Gravitational Lensing by Elliptical Galaxies, and the Schwarz Function

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Abstract. We discuss gravitational lensing by elliptical galaxies with some particular mass distributions. Using simple techniques from the theory of quadrature domains and the Schwarz function (cf. [18]) we show that when the mass density is constant on confocal ellipses, the total number of lensed images of a point source cannot exceed 5 (4 bright images and 1 dim image). Also, using the Dive–Nikliborc converse of the celebrated Newton's theorem concerning the potentials of ellipsoids, we show that "Einstein rings" must always be either circles (in the absence of a tidal shear), or ellipses.

1. Basics of gravitational lensing

Imagine n co-planar point-masses (e.g., condensed galaxies, stars, black holes) that lie in one plane, the lens plane. Consider a point light source S (a star, a quasar, etc.) in a plane (a source plane) parallel to the lens plane and perpendicular to the line of sight from the observer, so that the lens plane is between the observer and the light source. Due to deflection of light by masses multiple images $S_1, S_2, \ldots$ of the source may form (cf. Fig. 1). Fig. 2 and Fig. 3 illustrate some further aspects of the lensing phenomenon.

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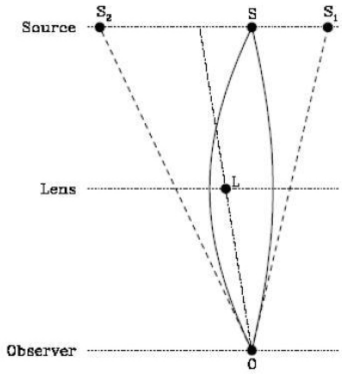

FIGURE 1. The lens L located between source S and observer O produces two images S_1, S_2 of the source S.

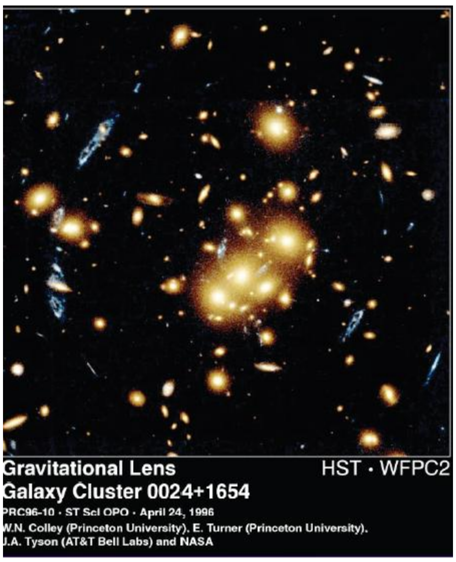

FIGURE 2. Lensing of a galaxy by a cluster of galaxies; the blue spots are all images of a single galaxy located behind the huge cluster of galaxies. (Credit: NASA, W. N. Cooley (Princeton), E. Turrer (Princeton) and J. A. Tyson (AT&T and Bell Labs).)

FIGURE 3. The bluish bright spots are the lensed images of a quasar (i.e., a quasi-stellar object) behind a bright galaxy in the center. There are 5 images (4 bright + 1 dim), but one cannot really see the dim image in this figure. (Credit: ESA, NASA, K. Sharon (Tel Aviv University) and E. Ofek (Caltech).)

2. Lens equation

In this section we are still assuming that our lens consists of n point masses. Suppose that the light source is located in the position w (a complex number) in the source plane. Then, the lensed image is located at z in the source plane while the masses of the lens L are located at the positions z_j, $j = 1, \ldots, n$ in the lens plane. The following simple equation, obtained by combining Fermat's Principle of Geometric Optics together with basic equations of General Relativity, connects then the positions of the lensed images, the source and the positions of the masses which cause the lensing effect

$$w = z - \sum_{1}^{n} \frac{\sigma_j}{\bar{z} - \bar{z}_j}, \tag{2.1}$$

where σ_j are some physical (real) constants. For more details on the derivation and history of the lensing equation (2.1) we refer the reader to [19], [12], [14], [21]. Sometimes, to include the effect caused by an extra ("tidal") gravitational pull by an object (such as a galaxy) far away from the lens masses, the right-hand side of (2.1) includes an extra linear term $\gamma\bar{z}$, thus becoming

$$w = z - \sum_{1}^{n} \frac{\sigma_j}{\bar{z} - \bar{z}_j} - \gamma\bar{z}, \tag{2.2}$$

where γ is a real constant. The right-hand side of (2.1) or (2.2) is called the lensing map. The number of solutions z of (2.1) (or (2.2)) is precisely the number of images of the source w generated by the lens L. Letting $r(z) = \sum_1^n \frac{\sigma_j}{z-z_j} + \gamma z + \bar{w}$, the lens equations (2.1) and (2.2) become

$$z - \overline{r(z)} = 0, \tag{2.3}$$

where $r(z)$ is a rational function with poles at z_j, $j = 1, \ldots, n$ and infinity if $\gamma \neq 0$.

3. Historical remarks

The first calculations of the deflection angle by a point mass lens, based on Newton's corpuscular theory of light and the Law of Gravity, go back to H. Cavendish and J. Michel (1784), and P. Laplace (1796)— cf. [20]. J. Soldner (1804) — cf. [21] is usually credited with the first published calculations of the deflection angle and, accordingly, with that of the lensing effect. Since Soldner's calculations were based on Newtonian mechanics they were off by a factor of 2. A. Einstein is usually given credit for calculating the lensing effect in the case of $n = 1$ (one mass lens) around 1933. Yet, some evidence has surfaced recently that he did some of these calculations earlier, around 1912 — cf. [17] and references therein. The recent outburst of activity in the area of lensing is often attributed to dramatic improvements of optics technology that make it possible to check many calculations and predictions by direct visualization.

H. Witt [24] showed by a direct calculation that for $n > 1$ the maximum number of observed images is $\leq n^2 + 1$. Note that this estimate can also be derived from the well known Bezout theorem in algebraic geometry (cf. [3, 8, 9, 22]). In [11] S. Mao and A. O. Petters and H. J. Witt showed that the maximum possible number of images produced by an n-lens is at least $3n + 1$. A. O. Petters in [13], using Morse's theory, obtained a number of estimates for the number of images produced by a non-planar lens. S. H. Rhie [15] conjectured that the upper-bound for the number of lensed images for an n-lens is $5n - 5$. Moreover, she showed in [16] that this bound is attained for every $n > 1$ and, hence, is sharp. Rhie's conjecture was proved in full in [8]. Namely, we have the following result.

Theorem 3.1. *The number of lensed images by an n-mass, $n > 1$, planar lens cannot exceed $5n - 5$ and this bound is sharp [16]. Moreover, the number of images is an even number when n is odd and odd where n is even.*

The proof of the above result rests on some simple ideas from complex dynamics (cf. [9, 10]).

4. "Thin" lenses with continuous mass distributions

If we to replace point masses by a general, real-valued mass distribution μ, a compactly supported Borel measure in the lens plane, the lens equation with shear

(2.2) becomes

$$w = z - \int_{\Omega} \frac{d\mu(\zeta)}{\bar{z} - \bar{\zeta}} - \gamma \bar{z}. \tag{4.1}$$

Here Ω is a bounded domain containing the support of μ. The case of the atomic measure $\mu = \sum_{1}^{n} \sigma_j \delta_{z_j}$, $\sigma_i \in \mathbb{R}$ is covered by Theorem 3.1. Also, as noted in [8], if we replace n-point-masses by n non-overlapping radially symmetric masses, the total number of images outside of the region occupied by n-masses is still $5n - 5$ when $\gamma = 0$, and $\leq 5n$ when $\gamma \neq 0$. The reason for that, of course, is that the Cauchy integral

$$\int_{|\zeta - z_j| < R} \frac{d\mu(\zeta)}{z - \zeta}, \quad |z - z_j| > R$$

for any radially symmetric measure $\mu = \mu(|\zeta - z_j|)$ is immediately calculated to be equal $\frac{c}{z - z_j}$, where c is the total mass μ of the disk $\{\zeta : |\zeta - z_j| < R\}$, hence reducing this new situation to the one treated in Theorem 3.1.

Here is another situation that can be treated with help from Theorem 3.1.

Recall that a simply-connected domain Ω is called a *quadrature domain* (of order n) if Ω is obtained from the unit disk $\mathbb{D} := \{z : |z| < 1\}$ via a conformal mapping φ that is a rational function of degree n, $\Omega = \varphi(\mathbb{D})$. Of course, all poles β_j, $j = 1, \ldots, n$ of φ will lie outside $\mathbb{D}$. Then if, say, μ is a uniform mass distribution in Ω, i.e., $\mu = \text{const}\, dx\, dy$, the Cauchy potential term in (4.1) for $z \notin \overline{\Omega}$ becomes

$$\sum_{j=1}^{n} \frac{c_j}{z - z_j}, \quad z_j = \varphi\left(\frac{1}{\overline{\beta_j}}\right), \tag{4.2}$$

where the coefficients c_j are determined by the quadrature formula associated with Ω (cf. [18] for details).

Hence, substituting (4.2) into (4.1) we again obtain that for such thin lens Ω with a uniform density distribution, the number of "bright" images outside Ω cannot exceed $5n - 5$ when no shear is present, or $5n$ otherwise.

In this general context the only previously known (to the best of our knowledge) result is the celebrated Burke's theorem [2]

Theorem 4.1. *A (finite) number of images produced by a* **smooth** *mass distribution μ is always odd, provided that $\gamma = 0$ (no shear).*

An elegant complex-analytic proof of Burke's theorem can be found in [19]. The crux of the argument is this. Take $w = 0$ and let n_+, n_- denote, respectively, the number of sense-preserving and sense-reversing zeros of the lens map in (4.1) ($\gamma = 0$).

The argument principle applies to harmonic complex-valued functions in the same way it does to analytic functions. Since the right-hand side of (4.1) behaves like $O(z)$ near ∞, the argument principle then yields that $1 = n_+ - n_-$. Thus, giving us the total number of zeros $N := n_+ + n_- = 2n_- + 1$, an odd number.

5. Ellipsoidal lens

Suppose the lens $\Omega := \left\{\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1, a>b>0\right\}$ is an ellipse. First assume the mass density to be constant, say 1. Let $c : c^2 = a^2 - b^2$ be the focal distance of Ω. The lens equation (4.1) can be rewritten as

$$\bar{z} - \frac{1}{\pi}\int\limits_{\Omega} \frac{dA(\zeta)}{z-\zeta} - \gamma z = \bar{w}, \tag{5.1}$$

where dA denotes the area measure. Using complex Green's formula (cf. e.g., [19]), we can rewrite (5.1) for $z \in \mathbb{C}\setminus\overline{\Omega}$ as follows:

$$\bar{z} - \frac{1}{2\pi i}\int\limits_{\partial\Omega} \frac{\bar{\zeta}\, d\zeta}{z-\zeta} - \gamma z = \bar{w}. \tag{5.2}$$

As is well-known [18], the (analytic) Schwarz function $S(\zeta)$ for the ellipse defined by $S(\zeta) = \bar{\zeta}$ on $\partial\Omega$ can be easily calculated and equals

$$\begin{aligned} S(\zeta) &= \frac{a^2+b^2}{c^2}\,\zeta - \frac{2ab}{c^2}\left(\zeta - \sqrt{\zeta^2-c^2}\right) \\ &= \frac{a^2+b^2-2ab}{c^2}\,\zeta + \frac{2ab}{c^2}\left(\zeta - \sqrt{\zeta^2-c^2}\right) \\ &= S_1(\zeta) + S_2(\zeta). \end{aligned} \tag{5.3}$$

Note that S_1 is analytic in $\overline{\Omega}$, while S_2 is analytic outside Ω and $S_2(\infty) = 0$. This is, of course, nothing else but the Plemelj–Sokhotsky decomposition of the Schwarz function $S(\zeta)$ of $\partial\Omega$. From (5.3) and Cauchy's theorem we easily deduce that for $z \in \mathbb{C}\setminus\overline{\Omega}$ the lens equation (5.2) reduces to

$$\bar{z} + \frac{2ab}{c^2}\left(z - \sqrt{z^2-c^2}\right) - \gamma z = \bar{w}. \tag{5.4}$$

Squaring and simplifying, we arrive from (5.4) at a complex quadratic equation

$$\left[\bar{z} + \left(\frac{2ab}{c^2}\,z - \gamma\right) z\bar{w}\right]^2 = \frac{2a^2b^2}{c^2}\left(z^2-c^2\right)$$

which is equivalent to a system of two irreducible real quadratic equations. Bezout's theorem (cf. [8,9], [10], [3]) then implies that (5.1) may only have 4 solutions $z \notin \Omega$. For $z \in \Omega$, using Green's formula and (5.3) we can rewrite the area integral in (5.1)

$$\begin{aligned} -\frac{1}{\pi}\int\limits_{\Omega}\frac{dA(\zeta)}{z-\zeta} &= -\bar{z} + \frac{1}{2\pi i}\int\limits_{\partial\Omega}\frac{\bar{\zeta}\,d\zeta}{\zeta - z} \\ &= -\bar{z} + \frac{1}{2\pi i}\int\limits_{\partial\Omega}\frac{[S_1(\zeta)+S_2(\zeta)]}{\zeta - z} \\ &= -\bar{z} + S_1(z) = -\bar{z} + \frac{a^2+b^2-2ab}{c^2}\,z \end{aligned} \tag{5.5}$$

We have used here that the Cauchy transform of $S_2 \,|_{\partial\Omega}$ vanishes in Ω since S_2 is analytic in $\overline{\mathbb{C} \setminus \Omega}$ and vanishes at infinity. Substituting (5.5) into (5.1), we arrive at a linear equation

$$\left(\frac{a^2+b^2-2ab}{c^2}-\gamma\right) z = \bar{w} \tag{5.6}$$

for $z \in \Omega$. Equation (5.6), of course, may only have one root in Ω. Thus, we have proved the following

Theorem 5.1. *An elliptic lens Ω (say, a galaxy) with a uniform mass density may produce at most four "bright" lensing images of a point light source outside Ω and one ("dim") image inside Ω, i.e., at most 5 lensing images altogether.*

This type of result has actually been observed experimentally - cf. Fig. 4, where four bright images are clearly present. It is conceivable that the dim image is also there but we can't see it because it is perhaps too faint compared with the galaxy. Of course, one has to accept Fig. 4 with a grain of salt since we do not expect "real" galaxies to have uniform densities. A model of an elliptical lens, with shear, that produces five images (4 bright + 1 dim) is given in Fig. 5.

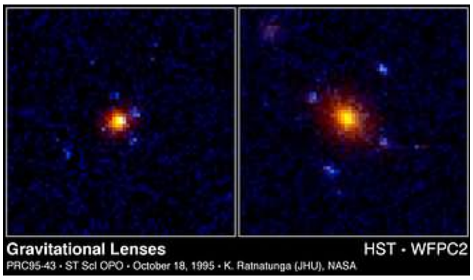

FIGURE 4. Four images of a light source behind the elliptical galaxy. (Credit: NASA, Kavan Ratnatunga (Johns Hopkins University).)

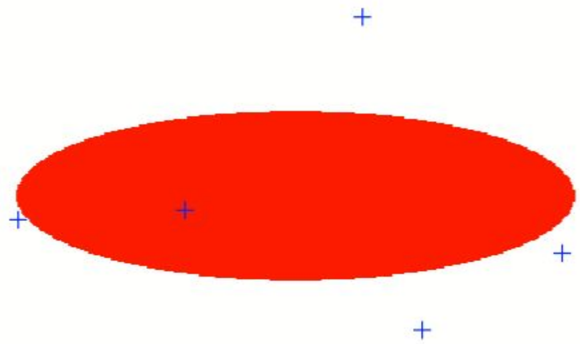

FIGURE 5. A model with five images of a source behind an elliptical lens with axis ratio 0.5 and uniform density 2.

We can extend the previous theorem for a larger class of mass densities. Denote by $q(x,y) := \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$ the equation of $\Gamma := \partial\Omega$. Let $q_\lambda(x,y) = \frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} - 1$, $-b^2 < \lambda < 0$ stand for the equation of the boundary $\Gamma_\lambda := \partial\Omega_\lambda$ of the ellipse Ω_λ confocal with Ω.

The celebrated MacLaurin's theorem (cf. [7]) yields that for any $z \in \mathbb{C} \setminus \overline{\Omega}$

$$\frac{1}{\text{Area}\,(\Omega_\lambda)} \int\limits_{\Omega_\lambda} \frac{dA(\zeta)}{\zeta - z} = \frac{1}{\text{Area}(\Omega)} \int\limits_{\Omega} \frac{dA(\zeta)}{\zeta - z}. \tag{5.7}$$

Thus, if we denote by $u(z,\lambda)$ the Cauchy potential of Ω_λ evaluated at $z \in \mathbb{C} \mid \overline{\Omega}$ we obtain from (5.7)

$$u\,(z,\lambda) = c(\lambda)u_\Omega(z,0), \tag{5.8}$$

where

$$c(\lambda) = \frac{\text{Area}\,(\Omega_\lambda)}{\text{Area}(\Omega)} = \frac{\left(a^2+\lambda\right)^{1/2}\left(b^2+\lambda\right)^{1/2}}{ab}. \tag{5.9}$$

Hence,

$$\frac{\partial u_\lambda(z,\lambda)}{\partial\lambda} = c'(\lambda)u_\Omega(z). \tag{5.10}$$

So, if the mass density $\mu(\lambda)$ in Ω only depends on the elliptic coordinate λ, i. e., is constant on ellipses confocal with Ω inside Ω, its potential outside Ω equals

$$u_{\mu,\Omega}(z) = cu_\Omega(z). \tag{5.11}$$

The constant c is easily calculated from (5.9)–(5.10) and equals

$$c = \int\limits_{-b^2}^{0} \mu(\lambda)c'(\lambda)\,d\lambda. \tag{5.12}$$

It is, of course, natural for physical reasons to assume that $\mu(\lambda) \uparrow \infty$ at the "core" of Ω (i.e., when $\lambda \downarrow -b^2$), the focal segment $[-c, c]$. Yet, from (5.12) since (5.9) yields $c'(\lambda) = O\left((b^2+\lambda)^{-1/2}\right)$ near $\lambda_0 = -b^2$, it follows that $\mu(\lambda)$ should not diverge at the core faster than say $O\left((b^2+\lambda)^{-1/2+\epsilon}\right)$ for some positive ϵ, so the integral (5.12) converges. Substituting (5.11) into the lens equation (5.1) with constant density replaced by the density $\mu(\lambda)$ and following again the steps in (5.2)–(5.4) we arrive at the following corollary.

Corollary 5.1. *An elliptic lens Ω with mass density that is constant inside Ω on the ellipses confocal with Ω may produce at most four "bright" lensing images of a point light source outside Ω.*

6. Einstein rings

For a one-point mass at z_1 lens with the source at $w = 0$ the lens equation (2.1) without shear becomes

$$z - \frac{c}{\bar{z} - \bar{z}_1} = 0. \tag{6.1}$$

As was already noted by Einstein (cf. [12,19,21] and references cited therein), (6.1) may have two solutions (images) when $z_1 \neq 0$ and a whole circle ("Einstein ring") of solutions when $z_1 = 0$, in other words when the light source, the lens and the observer coalesce - cf. Fig. 6 and Fig. 7.

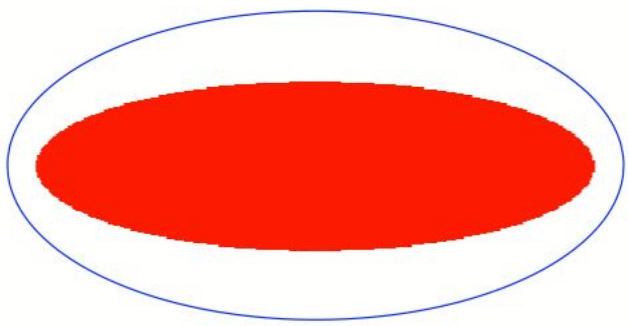

FIGURE 6. A model of an elliptical Einstein ring surrounding an elliptical lens with axis ratio 0.5 and uniform density 2. The shear in this case must be specially chosen to produce the ring instead of point images. Note that the ring is an ellipse confocal with the lens - cf. Thm. 6.1 .

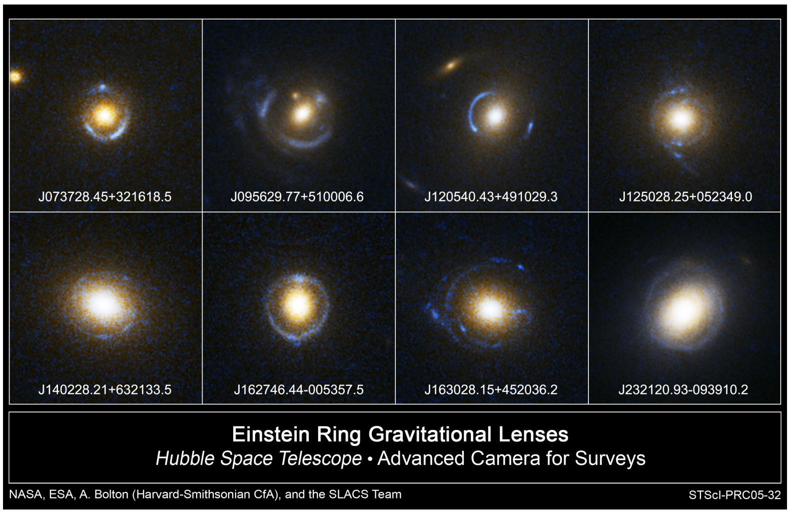

FIGURE 7. Einstein rings. The sources in these observed "realistic" lenses are actually extended, and that is why we see sometimes arcs rather than whole rings. (Credit: ESA, NASA and the SLACS survey team: A. Bolton (Harvard / Smithsonian), S. Burles (MIT), L. Koopmans (Kapteyn), T. Treu (UCSB), and L. Moustakas (JPL/Caltech).)

As the following simple theorem shows the "ideal" Einstein rings are limited to ellipses and circles in much more general circumstances.

Theorem 6.1. *Let Ω be any planar ("thin") lens with mass distribution μ_e. If lensing of a point source produces a bounded "image" curve outside of the lens Ω, it must either be a circle when the external shear $\gamma = 0$ or an ellipse.*

Proof. First consider a simpler case when $\gamma = 0$. If the lens produces an image curve Γ outside Ω, the lens equation (4.1) becomes

$$\bar{z} - \bar{w} = \int_{\Omega} \frac{d\mu(\zeta)}{z - \zeta}, \tag{6.2}$$

for all $z \in \Gamma$. Note that Γ being bounded and also being a level curve of a harmonic function must contain a closed loop surrounding Ω [22]. Without loss of generality, we still denote that loop by Γ. The right-hand side $f(z)$ of (6.2) is a bounded analytic function in the unbounded complement component $\tilde{\Omega}_\infty$ of Γ that vanishes at infinity. Hence $(z - w)f(z)$ is still a bounded and analytic function in $\tilde{\Omega}_\infty$ equal to $|z - w|^2 > 0$ on $\Gamma := \partial\tilde{\Omega}_\infty$. Hence $(z - w)f(z) = \text{const}$ and Γ must be a circle centered at w.

Now suppose $\gamma \neq 0$. Once again we shall still denote by Γ a closed Jordan loop surrounding Ω. Denote by $\tilde{\Omega}$ the interior of Γ, $\tilde{\Omega}_\infty = \mathbb{C} \setminus \operatorname{clos}\left(\tilde{\Omega}\right)$. Also, by translating we can assume that the position of the source w is at the origin.

The equation (4.1) now reads

$$\bar{z} = \int\limits_{\Omega} \frac{d\mu(\zeta)}{z-\zeta} + \gamma z, \quad z \in \Gamma. \tag{6.3}$$

In other words the right-hand side of (6.3) represents the Schwarz function S of Γ, analytic in $\mathbb{C} \setminus \operatorname{supp} \mu$ with a simple pole at ∞. It is well-known that this already implies that Γ must be an ellipse (cf. [4,18]) and references therein. For the reader's convenience we supply a simple proof.

Applying Green's formula to (6.3) yields (cf. (5.1)–(5.2)) that for all $z \in \tilde{\Omega}_\infty$

$$\int\limits_{\Omega} \frac{d\mu(\zeta)}{z-\zeta} = \frac{1}{\pi} \int\limits_{\tilde{\Omega}} \frac{dA(\zeta)}{z-\zeta}, \quad z \in \tilde{\Omega} := \mathbb{C} \setminus \tilde{\Omega}_\infty. \tag{6.4}$$

Let

$$h(z) := \frac{1}{\pi} \int\limits_{\tilde{\Omega}} \frac{dA(\zeta)}{z-\zeta} - \bar{z}, \quad z \in \tilde{\Omega}. \tag{6.5}$$

Then, $h(z)$ is analytic in $\tilde{\Omega}$ (cf. (5.2)) and, in view of (6.3) and (6.4)

$$h \,|_\Gamma = \int\limits_{\Omega} \left. \frac{d\mu(\zeta)}{z-\zeta} \right|_\Gamma - \bar{z}|_\Gamma = \gamma z \,|_\Gamma \,. \tag{6.6}$$

Thus, $h(z)$ is a linear function and since (6.5) implies for $z \in \tilde{\Omega}$

$$\overline{h(z)} := \frac{1}{2} \operatorname{grad} \left[\frac{1}{\pi} \int\limits_{\tilde{\Omega}} \log|z-\zeta| \, dA(\zeta) - |z|^2 \right], \tag{6.7}$$

we conclude from (6.7) that the potential of $\tilde{\Omega}$

$$u_{\tilde{\Omega}}(z) = \frac{1}{2\pi} \int\limits_{\tilde{\Omega}} \log|z-\zeta| \, dA(\zeta), \quad z \in \tilde{\Omega}$$

equals to a quadratic polynomial inside $\tilde{\Omega}$. The converse of the celebrated theorem of Newton due to P. Dive and N. Nikliborc (cf. [7, Ch. 13–14] and references therein) now yields that $\tilde{\Omega}$ must be an interior of an ellipse, hence $\Gamma := \partial\tilde{\Omega} = \partial\tilde{\Omega}_\infty$ is an ellipse. $\square$

Remark 6.1. One immediately observes that since the converse to Newton's theorem holds in all dimensions the last theorem at once extends to higher dimensions if one replaces the words "image curve" by "image surface".

7. Final remarks

1. The densities considered in §5 are less important from the physical viewpoint than so-called "isothermal density" which is obtained by projecting onto the lens plane the "realistic" three-dimensional density $\sim 1/\rho^2$, where ρ is the (three-dimensional) distance from the origin. This two-dimensional density could be included into the whole class of densities that are constant on all ellipses *homothetic* rather than confocal with the given one. The reason for the term "isothermal" is that when a three-dimensional galaxy has density $\sim 1/\rho^2$ the gas in the galaxy has constant temperature (cf. [5] and the references therein).

   Recall that the Cauchy potential of the ellipse $\Omega := \left\{\frac{x^2}{a^2}+\frac{y^2}{b^2}\leq 1, a>b>0\right\}$ outside of Ω (cf. (5.2)–(5.4)) equals

$$u_0(z) := k\left(z-\sqrt{z^2-c^2}\right), \quad z\in\mathbb{C}\setminus\overline{\Omega}, \quad c^2=a^2-b^2, \tag{7.1}$$

   where $k = 2ab/c^2$ is a constant. Replacing the ellipse Ω by a homothetic ellipse $\Omega_t := t\Omega := \frac{x^2}{a^2}+\frac{y^2}{b^2}\leq t^2,\ 0<t<1$. We obtain using (7.1) for $z\notin\overline{\Omega}$:

$$\begin{aligned} u(z,t) :&= \int_{t\Omega}\frac{dA(\zeta)}{\zeta-z} = t^2\int_{\Omega}\frac{dA(\zeta)}{t\zeta-z} \\ &= tu\left(\frac{z}{t};1\right) = k\left(z-\sqrt{z^2-c^2t^2}\right). \end{aligned}\tag{7.2}$$

   Thus,

$$\frac{\partial u(z,t)}{\partial t} = k\,\frac{c^2t}{\sqrt{z^2-c^2t^2}}. \tag{7.3}$$

   So, if the "isothermal" density $\mu=\frac{1}{t}$ on $\partial\Omega_t$ inside Ω (ignoring constants), we get from (7.3) that the Cauchy potential of such mass distribution outside Ω equals

$$u_\mu(z) := C_0\int_0^1\frac{dt}{\sqrt{z^2-c^2t^2}}, \tag{7.4}$$

   where the constant C_0 depends on Ω only. This is a transcendental function (one of the branches of $\arcsin\frac{c}{z}$), dramatically different from the algebraic potential in (7.1). The lens equation (4.1)now becomes

$$z - C_0\int_0^1\frac{dt}{\sqrt{\bar{z}^2-c^2t^2}} - \gamma\bar{z} = w. \tag{7.5}$$

   To the best of our knowledge the precise bound on the maximal possible number of solutions (images) of (7.5) is not known. Up to today, no more than 5 images (4 bright +1 dim) have been observed. However, in [5] there have been constructed explicit models depending on parameters a, b and $0<\gamma<1$ having 9 (i. e., $8+1$) images. The equation (7.5) essentially differs from all the lens equations considered in this paper since it involves estimating the number of zeros of a transcendental harmonic function with a simple pole at ∞. At this point, we are even reluctant to make a conjecture regarding what this maximal number might be.

Note, that in case of a circle $\Omega = \{x^2 + y^2 < 1\}$ with any radial density $\mu := \varphi(r)$, $r = \sqrt{x^2 + y^2} < 1$, the situation is very simple. The Cauchy potential $u(z)$ outside Ω, as was noted earlier, equals

$$\frac{c}{z}, \quad |z| > 1, \tag{7.6}$$

where c is a constant. Hence, outside Ω the lens equation becomes

$$z - \frac{c}{\bar{z}} - \gamma\bar{z} = w, \tag{7.7}$$

a well-known Chang–Refsdal lens (cf., e.g., [1]) that may have at most 4 solutions except for the degenerate case $\gamma = w = 0$, when the Einstein ring appears. In particular, when $\gamma = 0$, $w \neq 0$, such mass distribution may only produce two bright images outside Ω. For $z : |z| < 1$ inside the lens the potential is still calculated by switching to polar coordinates:

$$\begin{aligned} u(z) &:= \int_0^1 \int_0^{2\pi} \frac{\varphi(r) r dr d\theta}{re^{i\theta} - z} \\ &= \int_{|z|}^1 \varphi(r) dr \int_0^{2\pi} \frac{r d\theta}{re^{i\theta} - z} + \int_0^{|z|} \varphi(r) r dr \int_0^{2\pi} \frac{d\theta}{re^{i\theta} - z} \\ &= \int_{|z|}^1 \varphi(r) dr \int_0^{2\pi} \left(\sum_0^\infty \left(\frac{z}{r}\right) e^{-i(n+1)\theta} \right) d\theta \\ &\quad + \frac{1}{z} \int_0^{|z|} \varphi(r) r dr \int_0^{2\pi} \left(\sum_0^\infty \left(\frac{re^{i\theta}}{z}\right)^n \right) d\theta \\ &= \frac{2\pi}{z} \int_0^{|z|} \varphi(r) r dr. \end{aligned} \tag{7.8}$$

In particular, for the "isothermal" density $\varphi(r) \sim \frac{1}{r}$, (7.8) yields for $z : |z| < 1$

$$u(z) = \frac{2\pi}{z} |z|,$$

so the lens equation (7.7) becomes

$$\bar{z} - \frac{c}{z} |z| - \gamma z = \bar{w}, \tag{7.9}$$

where c is a real constant. Equation (7.9) can have at most two solutions *inside* Ω (only one, if $\gamma = 0$), again, excluding the degenerate case of the Einstein ring. Furthermore, since Burke's theorem allows only an odd number of images, the total maximal number of images for an isothermal sphere cannot exceed 5 (4 bright+1 dim) as before (or ≤ 3, i. e.,(2+1) if $\gamma = 0$). Note, that strictly speaking, Burke's theorem cannot be applied to the isothermal density because of the singularity at the origin. Yet, since the density is radial and smooth everywhere excluding the origin and because it is clear from (7.9) that the origin cannot be a solution, Burke's theorem does apply yielding the above conclusion.

2. The problem of estimating the maximal number of "dim" images inside the lens formed by a uniform mass-distribution inside a quadrature domain Ω (cf. §4) of order n is challenging. In this case the Cauchy potential in (4.1) inside Ω equals to the "analytic" part of the Schwarz function $S(z)$. It is known that $S(z)$ is an algebraic function of degree at most $2n$. Yet, the sharp bounds, similar to those in Theorem 3.1, for the number of zeros of harmonic functions of the form $\bar{z} - a(z)$, where $a(z)$ is an algebraic function, are not known.
3. Another interesting and difficult problem would be to study the maximal number of images by a lens consisting of several elliptical mass distributions. Some rough estimates based on Bezout's theorem can be made by imitating the calculations in §5. Yet, even for 2 uniformly distributed masses these calculations give a rather large possible number of images (≤ 15) while, so far, only 5 images by a two galaxies lens and 6 images by a three galaxies lens have been observed - cf. [6, 23].

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