\ell^* \in E_0^\perp$ for which the infimum in the (2.1) is attained.

Proof. For an arbitrary $\ell \in E_0^\perp$, we have

$$\sup_{\substack{f \in E_0 \\ \|f\| \le 1}} |\ell_0(f)| = \|\ell_0\|_{E_0^*} = \|\ell_0 - \ell\|_{E_0^*} \le \|\ell_0 - \ell\|_{E^*}$$

Hence,

$$\| \ell_0 \|_{E_0^*} \le \inf_{\ell \in E_0^\perp} \| \ell_0 - \ell \|$$

On the other hand, by the Hahn-Banach Theorem we can find $L \in E^*$ such that

$$L(f) = \ell_0(f)\ , \forall\ f \in E_0\ , \|L\|_{E^*} = \|\ell_0\|_{E_0^*}$$

Let $\ell^* = \ell_0 - L$. Then, for any $f \in E_0$ we have

$$\ell^*(f) = \ell_0(f) - L(f) = 0$$

Therefore, $\ell^* \in E_0^\perp$ and $\|\ell_0\|_{E_0^*} = \|\ell_0 - \ell^*\|_{E^*}$. The Proposition is proved.

We will also make use of the following Proposition which is again a straightforward corollary of the Hahn-Banach Theorem. For the sake of brevity we omit the proof.

<u>Proposition 2.2.</u> Let E , E_0 be the same as in Proposition 2.1. Let $\omega \in E$. Then,

$$\inf_{f \in E_0} \|\omega - f\| = \sup_{\substack{\ell \in E_0^\perp \\ \|\ell\| \le 1}} |\ell(\omega)| \tag{2.2}$$

Moreover, there exists $\ell^* \in E_0^\perp$ for which the supremum in the right-hand side of (2.2) is attained.

A proof of Proposition 2.2 and a lot of applications to the function theory in multiply-connected regions can be found in S. Ya. Khavinson's paper [31].

Proposition 2.3. (S. Ya. Khavinson - see [31], [32], [34]). Let G be a finitely connected region with the rectifiable boundary Γ. Let $\omega(\zeta) \in L^{\infty}(\Gamma, d\zeta) = L^{\infty}(\Gamma)$.

Then,

$$\sup_{\substack{f \in E_1(G) \\ \|f\|_{E_1(G)} \leq 1}} \left| \int_{\Gamma} f(\zeta)\omega(\zeta)d\zeta \right| = \min_{\phi \in H^{\infty}(G)} \|\omega(\zeta) - \phi(\zeta)\|_{L^{\infty}(\Gamma)} \tag{2.3}$$

Proof. Let us apply Proposition 2.1 to $E = L^1(\Gamma)$ and $E_0 = E_1(G)|_{\Gamma}$. Let $\ell \in E^* \cap E_0^{\perp}$. Then, $\exists\, \psi \quad L^{\infty}(\Gamma)$ representing ℓ such that for all $f \in E_1(G)$, we have

$$\int_{\Gamma} f(\zeta)\, \psi(\zeta) d\zeta = 0$$

Let $w = \alpha(z)$ map G conformally onto the canonical circular domain K. (See [27]). Then, (See [27], [36]), $\alpha'(z) \in E_1(G)$ and, hence, for any $z \notin \overline{G}$, $\frac{\alpha'(\zeta)}{\zeta - z}$ also belongs to $E_1(G)$. Therefore,

$$\int_{\Gamma} \frac{\alpha'(\zeta)\psi(\zeta)}{\zeta - z} d\zeta \equiv 0 \,, \quad z \notin \overline{G}$$

Hence, according to the F. and M. Riesz Theorem (see [27]), the function $Q(\zeta) = \alpha'(\zeta)\psi(\zeta)$ belongs to $E_1(G)$. Then, applying the generalized version of the M. Keldysh-Lavrentjev Theorem for finitely-connected regions due to G.C. Tumarkin and S. Ya. Khavinson (see [36], [45]) we obtain that

$$\frac{Q(\zeta)}{\alpha'(\zeta)} = \psi(\zeta) \in H_1(G)$$

Therefore, $\psi(\zeta)$ is representable by the Green integral in G by means of its boundary values (see [36]). But $\psi(\zeta)|_\Gamma \in L^\infty(\Gamma)$. Hence, $\psi(\zeta) \in H^\infty(G)$.

The remaining part now follows directly from this observation and Proposition 2.1.

We shall keep the same notation as above.

Proposition 2.4. (S. Ya. Khavinson - see [31], [34]).

Let us assume that in (2.3) $\omega(\zeta) \in C(\Gamma)$. Then, there exists $f^* \in E_1(G)$ for which the supremum in the left-hand side of (2.3) is attained. Moreover, if λ is the value of this supremum and ϕ^* is an extremal function for the right-hand side of (2.3) (which always exists by Proposition 2.1), then almost everywhere on Γ with respect to $|d\zeta|$ the following holds:

$$f^*(\zeta)[\omega(\zeta) - \phi^*(\zeta)]d\zeta = \lambda\, e^{i\delta}\, |f^*(\zeta)|\,|d\zeta| \tag{2.4}$$

where $\delta \in \mathbb{R}$ is a certain constant.

Proof. Let $f_n(\zeta)$ be a sequence of functions in $E_1(G)$ such that $\| f_n \|_{E_1(G)} \leq 1$ and

$$\left| \int_\Gamma f_n(\zeta)\, \omega(\zeta) d\zeta \right| \longrightarrow \lambda$$

Taking a subsequence, if necessary, we can assume that $\{f_n\}_1^\infty$ converges uniformly on the compact subsets of G to the function $f^* \in E_1(G)$. According to S. Ya. Khavinson's theorem on the weak convergence of the boundary values (see [34], [36]) we obtain, since $\omega \in C(\Gamma)$:

$$\left| \int_\Gamma f^*(\zeta)\omega(\zeta) d\zeta \right| = \lim_{n\to\infty} \left| \int_\Gamma f_n(\zeta)\, \omega(\zeta) d\zeta \right| = \lambda$$

At the same time for any $\omega'(\zeta) \in C(\Gamma)$

$$\int_\Gamma f^*(\zeta)\omega'(\zeta) d\zeta = \lim_{n\to\infty} \int_\Gamma f_n^*(\zeta)\, \omega'(\zeta) d\zeta$$

in view of the same result of Khavinson. Hence, $f_n(\zeta)d\zeta$ converges to $f^*(\zeta)d\zeta$ in the weak (*) topology of $(C(\Gamma))^*$. Therefore

$$\| f^* \|_{L^1(\Gamma)} \leq \underline{\lim}_{n\to\infty} \| f_n \|_{L^1(\Gamma)} \leq 1 \quad .$$

Hence, f^* is an extremal function for (2.3) .

Furthermore, by Cauchy's Theorem we have:

$$\left|\int_\Gamma f^*(\zeta)\omega(\zeta)d\zeta\right| = \left|\int_\Gamma f^*(\zeta)(\omega(\zeta) - \phi^*(\zeta))d\zeta\right| \le$$

$$\le \int_\Gamma |f^*(\zeta)||\omega(\zeta) - \phi^*(\zeta)||d\zeta| \le$$

$$\le ||\omega(\zeta) - \phi^*(\zeta)||_{L^\infty(\Gamma)} \; ||f^*||_{E_1(G)} \le \lambda$$

But f^* , ϕ^* are extremal functions in (2.3). Therefore, at each step above we have equalities. Then, (2.4) holds a.e. or Γ . Proposition is proved.

We recall that the function $f(z)$ is said to belong to the class $D(G)$ (or $N_+(G)$) if $f(z)$ is analytic in G and there exists the sequence of domains $\{G^i\}$ such that

$$G^1 \subset G^2 \subset \ldots\ldots , \quad \bigcup_{i=1}^{\infty} G^i = G \quad \text{and}$$

the integrals $\{\int_{\partial G^i} \ln^+|f|d\omega^i\}$ are uniformly absolutely continuous with respect to harmonic measures $d\omega^i$ on ∂G_i taken at the fixed point $z_0 \in G$.

More details about the classes A and D (N and N_+) of analytic functions in multiply-connected domains can be found in [27], [35], [36], [45].

We also point out the papers [8], [30] where there are given factorization theorems for the different classes in multiply-connected regions similar to the classical results of R. Nevaulinna, and V.I. Smirnov on the inner-outer factorization in the unit disk.

<u>Proposition 2.5</u>. (S. Ya. Khavinson - see [31], [34]).

Let Γ be analytic and let $\omega(\zeta)$ be analytic in a neighborhood of Γ. Moreover, we assume that $\omega(\zeta)$ can be extended into G as a meromorphic function with finitely many poles. Then, the extremal functions f^* and ϕ^* in the problem (2.3) can be analytically continued in the neighborhood of Γ.

<u>Sketch of the proof</u>. By (2.4) we have

$$e^{i\delta} f^*(\zeta)[\omega(\zeta) - \phi^*(\zeta)] [\frac{d\zeta}{ds}] \geq 0 \text{ a.e. on } \Gamma \qquad (2.5)$$

Let $\tilde{G}$ be an annular neighborhood of Γ, $\tilde{G} \subset G$.
Then, $f^*(\zeta)[\omega(\zeta)-\phi^*(\zeta)] \in E_1(\tilde{G})$.

Since Γ is analytic, from (2.5) it follows that $f^*(\zeta)[\omega(\zeta) - \phi^*(\zeta)]$ can be analytically continued across Γ by the symmetry principle for E_1 - functions. Put

$$R(\zeta) = f^*(\zeta)[\omega(\zeta) - \phi^*(\zeta)]$$

Since $R(\zeta)$ is analytic near Γ it has finitely many zeros there. Let $z_1, \ldots, z_k$ be zeros of $\omega - \phi^*$ in $\tilde{G}$. Let $g(z,\zeta)$ be the Green function of $\tilde{G}$. Then, from the Poisson-Jensen formula we obtain

$$\sum_{i=1}^{k} g(z,z_i) + \ln|\omega(z) - \phi^*(z)| =$$

$$= \frac{1}{2\pi} \int_{\partial\tilde{G}} \frac{\partial g(\zeta,z)}{\partial n_\zeta} [\ln|\omega(\zeta)-\phi^*(\zeta)|ds + d\mu]$$

where $\partial/\partial n_\zeta$ is the derivative in the inner normal direction, $d\mu \le 0$ and $d\mu$ is a singular measure.

If $z_{k+1}, \ldots, z_{k+m}$ are zeros of f^* in $\tilde{G}$, then in the same way we obtain

$$\sum_{i=1}^{m} g(z,z_{k+i}) + \ln|f^*(z)| = \frac{1}{2\pi} \int_{\partial\tilde{G}} \frac{\partial g(\zeta,z)}{\partial n_\zeta} [\ln|f^*(\zeta)|ds + d\mu_1]$$

where $d\mu_1 \le 0$ and singular. Since R can be extended across $\partial\tilde{G}$, the representation of $\ln|R(z)|$ has the form

$$\sum_{i=1}^{k+m} g(z,z_i) + \ln|R(z)| = \frac{1}{2\pi} \int_{\partial\tilde{G}} \frac{\partial g(\zeta,z)}{\partial n_\zeta} \ln|R(\zeta)|ds$$

As $\ln|R| = \ln|\omega - \phi^*| + \ln|f^*|$, we conclude that $\mu \equiv \mu_1 \equiv 0$. Therefore, in $\tilde{G}$ we have

$$\omega(z) - \phi^*(z) = \exp\{\frac{1}{2\pi} \int_{\partial\tilde{G}} \ln|\omega(\zeta) - \phi^*(\zeta)|\frac{\partial P}{\partial n} ds - \sum_{1}^{k} P(z,z_i)\}$$

where $P(z,\zeta) = g(z,\zeta) + i\,\tilde{g}(z,\zeta)$ is the complexified Green function in $\tilde{G}$. By (2.4) $|\omega(\zeta) - \phi^*(\zeta)| = \lambda$ a.e. on Γ.

Therefore, by the symmetry principle $\omega(\zeta) - \phi^*(\zeta)$ can be analytically continued across Γ . Since $\omega(\zeta)$ is analytic near Γ we obtain that ϕ^* also can be analytically continued across Γ .

Then, it follows that

$$f^*(z) = \frac{R(z)}{\omega(z) - \phi^*(z)}$$

can be also analytically continued in a neighborhood of Γ . Proposition is proved.

§3. Properties of Analytic Measures and λ_f .

Proposition 2.6. Let μ be an analytic measure. Then $\mu \perp R(X)$.

Proof. Take any $z \in C \setminus X$. Then, $\frac{1}{\zeta - z}$, $\zeta \in X$ is continuous in the neighborhood of X . Let $\{f_n\}_1^\infty$ be the sequence of $E_1(X_n)$-functions defining the measure μ . Then,

$$\hat{\mu}(z) = \int_{\partial X} \frac{d\mu}{\zeta - z} = \lim_{k\to\infty} \int_{\partial X_{n_k}} \frac{f_{n_k}(\zeta)d\zeta}{\zeta - z} = 0 ,$$

since $z \notin X_{n_k}$ for $n_k > n_0$. Thus, $\hat{\mu}(z) \equiv 0$ in $C \setminus X$. This is equivalent to the fact that $\mu \perp R(X)$.

We recall that $\lambda(X) = \lambda_{\overline{\zeta}}(X)$.

<u>Proposition 2.7</u>. $\lambda(X) = 0 \iff C(X) = R(X)$.

<u>Proof</u>. $\Longrightarrow$ $\lambda(X) = 0$ implies that $\bar{\zeta} \in R(X)$.

Since $\zeta \in R(X)$, it follows that $\mathrm{Re}\,\zeta$, $\mathrm{Im}\,\zeta \in R(X)$
Hence, $C[x] \subset R(X)$.

According to the Stone-Weierstrass Theorem this implies that $R(X) = C(X)$.

$\Longleftarrow$ is clear.

<u>Proposition 2.8</u>. Let $h \in H(X)$. Then

$$\lambda_h(X) = \inf_{\phi \in R(X)} \| h - \phi \|_{\partial X} .$$

($\| \ \|_K$ means the uniform norm on the compact set K) .

<u>Proof</u>: Take any $\phi_0 \in R(X)$. Then,

$$\| h - \phi_0 \|_{\partial X} \le \| h - \phi_0 \|_X$$

Hence,

$$\inf_{\phi \in R(X)} \| h - \phi \|_{\partial X} \le \| h - \phi_0 \|_X$$

Taking infimum in the right-hand side we obtain

$$\inf_{\phi \in R(X)} \| h - \phi \|_{\partial X} \le \lambda_h(X) \qquad (2.6)$$

On the other hand, for any $\phi \in R(X)$ the function $h(z) - \phi(z)$ is a complex-valued harmonic function on X .

Then, the function $|h(z) - \phi(z)|$ is subharmonic in X . Hence, $|h(z) - \phi(z)|$ attains its maximum on the boundary ∂X . So,

$$\| h - \phi \|_X = \sup_{z \in X} |h(z) - \phi(z)| = \sup_{z \in \partial X} |h(z) - \phi(z)| =$$

$$= \| h - \phi \|_{\partial X} .$$

Thus, for any $\phi \in R(X)$ we have

$$\lambda_h(X) \le \| h - \phi \|_{\partial X}$$

Therefore, taking the infimum we obtain

$$\lambda_h(X) \le \inf_{\phi \in R(X)} \| h - \phi \|_{\partial X} \tag{2.7}$$

From (2.6) and (2.7) our Proposition follows.

<u>Proposition 2.9</u>. Let $X = \bigcap_{n=1}^{\infty} X_n$, where $\{X_i\}$ is a decreasing sequence of compact finitely connected sets bounded by analytic Jordan curves. Let $h(\zeta)$ be a harmonic function in the neighborhood of X_1 . Then,

$$\lambda_h(X_1) \geq \lambda_h(X_2) \geq \ldots \geq \lambda_h(X) \qquad (2.8)$$

and

$$\lim_{n\to\infty} \lambda_h(X_n) = \lambda_h(X) \quad .$$

<u>Proof</u>. Fix n . Take any $\phi_0 \in R(X_n)$.

Then,

$$\lambda_h(X_{n+1}) \stackrel{\text{def}}{=} \inf_{\phi\in R(X_{n+1})} \| h - \phi \|_{X_{n+1}} \leq \| h - \phi_0 \|_{X_{n+1}} \leq$$

$$\leq \| h - \phi_0 \|_{X_n}$$

Taking an infimum over $\phi_0 \in R(X_n)$ we obtain, that

$$\lambda_h(X_{n+1}) \leq \lambda_h(X_n) \quad .$$

In the same way one can verify that for any n, $\lambda_h(X_n) \geq \lambda_h(X)$.

Now, fix $\varepsilon > 0$. Choose ϕ_1 analytic in the neighborhood of X such that

$$\| h - \phi_1 \|_X = \max_{\zeta\in\partial X} |h(\zeta) - \phi_1(\zeta)| \leq \lambda_h(X) + \varepsilon$$

As $h - \phi_1$ is uniformly continuous in the neighborhood of ∂X we can find neighborhood U of X such that $\phi_1 \in R(\overline{U})$ and

$$\| h - \phi_1 \|_X \le \| h - \phi_1 \|_{\overline{U}} \le \| h - \phi_1 \|_X + \varepsilon$$

We can choose $n_0 : \forall n \ge n_0, \partial X_n \subset U$. Then, $\phi_1(\zeta) \in R(X_{n_0})$. Moreover, according to the choice of U , ϕ_1 , we have for $n \ge n_0$.

$$\| h - \phi_1 \|_{X_n} = \| h - \phi_1 \|_{\partial X_n} \le \| h - \phi_1 \|_{\overline{U}} \le$$

$$\le \| h - \phi_1 \|_X + \varepsilon \le \lambda_h(X) + 2\varepsilon$$

Hence,

$$\lambda_h(X_n) \le \| h - \phi_1 \|_{X_n} \le \lambda_h(X) + 2\varepsilon$$

Therefore,

$$\lim_{n\to\infty} \lambda_h(X_n) \le \lambda_h(X) + 2\varepsilon$$

Since ε was arbitrary, we obtain that

$$\lim_{n\to\infty} \lambda_h(X_n) \le \lambda(X)$$

Together with (2.8) which has already been proved this finishes the proof of our Proposition.

In the following Proposition we summarize some of the results in §2 and 3.

Proposition 2.10. Let $f \in C(X)$. Then

$$\sup_{\substack{\mu \perp R(X) \\ \|\mu\| \le 1}} \left| \int_X f d\mu \right| = \lambda_f(X) \overset{\text{def}}{=} \inf_{\phi \in R(X)} \|f - \phi\|_X \tag{2.9}$$

Moreover, if in addition to this $f \in H(X)$, then

$$\sup_{\substack{\mu \perp R(X) \\ \|\mu\| \le 1 \\ \operatorname{supp} \mu \subset \partial X}} \left| \int_X f d\mu \right| = \inf_{\phi \in R(X)} \|f - \phi\|_{\partial X} = \lambda_f(X) \tag{2.10}$$

There always exists an extremal measure μ^* , $\|\mu^*\| = 1$ for which the supremum in (2.9) (or in (2.10)) is attained.

Proof. (2.9) follows directly from Proposition 2.2 and F. Riesz's representation Theorem for $C(X)^*$ if we put $E = C(X)$, $E_0 = R(X)$ (2.10) also follows immediately from Proposition 2.2 and Proposition 2.7 if we set $E = C(\partial X)$, $E_0 = R(X)|_{\partial X}$.

§4. Analytic Measures as Solutions of Extremal Problems.

Lemma 2.1. Let X be a finitely connected compact set and let ∂X consists of finitely many analytic closed Jordan curves. Let $H^\infty(X) = \sum_{i=1} \oplus H^\infty(X_i)$, where X_i are connected components of X . Let $\omega(\zeta) \in C(\partial X)$. Then,

$$\min_{\phi\in H^\infty(X)} \|\omega - \phi\|_{L^\infty(\partial X)} = \inf_{\phi\in R(X)} \|\omega - \phi\|_{\partial X} \qquad (2.11)$$

<u>Proof</u>. For reader's convenience we assume that X has only one component. All generalizations for the case $n > 1$ are straightforward and we omit them.

Fix $\varepsilon > 0$. Since $\omega(\zeta) \in C(\partial X)$ and as it is known $R(\partial X) = C(\partial X)$, there exists $\omega_0(\zeta)$ such that $\omega_0(\zeta)$ is analytic in a neighborhood of ∂X , meromorphic on X and

$$\|\omega(\zeta) - \omega_0(\zeta)\|_{\partial X} < \varepsilon \qquad (2.12)$$

Let us consider the following extremal problem:

$$\min_{\phi\in H^\infty(X)} \|\omega_0(\zeta) - \phi\|_{L^\infty(\partial X)}$$

According to Proposition 2.3, there exists $\phi_0^* \in H^\infty(X)$ which is an extremal function for this problem. Since $\omega_0(\zeta)$ is analytic in the neighborhood of ∂X_n , the Proposition 2.5 implies that ϕ_0^* is analytic in the neighborhood of ∂X .

As X is finitely connected, we obtain from this that $\phi_0^* \in R(X)$. (see [6], [24]).

Take an arbitrary $\phi_1 \in H^\infty(X)$. Then, according to (2.12) we have

$$\|\omega - \phi_1\|_{L^\infty(\partial X)} = \|\omega - \omega_0 + \omega_0 -$$

$$- \phi_1\|_{L^\infty(\partial X)} \geq \|\omega_0 - \phi_1\|_{L^\infty(\partial X)} - \|\omega_0 - \omega\|_{\partial X} \geq$$

$$\geq \min_{\phi \in H^\infty(X)} \|\omega_0 - \phi\|_{L^\infty(\partial X)} - \varepsilon =$$

$$= \|\omega_0 - \phi_0^*\|_{L^\infty(\partial X)} - \varepsilon =$$

$$= \|\omega_0 + \omega - \phi_0^* - \omega\|_{L^\infty(\partial X)} - \varepsilon \geq$$

$$\geq \|\omega - \phi_0^*\|_{L^\infty(\partial X)} - \|\omega - \omega_0\|_{\partial X} - \varepsilon \geq$$

$$\geq \inf_{\phi \in R(X)} \|\omega - \phi\|_{\partial X} - 2\varepsilon \quad .$$

Thus, for any $\phi_1 \in H^\infty(X)$ we have

$$\|\omega - \phi_1\|_{L^\infty(\partial X)} \geq \inf_{\phi \in R(X)} \|\omega - \phi\|_{\partial X} - 2\varepsilon \quad .$$

This implies that

$$\inf_{\phi \in H^\infty(X)} \|\omega - \phi_1\|_{L^\infty(\partial X)} \geq \inf_{\phi \in R(X)} \|\omega - \phi\|_{\partial X} - 2\varepsilon \quad .$$

Since ε was arbitrary we obtain that

$$\inf_{\phi\in H^\infty(X)} \|\omega-\phi\|_{L^\infty(\partial X)} \geq \inf_{\phi\in R(X)} \|\omega-\phi\|_{\partial X}$$

On the other hand $R(X) \subset H^\infty(X)$. So,

$$\min_{\phi\in H^\infty(X)} \|\omega-\phi\|_{L^\infty(\partial X)} \leq \inf_{\phi\in R(X)} \|\omega-\phi\|_{\partial X}$$

From the last two inequalities our lemma follows.

<u>Theorem 2.1</u>. Let X be an arbitrary compact set. Let $\{X_n\}_1^\infty$ be a decreasing sequence of finitely connected compact sets with analytic boundaries such that $\bigcap_{n=1}^\infty X_n = X$. Let $h(\zeta)$ be a function harmonic in the neighborhood U of X . Then,

$$\sup_{\substack{\|\mu\|\leq 1 \\ \mu\in R(X)^\perp}} \left|\int_X h\,d\mu\right| = \sup_{\substack{\|\mu\|\leq 1 \\ \mu\in R(X)^\perp \\ \operatorname{supp}\mu\subset\partial X}} \left|\int_{\partial X} h\,d\mu\right| = \tag{2.13}$$

$$= \inf_{\phi\in R(X)} \|h-\phi\|_{\partial X} = \inf_{\phi\in R(X)} \|h-\phi\|_X \overset{\text{def}}{=} \lambda_h(X) .$$

and there exists the measure μ^* <u>analytic</u> with respect to the sequence $\{X_n\}$ for which the supremum in (2.13) is attained.

Proof. (2.13) follows immediately from Proposition 2.10, since $h \in H(X)$. So, it remains to show that there exists the analytic measure μ^* giving the supremum in (2.13).

$\exists n_0 : \forall\, n > n_0 \quad \partial X_n \subset U$. Thus, without loss of generality we can assume that $U \supset X_1 \supset \ldots$. According to Proposition 2.8 for all n $\lambda_h(X_n) = \inf_{\phi \in R(X_n)} \| h - \phi \|_{\partial X_n}$. As $h \in C(\partial X_n)$ for all n, lemma 2.1 implies that

$$\lambda_h(X_n) = \min_{\phi \in H^\infty(X_n)} \| h - \phi \|_{L^\infty(\partial X_n)} \tag{2.14}$$

Fix n . Let $X_n = \bigcup_{i=1}^{j_n} X_n^i$, where X_n^i are connected components of X_n . Then, $E_1(X_n) = \bigoplus_{i=1}^{j_n} E_1(X_n^i)$ with the norm

$$\| \ \|_{E_1(X_n)} = \sum_{i=1}^{j_n} \| \ \|_{E_1(X_n^i)}$$

Consider the following extremal problem in X_n :

$$\sup_{\substack{f \in E_1(X_n) \\ \|f\|_{E_1(X_n)} \le 1}} \left| \int_{\partial X_n} f\, h\, d\zeta \right| \tag{2.15}$$

Denote this supremum by Λ_n .

Claim. $\Lambda_n = \min_{\phi \in H^\infty(X_n)} \| h - \phi \|_{L^\infty(\partial X_n)} = \lambda_h(X_n)$.

Proof of Claim. Let us set $E = E_1(X_n)|_{\partial X_n} \subset L^1(\partial X_n, d\zeta)$ in Proposition 2.1. Since $(L^1)^* = L^\infty$, we obtain that

$$\Lambda_n = \min_{\substack{\phi \in L^\infty(\partial X_n) \\ \phi \perp E_1(X_n)}} \| h - \phi \|_{L^\infty(\partial X_n)}$$

Let $\phi \in L^\infty(\partial X_n)$ and $\phi \perp E_1(X_n)$. Then, $\phi|_{\partial X_n^i} \perp R(X_n^i)|_{\partial X_n^i}$ for $i = 1, \ldots, j_n$. By F. and M. Riesz Theorem we obtain that $\phi|_{\partial X_n^i}$ represents the boundary values of a function ϕ^i belonging to $H^\infty(X_n^i)$ for all i.

Therefore, $\phi \in H^\infty(X)|_{\partial X}$ and in virtue of (2.16) the proof of the claim is complete.

Since $h \in C(\partial X)$, then by Proposition 2.4. it follows that there exists an extremal function $f_n^* \in E_1(X_n)$, $\|f_n^*\|_{E_1(X_n)} = 1$ in the problem (2.15). Consider the sequence of the differentials $\mu_n = f_n^*(\zeta)d\zeta|_{\partial X_n}$.

For all n $f_n^* \in E_1(X_n)$ and

$$\int_{\partial X_n} |f_n^*(\zeta)||d\zeta| = \|f_n^*\|_{E_1(X_n)} = 1 \quad ,$$

since f_n^* is an extremal function in (2.15). So, there exists a subsequence $\{\mu_{n_k}\}$ converging weak (*) to an analytic measure μ^*. Furthermore, according to the claim and Proposition 2.9., we obtain:

$$\left| \int_{\partial X} h d\mu^* \right| = \lim_{k\to\infty} \left| \int_{\partial X_{n_k}} h f^*_{n_k} \, d\zeta \right| =$$

$$= \lim_{k\to\infty} \Lambda_{n_k} = \lim_{k\to\infty} \lambda_h (X_{n_k}) = \lambda_h(X)$$

Thus, μ^* is an extremal measure for the problem (2.13). The proof is complete.

Corollary 2.1. Let X be a compact set in C, such that $R(X) \neq C(X)$. Then, there always exist nontrivial ($\not\equiv 0$) analytic measures on ∂X orthogonal to $R(X)$.

Proof. In view of Proposition 2.7 $\lambda(X) > 0$.

According to Theorem 2.1 there exists an analytic measure μ^* which is an extremal measure in (2.13) for $h \equiv \overline{\zeta}$. Since

$$\left| \int_{\partial X} \overline{\zeta} d\mu^* \right| = \lambda(X) > 0 ,$$

we obtain that $\mu^* \not\equiv 0$.

§5. Further remarks on analytic measures and $\lambda(X)$.

We start out this section with the following remark.

Observation. Unfortunately, the definition of analytic measures depends on a choice of the sequence $\{X_n\}_1^\infty$. Namely, let $\{X'_n\}_1^\infty$ be another decreasing sequence of the finitely connected compact sets with analytic boundaries converging to X (i.e. $X = \bigcap_{n=1}^\infty X'_n$)

Let μ be an analytic measure on ∂X defined by the sequence of the analytic differentials $\mu_n = f_n \, d\zeta|_{\partial X_n}$, $f_n \in E_1(X_n)$. Then, there is no guarantee that the measures $\mu_n' = f_n \, d\zeta|_{\partial X_n'}$ (assuming $X_n' \subset X_n$) even have uniformly bounded total variations. Thus, we can not talk about the weak (*) convergence of μ_n' to μ . At the same time in the following Proposition we discuss the situation when some information can be obtained.

<u>Proposition 2.11</u>. Let X, $\{X_n\}_1^\infty$, $\{X_n'\}_1^\infty$, μ , $\{f_n\}_1^\infty$, μ_n , μ_n' be as above. Assume,

$$\|\mu_n'\| = \int_{\partial X_n'} |f_n||d\zeta| \leq \text{const}$$

Then, there exists a subsequence $\{n_k\}^\infty$, such that $\mu_{n_k}' = f_{n_k} \, d\zeta|_{\partial X_{n_k}'}$ converges weak (*) to the measure μ .

Moreover, if any subsequence $\{\mu_{n_k}'\}$ converges weak (*) to a certain measure μ^+ , then $\mu^+ \equiv \mu$.

<u>Proof</u>. Choose a subsequence $\{n_k\}$ such that μ_{n_k}' weak (*) converges to a certain measure μ' as $k \to \infty$. (We can do it since $\|\mu_n'\|$ are uniformly bounded.) It remains to show that $\mu' \equiv \mu$. Let $\sigma \equiv \mu' - \mu$. Then, $\hat{\sigma}(z) \equiv 0$, $\forall_z \in C\setminus X$, since μ, μ' are orthogonal to $R(X)$. At the same time for each $z \in X$, we have

$$\frac{1}{2\pi i}\int_{\partial X'_{n_k}} \frac{f_{n_k}\, d\zeta}{\zeta - z} - \frac{1}{2\pi i}\int_{\partial X_{n_k}} \frac{f_{n_k}\, d\zeta}{\zeta - z} =$$

$$= f_{n_k}(z) - f_{n_k}(z) \equiv 0 \quad ,$$

since $f_{n_k} \in E_1(X_{n_k})$, and $f_{n_k} \in E_1(X'_{n_k})$ and, therefore, f_{n_k} is representable by the Cauchy integral in X_{n_k} and X'_{n_k} . (We are assuming here that $X'_{n_k} \subset X_{n_k}$) . So, $\hat{\mu}_{n_k}(z) - \hat{\mu}'_{n_k}(z) \equiv 0$ on X for all n_k .

At the same time by Proposition 1.4 $\hat{\mu}_{n_k}(z) - \hat{\mu}'_{n_k}(z) \to \hat{\sigma}(z)$ in the weak topology of L^1 . Hence $\hat{\sigma}(z) \equiv 0$ a.e. on X . So, $\hat{\sigma}(z) \equiv 0$ a.e. in C . This proves that $\sigma \equiv 0$, a.e., $\mu \equiv \mu'$ Proposition is proved.

<u>Proposition 2.12</u>. Let μ be an analytic measure on ∂X defined by the sequence $\{f_n\}_1^\infty$, $f_n \in E_1(X_n)$. Then, the following statements hold:

(i) supp $\hat{\mu}(z) \subset X$

(ii) there exists a subsequence $\{f_{n_k}\}$ ∘ such that $f_{n_k} \longrightarrow \hat{\mu}(z)$ in the weak topology of L^1 ∘

(iii) there exists the subsequence $\{n'_k\}$ and the sequence of functions $\{\phi_{n'_k}\}_{k=1}^\infty$ such that

$$\phi_{n'_k} \in E_1(X'_{n_k}) \quad \text{and}$$

$$\phi_{n_k} \longrightarrow \hat{\mu}(z) \quad \text{a.e.}$$

On the other hand, let $f(z) \in L^1$, $\operatorname{supp} f \subset X$. Let us also assume that one of the following conditions holds:

(1) $\exists \{f_n\}_1^\infty$, $f_n \in E_1(X_n)$, $\|f_n\|_{E_1(X_n)} \le$

$\le M_1 < +\infty$ and $f_n \longrightarrow f$ in the weak topology of L^1.

(2) $\exists \{\phi_n\}_1^\infty$, $\phi_n \in E_1(X_n)$, $\|\phi_n\|_{E_1(X_n)} \le$

$\le M_2 < +\infty$, $\operatorname{supp} \phi_n \subset X_n$ and $\phi_n \longrightarrow f$ a.e.

Then, there exists an analytic measure μ such that $\hat{\mu}(z) = f(z)$ a.e.

Proof. Part I. (i) follows from the fact that $\mu \perp R(X)$, so, $\hat{\mu}(z) \equiv 0$ outside X.

(ii) Take a subsequence $\{f_{n_k}\}_{k=1}^\infty$ such that the measures $d\mu_{n_k} = f_{n_k}\, d\zeta|_{\partial X_{n_k}}$ converge to μ weak (*). Then, according to Proposition 1.4 $\hat{\mu}_{n_k}(z)$ converge to $\hat{\mu}(z)$ weakly in L^1.

$$\hat{\mu}_{n_k}(z) = \frac{1}{2\pi i} \int_{\partial X_{n_k}} \frac{f_{n_k}(\zeta)\, d\zeta}{\zeta - z} \equiv f_{n_k}(z)$$

for all $z \in X_{n_k}$. In particular, $f_{n_k} \equiv \hat{\mu}_{n_k}(z)$ on X .

This proves (ii).

(iii) From (ii) it follows that there exists a sequence of functions $\{\phi_{n_k}\}_{k=1}^{\infty}$ such that

$$\forall k : \phi_{n_k} = \sum_{i=1}^{j_{n_k}} \alpha_{n_k}^{(i)} f_{n_i} , \quad \text{where} \quad \alpha_{n_k}^{(i)} \geq 0 ,$$

$$\sum_{i=1}^{j_{n_k}} \alpha_{n_k}^{(i)} = 1 \quad \text{and} \quad \phi_{n_k} \longrightarrow \hat{\mu}(z) \quad \text{in}$$

L^1 - norm (see for instance [47]) .

Then, $\phi_{n_k} \in E_1 (X_{j_{n_k}})$. By taking a subsequence if necessary which converges to $\hat{\mu}(z)$ a.e., we finish the proof.

<u>Part II</u>. At first we note the following. Consider any set E on the complex plane. Let $\{\phi_n\}$ be such that (2) holds. Then,

$$\left| \int_E \phi_n \, dxdy \right| = \left| \int_E \left\{ \int_{\partial X_n} \frac{1}{2\pi i} \frac{\phi_n(\zeta)d\zeta}{\zeta - z} \right\} dxdy \right| \leq$$

$$\leq M_2 \int_E \frac{1}{|z|} \, dxdy$$

Since $\frac{1}{z}$ is locally integrable, this implies that the indegrals $\{\int \phi_n \, dxdy\}$ are uniformly absolutely continuous with respect to the area measure.

Hence, for any $\phi_0 \in L^\infty$ we have

$$\int \phi_n \phi \, dxdy \longrightarrow \int f \phi \, dxdy$$

Thus, (2) implies (1). So, it suffices to prove our statement assuming that (1) holds.

The weak convergence in L^1 implies that $f_n \longrightarrow f$ in the distribution sence. Moreover, for any n and $z \in X_n$, we have,

$$f_n(z) = \frac{1}{2\pi i} \int_{\partial X_n} \frac{f_n(\zeta) \, d\zeta}{\zeta - z} \tag{2.16}$$

since $f_n \in E_1(X_n)$. Denote the measures $f_n(\zeta)d\zeta|_{\partial X_n}$ by μ_n . So,

$$\hat{\mu}_n(z) \longrightarrow f(z) \quad \text{weakly in} \quad L^1$$

and, therefore, $\hat{\mu}_n \longrightarrow f$ in the distribution sence.

Hence, $\frac{\partial}{\partial z} \hat{\mu}_n \longrightarrow \frac{\partial}{\partial z} f$ in the distribution sence. According to Proposition 1.3 in Chapter I, we have

$$-2i \frac{\partial}{\partial z} \hat{\mu}(z) = f_n(\zeta)d\zeta|_{\partial X_n} = \mu_n \quad .$$

Since $\|\mu_n\| \le M_1 < +\infty$ we can choose a subsequence $\{\mu_{n_k}\}$ converging weak (*) to the measure μ. Then, $\{\mu_{n_k}\}$ converges to μ in the distribution sense. Therefore, $\frac{i}{2}\mu \equiv \frac{\partial}{\partial \bar z} f$. Applying Proposition 1.3 again, we obtain that

$$f(z) = \frac{1}{2\pi i} \int_{\partial X} \frac{d\mu(\zeta)}{\zeta - z} \quad \text{a.e.}$$

Clearly, $\mu \perp R(X)$ and μ is an analytic measure. The proof is complete.

Remark. Unfortunately, we cannot claim in (iii) that the norms $\|\phi_{n_k}\|_{E_1(X_{j_{n_k}})}$ are uniformly bounded. At the same time, if we considered the measures

$$d\psi_{n_k} = \sum_{i=1}^{j_{n_k}} \alpha_{n_k}^{(i)} \left(f_{n_i}\, d\zeta \,|_{\partial X_{n_i}}\right) ,$$

we would have,

$$\|d\psi_{n_k}\| \le \sum_{i=1}^{j_{n_k}} \alpha_{n_k}^{(i)} \|d\mu_{n_i}\| \le M < +\infty$$

Therefore, there exists a subsequence of the sequence $\{d\psi_{n_k}\}$ converging weak (*) to $d\mu$. The measures $\{d\psi_{n_k}\}$ do not define $d\mu$ as an analytic μ measure, since each $d\psi_{n_k}$ is supported on few boundaries ∂X_{n_i}, $i = 1, \ldots, j_{n_k}$. So here we deal with a more complicated object which is natural to call an "analytic current" following ideas of the geometric measure theory (cf. to [21], [23], [49]).

The following Proposition allows to estimate $\lambda(X)$ in terms of geometrical characteristics of X provided that X has a finite perimeter.

<u>Proposition 2.13</u>. Let X have the finite perimeter $P(X)$ and $m_2(X) > 0$. Then,

$$\lambda(X) \geq 2\frac{m_2(X)}{P(X)}$$

In case of $X = \{z : |z| \leq 1\}$, this inequality becomes an equality

<u>Proof</u>. Let B_X be a reduced boundary of X . Since X is compact, $B_X \subset X$. Take any $\phi \in R(X)$.

We have

$$P(X) \cdot \|\overline{\zeta} - \phi\|_X = \sup_{\zeta \in X} |\overline{\zeta} - \phi(\zeta)| \cdot P(X) \geq$$

$$\geq \sup_{\zeta \in B_X} |\overline{\zeta} - \phi(\zeta)| \cdot P(X) =$$

$$= \int_{B_X} \sup_{\zeta \in B_X} |\overline{\zeta} - \phi(\zeta)| \, |dz| \geq \left| \int_{B_X} (\overline{z} - \phi(z)) dz \right|$$

Since, $dz \big|_{B_X} \perp R(X)$, from (2.17) we obtain the following:

$$\| \overline{\zeta} - \phi(\zeta) \| \cdot P(X) \geq \left| \int_{B_X} \overline{z}\, dz \right|$$

According to the definition of the perimeter measure $\frac{1}{2\pi i}\, dz\big|_{B_X}$, we get

$$\left| \int_{B_X} \overline{z}\, dz \right| = 2 \left| \int_X \int \frac{\partial}{\partial \overline{z}}\, \overline{z}\, dxdy \right| = 2 \cdot m_2(X)$$

as $\overline{z}$ is a C^∞ -function.

So, finally we obtain

$$\| \zeta - \phi(\zeta) \|_X \geq 2 \cdot \frac{m_2(X)}{P(X)}$$

for all $\phi \in R(X)$.

Taking an infimum, we obtain

$$\lambda(X) \geq 2 \cdot \frac{m_2(X)}{P(X)}$$

If X is the unit disk, then $m_2(X) = \pi$, $P(X) = 2\pi$.

$$\lambda(X) \leq \| \zeta - 0 \|_{\partial X} = 1$$

Therefore $\lambda(X) = 1$. The proof is complete.

<u>Proposition 2.14</u>. Let X be a compact set in $\mathbb{C}$.

Then,

$$\lambda(X) = \frac{1}{\pi} \sup_{\substack{\mu \perp R(X) \\ \|\mu\| \le 1}} \left| \int_X \hat{\mu}(z)\,dxdy \right|$$

Moreover, there exists an analytic measure μ^* for which supremum in the right hand side is attained.

<u>Proof</u>. According to Proposition 2.2. we have

$$\lambda(X) \overset{\text{def}}{=} \inf_{\phi \in R(X)} \|\bar{\zeta} - \phi\|_{C(X)} = \sup_{\substack{\mu \perp R(X) \\ \|\mu\| \le 1}} \left| \int_X \bar{\zeta}\, d\mu \right|$$

Take any $\mu \in R(X)^{\perp}$. We can regard $\bar{\zeta}|_X$ as a restriction of a C_0^{∞} - function ψ on X . Then, according to Green's formula we have

$$\int_X \zeta\, d\mu = \frac{1}{\pi} \int_X \iint_C \frac{1}{z-\zeta} \frac{\partial\psi}{\partial z}\, dxdy\ \ d\mu(\zeta) =$$

$$= \frac{1}{\pi} \int_C \frac{\partial\psi}{\partial z}\ \hat{\mu}(z)\ dxdy = \frac{1}{\pi} \int_X \hat{\mu}(z)\ dxdy .$$

We recall, that since $\mu \in R(X)^{\perp}$, $\hat{\mu} \equiv 0$ outside of X .

Hence,

$$\lambda(X) = \frac{1}{\pi} \sup_{\substack{\mu \perp R(X) \\ \|\mu\| \le 1}} \left| \int_X \hat{\mu}(z)\, dxdy \right|$$

Taking into account that $\overline{\zeta}$ is harmonic in the neighborhood of X and applying Theorem 2.1. we finish the proof.

Remark. Proposition 2.14 explains why we called $\lambda(X)$ the rational capacity of X. According to the above proposition, $\lambda(X)$ shows "how much" of the Cauchy transform of measures orthogonal to $R(X)$ can be accumulated on X.

Proposition 2.15. Let X be any compact set in C. Then,

$$\sqrt{\frac{m_2(X)}{\pi}} \ge \lambda(X)$$

Moreover, this inequality is sharp; since for the unit disk it becomes an equality,

Remark. This inequality has been also observed by H. Alexander - see H. Alexander, Projections of Polynomial Hulls, J. Funct. Anal., Vol. 3, No. 1 (1973), pp. 13-19.

Proof. According to Proposition 2.14 we have

$$\lambda(X) = \frac{1}{\pi} \sup_{\substack{\mu \perp R(X) \\ \|\mu\| \leq 1}} \left| \int_X \hat{\mu}(z)\, dxdy \right|$$

Fix $\mu \in R(X)^{\perp}$, $\|\mu\| \leq 1$. Then, applying Fubini's Theorem we obtain

$$\left| \int_X \hat{\mu}(z) dxdy \right| = \left| \int_X d\mu(\zeta) \int_X \frac{dxdy}{z-\zeta} \right| \leq$$

$$\leq \max_{\zeta \in X} \left| \int_X \frac{dxdy}{z-\zeta} \right| \|\mu\| \leq \max_{\zeta \in X} \left| \int_X \frac{dxdy}{z-\zeta} \right|$$

To estimate the last term we follow the classical paper of L. Ahlfors and A. Beurling [1]. (Also see - [25]).

For the sake of completeness we repeat their beautiful argument. Let

$$F(\zeta) = \int_X \frac{dxdy}{z-\zeta} \qquad , \quad z = x + iy$$

Without loss of generality we can assume that

$$F(0) = \max_{\zeta \in X} |F(\zeta)|$$

Then, transferring to polar coordinates we obtain

$$F(0) = \iint_X \frac{dxdy}{z} = \iint_X \cos\theta \, drd\theta$$

Let $X^+ = X \cap \{x = \operatorname{Re} z > 0\}$. Put $\ell(\partial\ell)$ to be the length of the set of points $se^{i\theta} \in X^+$ with $s \leq r$.

Then, applying the Schwarz inequality, we obtian:

$$F(0) \leq \iint_{X^+} \cos\theta \, drd\theta = \int_{-\pi/2}^{\pi/2} \ell(\infty,\theta) \cos\theta \, d\theta \leq$$

$$\leq (\frac{\pi}{2} \int_{-\pi/2}^{\pi/2} (\ell(\infty,\theta))^2 d\theta)^{1/2}$$

Since $r - \ell(r,\theta)$ is a positive and increasing function,

$$\int_{X^+\cap\{re^{i\theta}\}} r\, dr \geq \int_{X^+\cap\{re^{i\theta}\}} \ell(r,\theta)\, d\ell(r,\theta) = \frac{\ell(\infty,\theta)^2}{2}$$

Thus,

$$F(0) \leq (\pi \int_{-\pi/2}^{\pi/2} d\theta \int_{X^+\cap\{re^{i\theta}\}} r\, dr)^{1/2} \leq (\pi\, m_2(X^+))^{1/2} \leq$$

$$\leq (\pi\, m_2(X))^{1/2}$$

Therefore,

$$\frac{1}{\pi} \left| \int_X \hat{\mu} \, d\, m_2 \right| \leq \sqrt{\frac{m_2(X)}{\pi}}$$

for any measure $\mu \perp R(X)$, $\|\mu\| \leq 1$. Thus,

$$\lambda(X) \leq \sqrt{\frac{m_2(X)}{\pi}}$$

If $X = \{z : |z| \leq 1\}$, then,

$$\lambda(X) = 1 = \sqrt{\frac{m_2(X)}{\pi}} \quad .$$

Proposition is proved.

Corollary 2.2. Let X be a compact set. Let $\gamma(X)$ denote the analytic capacity of X (see [24], [25], [48]).

Then,

$$\gamma(X) \geq \lambda(X)$$

The inequality is sharp since for the unit disk it becomes an equality.

Proof. This follows immediately from Proposition 2.15 and the result of L. Ahlfors and A. Beurling which says that

$$m_2(X) \leq \pi \gamma (X)^2$$

(see [1], [24], [25]).

Corollary 2.3. (Isoperimetric inequality - cf. [10], [21]) Let X be a compact set in C. Then

$$P(X)^2 \geq 4 \pi m_2(X)$$

This inequality is sharp since it becomes an equality for the unit disk.

Proof. The result follows immediately from Propositions 2.13 and 2.15.

§6. Annihilating measures of $R(X)$ and the space of analytic measures.

Let X be an arbitrary compact set. Let $\{X_n\}_1^\infty$, $X_1 \supset X_2 \supset \dots$, $\bigcap_{n=1}^{\infty} X_n = X$ be a fixed sequence of fintely connected compact sets with Jordan analytic boundaries converging to X.

Let $M = M(X,\{X_n\})$ denote the linear span of all analytic measures defined relative to the sequence $\{X_n\}$.

Definition. Define the space $\mathcal{M} = M(X,\{X_n\})$ to be a weak (*) closure of $M(X,\{X_n\})$.

The following theorem is an extension of Theorem 2.1.

Theorem 2.2. Let $f \in H(X)$. Then, there exists a measure $\mu^* \in \mathcal{M}$, $\|\mu^*\| = 1$ such that μ^* is an extremal measure in the left hand side of the problem (2.13).

Proof. Let $\{h_n\}$ be a sequence of functions such that each h_n is harmonic in a certain neighborhood of X and $\|f - h_n\|_{\partial X} < 1/n$. Put

$$\sup_{\substack{\|\mu\| \le 1 \\ \mu \perp R(X)}} \left| \int_X h_n \, d\mu \right| = L_n$$

Let μ_n^* be analytic measures such that $\|\mu_n^*\| = 1$ and μ_n^* is an extremal measure for this extremal problem. The existence of μ_n^* is guaranteed by Theorem 2.1.

By Proposition 2.10 there exists an extremal measure in the problem (2.13) for f . We denote this measure by μ_0^* . Then, applying Proposition 2.10 and Theorem 2.1 again, we obtain:

$$|\lambda_f(X) - L_n| = \left| \sup_{\substack{\|\mu\| \le 1 \\ \mu \perp R(X)}} \left| \int_X f \, d\mu \right| - \left| \int_X h_n \, d\mu_n^* \right| \right| =$$

$$= \left| \left| \int_X f \, d\mu_0^* \right| - \left| \int_X h_n \, d\mu_n^* \right| \right| \le$$

$$\begin{cases} \left|\int_X f\, d\mu_0^*\right| - \left|\int_X h_n\, d\mu_0^*\right|, & \text{if } \lambda_f(X) \ge \lambda_{h_n}(X) \\ \left|\int_X h_n\, d\mu_0^*\right| - \left|\int_X f\, d\mu_n^*\right|, & \text{if } \lambda_f(X) < \lambda_{h_n}(X) \end{cases}$$

$$\le \| f - h_n \|_X = 1/n \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty$$

At the same time, since $\| \mu_n^* \| = 1$, there exists a subsequence $\{\mu_{n_k}^*\}$ converging weak (*) to a measure μ^*. Therefore

$$\left|\int_X f\, d\mu^*\right| = \lim_{k\to\infty} \left|\int_X f\, d\mu_{n_k}^*\right| = \lim_{k\to\infty} \left|\int_X h_{n_k}\, d\mu_{n_k}^*\right| =$$

$$= \lim_{k\to\infty} L_{n_k} = \lambda_f(X)$$

Hence, μ^* is an extremal measure. But $\mu^* =$ weak (*) $\lim_{k\to\infty} \mu_{n_k}^*$, $\mu_{n_k}^* \in M(X,\{X_n\})$. So, $\mu^* \in \mathscr{M}(X,\{X_n\})$. Theorem is proved.

Notation. We recall, that if E_0 is a subspace of a linear topological space E then $E_0^\perp = \{ f \in E^* : f|_{E_0} \equiv 0\}$. If E_1 is a subspace of the dual space E^*, then ${}^\perp E_1 = \{x \in E : \forall f \in E_1\ f(x) = 0\}$.

<u>Theorem 2.3</u>. Let X , $\{X_n\}_1^\infty$ be as above. If $H(X)|_{\partial X} = C(\partial X)$, then $\mathcal{M} = (R(X)|_{\partial X})^\perp$.

Proof. Assume that the theorem is false. It is clear that $\mathcal{M} \subset (R(X)|_{\partial X})^\perp$. Then, there exists a measure μ_0 , such that $\text{supp}\ \mu_0 \subset \partial X$, $\mu_0 \perp (R(X)|_{\partial X})$ and $\mu_0 \notin \mathcal{M}$. Since $\mathcal{M}(X,\{X_n\})$ is weak (*) closed, it is closed in the strong (normed) topology on $C(\partial X)^*$. Hence,

$$\inf_{\mu\in\mathcal{M}} \|\mu_0 - \mu\| > 0 \tag{2.18}$$

Let us consider the following extremal problem in $C(\partial X)$.

$$\sup_{\substack{f\in{}^\perp\mathcal{M} \\ \|f\|_{\partial X} \le 1}} \left| \int_{\partial X} f\, d\mu_0 \right|$$

According to Proposition 2.1, we have

$$\sup_{\substack{f\in{}^\perp\mathcal{M} \\ \|f\|_{\partial X} \le 1}} \left| \int_{X} f\, d\mu_0 \right| = \inf_{\mu\in({}^\perp\mathcal{M})^\perp} \|\mu_0 - \mu\| \quad .$$

As it is known (see [47]) $({}^\perp\mathcal{M})^\perp$ is equal to the weak (*) closure of $\mathcal{M}$ which is $\mathcal{M}$ itself. So, from (2.18) we obtain that

$$\sup_{\substack{f \in {}^{\perp}\mathcal{M} \\ \|f\| \leq 1}} \left| \int_{\partial X} f \, d\mu_0 \right| = \inf_{\mu \in \mathcal{M}} \|\mu_0 - \mu\| > 0 \quad .$$

Therefore, there exits $f_0 \in {}^{\perp}\mathcal{M}$ such that

$$\left| \int_{\partial X} f_0 \, d\mu_0 \right| > 0 \tag{2.19}$$

Since $\mu_0 \in (R(X)|_{\partial X})^{\perp}$, (2.19) implies that $f_0 \notin R(X)|_{\partial X}$. At the same time $f_0 \in H(X)|_{\partial X}$. Then, according to Proposition 2.8.

$$\inf_{\phi \in R(X)} \|f_0 - \phi\|_{\partial X} = \inf_{\phi \in R(X)} \|f_0 - \phi\|_X =$$
$$\overset{\text{def}}{=} \lambda_{f_0}(X) > 0$$

Hence, in view of Theorem 2.2 there exists the measure $\mu^* \in \mathcal{M}$ such that $\|\mu^*\| = 1$. and

$$\left| \int_{\partial X} f_0 \, d\mu^* \right| = \lambda_{f_0}(X) > 0$$

But this is impossible, since $f_0 \in {}^{\perp}\mathcal{M}$. Thus, we arrived into the contradiction. Theorem is proved.

<u>Corollary 2.4</u>. If $H(X)_{\partial X} = C(\partial X)$, then the space $\mathcal{M}(X, \{X_n\})$ does not depend on the sequence $\{X_n\}$.

Remark 1. Theorem 2.3 is natural to consider as a generalization of F. and M. Riesz theorem for finitely connected domains to the compact sets with a "good" boundary in respect to the Dirichlet problem.

Remark 2. We want to point out that we never made any assumptions concerning the existence of the interior of a set X. So, the statement of Theroem 2.3 holds in particular for no where dense compact sets X for which $H(X) = C(X)$. At the same time we did not put any assumptions on the connectivity of the complement $C \backslash X$. The last hypothesis was assumed in E. Bishop's papers [3], [4], [5] and in the further generalization of his results due to B. Øksendal (see [40], [41]). At the same time, our definition of an analytic measure μ does not imply the control of $\|\mu\|$ over the norms of the analytic differentials $\{f_n \, d\zeta\}_1^\infty$ converging to μ. This is the loss in comparison with E. Bishop's definition for simply connected sets with connected complement. (cf. [2], [6]).

Now we state the corollary giving certain geometric conditions on the set X under which one can apply Theorem 2.3.

We recall that the point $z \in \partial X$ is said to satisfy the Lebesgue condition if

$$\int_S \frac{dr}{r} = \infty$$

where $S = \{r : 0 < r < 1, \{\zeta : |z - \zeta| = r\} \cap X^c \neq \emptyset\}$

<u>Corollary 2.5</u>. Let X be a compact set in C such that an each point on ∂X satisfies the Lebesgue condition. Then $\mathcal{M}(X,\{X_n\}) = (R(X)|_{\partial X})^{\perp}$ for any sequence $\{X_n\}$ converging to X .

<u>Proof</u>. The statement follows immediately from Theorem 2.3 and the well-known fact that for such X $H(X)|_{\partial X} = C(\partial X)$ (see [24], Chapter II, Lemma 3.2).

We want to finish this Chapter by posing some problems which, we think, are of some interest.

<u>Problem 1</u>. Let $\mathcal{M}(X)$ denot the weak (*) closure of a linear span of the spaces $\mathcal{M}_{\alpha}(X,\{X_n^{\alpha}\})$ where $\{X_n^{\alpha}\}_1^{\infty}$ $\alpha \in I$ are all possible sequences of finitely connected compact sets converging decreasingly to X . Is $\mathcal{M}(X) = (R(X)|_{\partial X})^{\perp}$? If it is false (which is very possible) what are the necessary and sufficient conditions on the set X for this statement to be true?

The natural continuation of Problem 1 is the following question.

<u>Problem 2</u>. To construct an example of a compact set X and a sequence of finitely connected compact sets $\{X_n\}$ converging to X such that $\mathcal{M}(X,\{X_n\}) \neq (R(X)|_{\partial X})^{\perp}$.

<u>Problem 3</u>. Let $\{X_n^{(1)}\}$, $\{X_n^{(2)}\}$ be two sequences of finitely connected compact sets converging to X . What are necessary and sufficient conditions on $\{\partial X_n^{(1)}\}$ and $\{\partial X_n^{(2)}\}$

for $\mathcal{M}(X,\{X_n^{(1)}\}) = \mathcal{M}(X,\{X_n^{(2)}\})$. Furthermore, if any decreasing sequence $\{X_n\}$ leads to the same space $\mathcal{M}(X,\{X_n\} \subset (R(X)|_{\partial(X)})^{\perp}$. What could we say about the geometry of ∂X ?

CHAPTER 3

LOCALIZATION OF THE CAUCHY TRANSFORM

§1. Introductory remarks.

Let u be any distribution in C, i.e. $u \in \mathcal{D}'(\mathbb{R}^2)$. If h is an arbitrary C_0^∞-function, then as it is well known the Leibniez's rule still holds for the product uh. (see [47]). Namely,

$$\frac{\partial}{\partial \bar{z}}(u \cdot h) = \frac{\partial}{\partial \bar{z}} u \cdot h + u \cdot \frac{\partial}{\partial \bar{z}} h$$

where the equality is understood in the distribution sence. In particular, let μ be any Borel measure in C. Put $u = \hat{\mu}(z) \in L^1$. Then, for any $h \in C_0^\infty$ we have (cf. to [24], [25]).

$$\frac{\partial}{\partial \bar{z}}(\hat{\mu} \cdot h) = \frac{\partial}{\partial \bar{z}}\hat{\mu} \cdot h + \hat{\mu}\frac{\partial h}{\partial \bar{z}} = -\pi \mu h + \hat{\mu}\frac{\partial h}{\partial \bar{z}} \tag{3.1}$$

Or, in other words, $\hat{\mu} \cdot h$ is a Cauchy transform of the measure $\mu h - \frac{1}{\pi}\hat{\mu}\frac{\partial h}{\partial \bar{z}} dxdy$. When $h \equiv 1$ on a small disk Δ_0 and $\equiv 0$ outside of a little bigger disk Δ_1, (3.1) allows us to "localize" the Cauchy transform $\hat{\mu}$ in Δ_0. In view of this (3.1) has many applications to the problems of rationals approximation (see [9], [24], [40], [41], [48]).

If h is not a C^1-function, or not even a continuous function then, generally speaking, we can not apply (3.1). For example let μ be a δ-mass and let h be an arbitrary L^∞-function. Then the right-hand side of (3.1) does not make any sense (in particular, any "distribution sense").

At the same time, it turns out that when h is a characteristic function of a nicely bounded domain, in particular, h is a characteristic function of a disk, (3.1) does hold for any μ.

More precisely, in §2 in Theorem 3.1 we prove that for any fixed $\zeta \in C$, for any μ and almost all $r > 0$ (3.1) holds for $h = \chi_{\Delta_r}$ where $\Delta_r = \{z : |z - \zeta| < r\}$. As an immediate corollary of this result we obtain the "splitting" theorem for measures orthogonal to $R(X)$ due to E. Bishop (for $P(X)$) and L. Kodama, (see [3], [37]). Moreover, we provide an explicit formula for the resulting measures. This has not been done in [3] or [37]. Furthermore, our extension of the formula (3.1) allows us to localize explicitly the Cauchy transform of a given measure in small disks. We develop certain techniques of working with such "localized" Cauchy transform. As a result in §3, we obtain series of statements which allows us to obtain results concerning the "thickness" of a measure μ provided that $\hat{\mu}$ has certain "nice" properties like boundedness, continuity, etc. This technique, and even the localization formula itself is, in spirit, very close to the ideas of localization of the perimeter measure contained in the E. De Giorgi paper [11] or, more general, to the theory of slices introduced in Geometric Measure Theory by H. Federer (see his papers [22], [23].

§2. The localization formula for the Cauchy transform

Theorem 3.1. Let μ be a finite complex measure in C with a compact support. Fix an arbitrary $\zeta_0 \in C$. Put $\Delta_r = \{z : |z - \zeta_0| < r\}$. Then, for almost all $r > 0$ the following holds,

$$\frac{\partial}{\partial \bar{z}}(\hat{\mu} \cdot \chi_{\Delta_r}) = (\frac{\partial}{\partial \bar{z}}\hat{\mu}) \cdot \chi_{\Delta_r} + \hat{\mu} \cdot (\frac{\partial}{\partial \bar{z}} \chi_{\Delta_r}) = \tag{3.2}$$

$$= -\pi\mu|_{\Delta_r} - \frac{1}{2i}\hat{\mu}(\zeta)\, d\zeta|_{\partial\Delta_r}$$

Remark (4.1) is understood in the distribution sense. $\mu|_{\Delta_r}$ means the restriction of the measure μ onto Δ_r.

Proof. Since $U_\mu(2) = \int_C \frac{d(\mu)(\zeta)}{|\zeta - z|} \in L^1_{loc}$, for almost all $r > 0$ we have

$$U_\mu(z)\Big|_{\partial\Delta_r} \in L^1(\partial\Delta_r, ds) \tag{3.3}$$

As μ is a finite measure the following also holds for almost $r > 0$

$$|\mu|(\partial\Delta_r) = 0 \tag{3.4}$$

From now on we assume that we have chosen $r > 0$ such that (3.3) and (3.4) are satisfied. Take an arbitrary $\phi \in C_0^\infty(C)$. Applying Fubini's theorem we obtain the following.

$$< \frac{\partial}{\partial \bar{z}}(\hat{\mu}\cdot\chi_{\Delta_r}),\ \phi > = -< \hat{\mu}\cdot\chi_{\Delta_r},\ \frac{\partial\phi}{\partial\bar{z}} > =$$

$$= -\int_{\Delta_r} \hat{\mu}\cdot\frac{\partial\phi}{\partial\bar{z}}\,dxdy = -\int_{\Delta_r}\frac{\partial\phi}{\partial\bar{z}}\,dxdy\int_C\frac{d\mu(\zeta)}{\zeta - z} =$$

$$= -\int_C d\mu(\zeta)\int_{\Delta_r}\frac{\partial\phi}{\partial\bar{z}}\cdot\frac{1}{\zeta - z}\,dxdy \tag{3.5}$$

Denote the inner integral in (3.5) by $F(\zeta)$. Note that $F(\zeta) \in C(\mathbb{C})$. Hence, from (3.5) we obtain

$$< \frac{\partial}{\partial\bar{z}}(\hat{\mu}\,\chi_{\Delta_r}),\ \phi > = -\int F(\zeta)d\mu(\zeta) =$$

$$= -\int_{C\setminus\bar{\Delta}_r} F(\zeta)d\mu(\zeta) - \int_{\partial\Delta_r} F(\zeta)d\mu(\zeta) - \int_{\Delta_r} F(\zeta)d\mu(\zeta) \tag{3.6}$$

The second integral in (3.6) is equal to zero, since $|\mu|(\partial\Delta_r) = 0$. Moreover, in accordance with Lemma 1.2 in Chapter 1 applying Fubini's theorem again we obtain

$$\int_{C\setminus\overline{\Delta}_r} F(\zeta)d\mu(\zeta) = \int_{C\setminus\overline{\Delta}_r} d\mu(\zeta) \int_{\Delta_r} \frac{\partial\phi}{\partial\bar{z}} \frac{1}{\zeta - z} dxdy =$$

$$= \int_{C\setminus\overline{\Delta}_r} d\mu(\zeta) \left\{ \frac{1}{2i} \int_{\partial\Delta_r} \frac{\phi(z)}{\zeta - z} dz \right\} = \tag{3.7}$$

$$= \frac{1}{2i} \int_{\partial\Delta_r} \phi(z) \left\{ \int_{C\setminus\overline{\Delta}_r} \frac{d\mu(\zeta)}{\zeta - z} dz \right\}$$

At the same time, applying Lemma 1.2 to the third integral in (3.6) we obtain

$$\int_{\Delta_r} F(\zeta)d\mu(\zeta) = \int_{\Delta_r} d\mu(\zeta) \int_{\Delta_r} \frac{\partial\phi}{\partial\bar{z}} \frac{1}{\zeta - z} dxdy =$$

$$= \int_{\Delta_r} \left\{ \pi\phi(\zeta) + \frac{1}{2i} \int_{\partial\Delta_r} \frac{\phi(z)dz}{\zeta - z} \right\} d\mu(\zeta) \tag{3.8}$$

$$= \int_{\Delta_r} \pi\cdot\phi(\zeta)d\mu(\zeta) + \int_{\Delta_r} \left\{ \frac{1}{2i} \int_{\partial\Delta_r} \frac{\phi(z)dz}{\zeta - z} \right\} d\mu(\zeta)$$

According to (3.3) we can apply Fubini's theorem to the second integral in (3.8). Then, we get,

$$\int_{\Delta_r} F(\zeta)d\mu(\zeta) = \int_{\Delta_r} \pi\phi(\zeta)d\mu(\zeta) + \frac{1}{2i} \int_{\partial\Delta_r} \phi(z) \left\{ \int_{\Delta_r} \frac{d\mu(\zeta)}{\zeta - z} \right\} dz \tag{3.9}$$

According to (3.4), we have

$$\int_{\Delta_r} \pi\, \phi(\zeta)d\mu(\zeta) = \int_{\overline{\Delta}_r} \pi\, \phi(\zeta)d\mu(\zeta)$$

$$\int_{\Delta_r} \frac{d\mu(\zeta)}{\zeta - z} = \int_{\overline{\Delta}_r} \frac{d\mu(\zeta)}{\zeta - z} \quad \text{for all} \quad z \in C \tag{3.10}$$

Therefore, combining (3.6), (3.7), (3.9) and (3.10) together we obtain

$$< \frac{\partial}{\partial z}(\mu\, \chi_{\Delta_r}), \phi > = - \frac{1}{2i} \int_{\partial \Delta_r} \phi(z) \left\{ \int_{C \setminus \overline{\Delta}_r} \frac{d\mu(\zeta)}{\zeta - z} \right\} dz$$

$$- \int_{\Delta_r} \pi\, \phi(\zeta)d\mu(\zeta) - \frac{1}{2i} \int_{\partial \Delta_r} \phi(z) \left\{ \int_{\overline{\Delta}_r} \frac{d\mu(\zeta)}{\zeta - z} \right\} dz$$

$$= - \int_{\Delta_r} \pi\, \phi(\zeta)\, d\mu(\zeta) - \frac{1}{2i} \int_{\partial \Delta_r} \phi(z) \left\{ \int_{C} \frac{d\mu(\zeta)}{\zeta - z} \right\} dz =$$

$$= - \int_{\Delta_r} \pi\, \phi(\zeta)\, d\mu(\zeta) - \frac{1}{2i} \int_{\partial \Delta_r} \phi(z)\hat{\mu}(z)dz$$

As ϕ was an arbitrary C_0^∞-funtion (3.2) has been verified .

Before we state and prove some immediate corollaries of Theorem 3.1 let us make a remark.

As one can easily notice that in the proof of Theorem 3.1 we have never used the fact that Δ_r is actually a disk. The only conditions we must have fulfilled on the boundary are (3.3) and (3.4). Therefore, (3.2) still holds for any smoothly bounded finitely connected domain G as long as (3.3) and (3.4) hold on ∂G.

Corollary 3.1. Let μ, Δ_r be as in Theorem 3.1. Then for almost all $r > 0$ the function $\hat{\mu}|_{\Delta_r} \equiv \hat{\mu}\cdot\chi_{\Delta_r}$ is the Cauchy transform of the measure

$$\mu_1 \overset{\text{def}}{\equiv} \mu|_{\Delta_r} + \frac{1}{2\pi i}\,\hat{\mu}(\zeta)d\zeta|_{\partial\Delta_r} .$$

Proof. According to Proposition 1.3 in Chapter 1 and Theorem 3.1 $\frac{\partial}{\partial\bar{z}}(\hat{\mu}\cdot\chi_{\Delta_r}) \equiv \frac{\partial}{\partial\bar{z}}\,\hat{\mu}_1$ in the distribution sense. Since $\lim_{z\to\infty} \hat{\mu}\cdot\chi_{\Delta_r}(Z) = \lim_{z\to\infty} \hat{\mu}_1(z) = 0$. Our corollary follows from the result in [25]. (Theorem 1.4, Ch. II).

Corollary 3.2. (cf. E. Bishop-L. Kodame-papers [3], [37]). Let X be a compact set in C. Let μ be a measure orthogonal to $R(X)$. Then the following statements hold:

(i) Fix an arbitrary $\zeta_0 \in C$. Then, for almost all $r > 0$ the measure $\mu_1 = \mu|_{\Delta_r} + \frac{1}{2\pi i}\,\hat{\mu}(\zeta)d\zeta|_{\partial\Delta_r}$ is orthogonal to $R(X \cap \overline{\Delta}_r)$. (Here, $\Delta_r = \{z : |z-\zeta_0| < r\}$).

(ii) For almost all real numbers x_0 the measure

$$\mu_2 = \mu\Big|_{\{\mathrm{Re}\,\zeta > x_0\}} - \frac{1}{2\pi i}\,\hat{\mu}(\zeta)d\eta\Big|_{\mathrm{Re}\,\zeta = x_0} ,$$

where $\zeta = \zeta + i\eta$

is orthogonal to $R(X \cap \{z : \operatorname{Re} z \geq x_0\})$.

Proof. We will only give a proof of (i). The proof of (ii) is similar, in view of the remark after Theroem 3.1.

Theorem 3.1. According to Corollary 3.1. $\hat{\mu}_1 \equiv \hat{\mu}\cdot\chi_{\Delta_r}$. Hence, $\hat{\mu}_1 \equiv 0$ outside of Δ_r . Also, $\hat{\mu} \equiv 0$ on $\Delta_r \setminus X$, since $\mu \perp R(X)$. Therefore, $\hat{\mu}_1 \equiv 0$ on $\Delta_r \setminus X$. Thus, $\hat{\mu}_1 \equiv 0$ on $C \setminus (X \cap \overline{\Delta}_r)$ which is equivalent to the fact that $\mu_1 \perp R(X \cap \overline{\Delta}_r)$.

§3. Estimates of measures induced by the local properties of their Cauchy transforms.

Let μ be an arbitrary compactly supported Borel measure in C .

Definition. Let $\zeta_0 \in C$. We call ζ_0 a regular point for the measure μ (μ-regular point) if there exists a real number θ such that

$$\lim_{r\to 0^+} \frac{\mu(\Delta_r)}{|\mu|(\Delta_r)} = e^{i\theta}$$

where $\Delta_r = \{z : |z - \zeta_0| < r\}$. (Here $|\mu|$ denotes the total variation of μ)

<u>Note</u>. It follows from standard results in Measure Theory that μ-almost every point ζ_0 is a μ-regular point.

Let $E_\mu = \{\zeta_0 \in C : \zeta_0$ is a μ-regular point$\}$

Clearly, $\mu(A) = \mu(A \cap E_\mu)$ for any Borel set $A \subset C$. We recall that if $f \in C(\overline{C})$, then the <u>modules of continuity of f</u> $\omega(\delta, f)$ is defined to be $\sup_{\substack{z,w \in C \\ |z-w|<\delta}} |f(z) - f(w)|$.

<u>Theorem 3.2</u>. Let μ be a Borel measure with compact support. Fix $\zeta_0 \in E_\mu$. Then the following statements hold.

(i) If $\|\hat{\mu}\|_{L^\infty(C)} \le M < +\infty$, then

$$\overline{\lim_{r\to 0^+}} \frac{|\mu|(\Delta_r)}{r} \le M_1 < +\infty$$

where M_1 does not depend on ζ_0.

(ii) If $\hat{\mu}(z) \in C(C)$ and $\omega(r) = \omega(r:\hat{\mu})$ is its modules of continuity, then

$$\overline{\lim_{r\to 0^+}} \frac{|\mu|(\Delta_r)}{r\,\omega(r)} \le M_2 < +\infty ,$$

where M_2 does not depned on ζ_0.

(Here, $\Delta_r = \{z : |z - \zeta_0| < r\}$).

Proof. (i) Since $\zeta_0 \in E_\mu$, we can find $r_0 = r_0(\zeta_0)$ such that for all $r < r_0$ the following inequality holds

$$|\mu| \ (\Delta_r) < 2 \ |\mu(\Delta_r)| \tag{3.11}$$

Moreover, since $\hat{\mu} \in L^\infty(C)$ and $\|\hat{\mu}\|_{L^\infty} \le M$, we can assume that for almost all $r < r_0$ the following is satisfied

$$\hat{\mu}\Big|_{\partial\Delta_r} \in L^\infty(\partial\Delta_r, |d\zeta|), \ \|\hat{\mu}\|_{L^\infty(\partial\Delta_r,|d\zeta|)} \le M \tag{3.12}$$

Also, according to Theorem 3.1 for almost all $r > 0$ we have

$$\frac{\partial}{\partial\bar{z}}(\hat{\mu}\cdot\chi_{\Delta_r}) = -\pi\mu\Big|_{\Delta_r} - \frac{1}{2i}\ \hat{\mu}(\zeta)d\zeta\Big|_{\partial\Delta_r} \tag{3.13}$$

From now on we consider only $r > 0$ such that (3.11),(3.12) and (3.13) hold. Fix r. Consider a function $\phi \in C_0^\infty(C)$ such that $\phi \equiv 1$ on $\Delta_{\frac{3}{2}r}$ and $\operatorname{supp}\phi \subset \Delta_{2r}$. Then, according to (3.13) we have

$$\int_{\Delta_r} \hat{\mu}\ \frac{\partial\phi}{\partial\bar{z}}\ dxdy = 0 = -<\frac{\partial}{\partial\bar{z}}(\hat{\mu}\,\chi_{\Delta_r})\ ,\ \phi>$$

$$= \pi\int_{\Delta_r}\phi\, d\mu + \frac{1}{2i}\int_{\partial\Delta_r}\phi\cdot\tilde{\hat{\mu}}(\zeta)d\zeta = \pi\,\mu\,(\Delta_r) + \frac{1}{2i}\int_{\partial\Delta_r}\hat{\mu}(\zeta)d\zeta \quad .$$

From this, making use of inequality (3.11) we obtain

$$|\mu|(\Delta_r) \le \frac{1}{\pi} \Big| \int_{\partial \Delta_r} \hat{\mu}(\zeta) d\zeta \Big| \le 2\ M \cdot r$$

Therefore, for almost all $r < r_0$ we have

$$\frac{|\mu|(\Delta_r)}{r} \le 2\ M$$

At the same time, for any $r_1 < r_0$, $\exists\, r_2 : r_1 < r_2 < r_0$ such that (3.11),(3.12) and (3.13) hold for r_2 and

$$\lim_{r_1 \to 0} \frac{r_2}{r_1} = 1$$

therefore, we have

$$2M \ge \frac{|\mu|(\Delta_{r_2})}{r_2} \ge \frac{|\mu|(\Delta_{r_1})}{r_1} \cdot \frac{r_1}{r_2}$$

Hence,

$$\overline{\lim_{r_1 \to 0}} \frac{|\mu|(\Delta_{r_1})}{r_1} \le 2M\ \overline{\lim_{r_1 \to 0}} \frac{r_2}{r_1} = 2M$$

So, (i) is proved.

(ii) Again, fix r_0 such that for $r < r_0$ (3.11) holds. Let us first restrict ourselves only onto those r for which (3.13) holds. Fix such an r . Applying (3.13) to the same test function ϕ as in part (i) and using (3.11) we obtain

$$|\mu|(\Delta_r) \le 2 \left| \frac{1}{2\pi i} \int_{\partial\Delta_r} \hat{\mu}(\zeta) d\zeta \right| = \frac{1}{\pi} \left| \int_{\partial\Delta_r} \{\hat{\mu}(\zeta) - \hat{\mu}(\zeta_0)\} d\zeta \right.$$

$$\left. + \int_{\partial\Delta_r} \hat{\mu}(\zeta_0) d\zeta \right| \le \frac{1}{\pi} \int \left| \hat{\mu}(\zeta_\zeta) - \hat{\mu}(\zeta_0) \right| |d\zeta| \le$$

$$\le 2\, r\, \omega(r) \quad .$$

So, for almost all $r < r_0$ we have

$$\frac{|\mu|(\Delta_r)}{r\omega(r)} \le \text{const.}$$

Since $\omega(r)$ is continuous, we can repeat the same argument as in part (i) and conclude that

$$\overline{\lim_{r\to 0^+}} \frac{|\mu|(\Delta_r)}{r\omega(r)} \le \text{const.}$$

Corollary 3.3. Let μ, ζ_0, Δ_r be the same as in Theorem 3.2. If $\hat{\mu} \in \text{Lip}(\alpha, C)$, $0 < \alpha \le 1$, then,

$$\overline{\lim_{r\to 0^+}} \frac{|\mu|(\Delta_r)}{r^{1+\alpha}} \le \text{const} < +\infty \quad ,$$

where constant does not depend on ζ_0 .

In particular, if $\hat{\mu} \in \mathrm{Lip}(1,C)$, then

$$\overline{\lim_{r\to 0^+}} \frac{|\mu|(\Delta_r)}{r^2} \leq \text{const} < +\infty$$

Remark. We note that in Theorem 3.2 we could replace disks Δ_r by squares with the side of size r, or more general by any piecewise smoothly bounded finitely connected sets with perimeters equal to $\text{const} \cdot r$.

We recall that if $h(r) > 0$ is an increasing continuous function and $h(0) = 0$, then m_h denotes the Hausdorff measure on C associated with the function $h(r)$. Results concerning Hausdorff measures and their properties can be found in [23], [21], [25], [28].

Proposition 3.1. Let μ be a measure in C with a compact support.

(i) If $\hat{\mu} \in L^{\infty}(C)$, then the measure $|\mu|$ is absolutely continuous with respect to 1-dimensional Hausdorff measure m_1, i.e., $m_1(E) = 0$ implies $|\mu|(E) = 0$ for any set $E \subset C$.

(ii) If $\hat{\mu} \in C(C)$ and $\omega(r) = \omega(r,\hat{\mu})$ is its modules of continuity, then $|\mu|$ is absolutely continuous with respect to the Hausdorff measure m_h, where $h(r) = r\omega(r)$.

(iii) Moreover, in case of (i) the Radon-Nikodym derivative of μ with repsect to dm_1 is bounded. Also, if (ii) holds then the Radon-Nikodym derivative of μ with respect to dm_h,

$h(r) = r\omega(r)$ is bounded.

Proof. Note that (i) is a particular case of (ii) if we define $\omega(r) \equiv \text{const} = 2\|\hat{\mu}\|_{L^\infty}$. In the following we use the method given in Ch. II of [21].

Since $\mu(A) = \mu(A \cap E_\mu)$ for all A we can only consider $A \subset E_\mu$. Let

$$B(\delta) = \{z \in A : |\mu| (A \cap \Delta_r) \le M h(r)$$

$$\forall r \le \delta \text{ , whenever } z \in \Delta_r\} .$$

Then clearly

$$|\mu|[B(\delta)] \le M \sum_j h(r_j)$$

for any countable system of disks Δ_{r_j}, $r_j \le \delta$ covering $B(\delta)$. Therefore

$$|\mu|[B(\delta)] \le M \inf_{\substack{\{\Delta_{r_j}\}, r_j \le \delta, \\ \bigcup_{j=1}^{\infty} \Delta_{r_j} \supset B(\delta)}} \{\sum_j h(r_j)\} \le$$

$$\le M\, m_h\, (B(\delta)) \le M\, m_h(A) .$$

According to Theorem 3.2 for each $\zeta_0 \in A \subset E_\mu$ we have $(\Delta_r = \Delta_r(\zeta_0))$

$$\overline{\lim_{r\to 0}} \frac{|\mu|(\Delta_r)}{h(r)} \leq M$$

Therefore, $A = \bigcup_{n=1}^{\infty} B(\frac{1}{n})$. Since $B(1) \subset B(\frac{1}{2}) \ldots$ we have

$$|\mu|(A) = \lim_{n\to\infty} B(\frac{1}{n}) \leq M\, m_h\,(A) \tag{3.14}$$

From (3.14) the proposition follows.

Corollary 3.4. Let μ, E_μ be as above. Assume $\mu \not\equiv 0$. Then, the following statements hold.

(i) If $\hat{\mu} \in L^\infty(C)$, then $m_1(E_\mu) > 0$.

(ii) If $\hat{\mu} \in C(C)$, then $m_1(E_\mu) = \infty$

(iii) If $\hat{\mu} \in Lip(\alpha, C)$, where $0 < \alpha \leq 1$, then

$m_{1+\alpha} \overset{def}{=} m_h(E_\mu) > 0$, where $h(r) = r^{1+\alpha}$. Moreover, μ is absolutely continuous with respect to $m_{1+\alpha}$ and

$$\frac{d\mu}{dm_{1+\alpha}} \in L^\infty(m_{1+\alpha}, C)$$

In particular, if $\hat{\mu} \in Lip(1, C)$, then $\mu = g\,dm_2$, $g \in L^\infty(C)$.

Remark. The results stated in Corollary 3.4 are closely related to E. Dolženko's theorem and its corollaries (see [25] for the detailed discussion).

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