AN INVERSE PROBLEM FOR THE DOUBLE LAYER POTENTIAL

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1. Introduction

1.1. The solution to the Dirichlet problem by means of a double layer potential was initiated by C. Neumann and H. Poincaré, and completed in a celebrated paper by I. Fredholm. What is involved, very briefly, is the following. (In these introductory remarks we consider the problem in $\mathbb{R}^n$, although most of the later analysis will deal with $n = 2$. Also, we suppose all data to be as smooth as needed to justify various assertions.)

If Ω is a smoothly bounded domain in $\mathbb{R}^n$, $y \in \Omega$, and $g(\cdot, y)$ denotes the Green function of Ω for the Laplace operator with pole at y, we have the well known formula

$$u(y) = \int_{\partial\Omega} u(x)(\partial/\partial N_x)g(x,y)dS_x \tag{1.1}$$

valid for every harmonic function u in Ω smooth up to the boundary. Here $\partial/\partial N$ denotes partial differentiation with respect to the outward directed normal, dS hypersurface measure on $\partial\Omega$, and g is so normalized that the *Poisson kernel*

$$P(x,y) := (\partial/\partial N_x)g(x,y) \tag{1.2}$$

satisfies

$$\int_{\partial\Omega} P(x,y)dS_x = 1. \tag{1.3}$$

If the Poisson kernel for Ω were known, the solution to the Dirichlet problem

$$\Delta u = 0 \qquad \text{in } \Omega \tag{1.4}$$

$$u = f \qquad \text{on } \partial\Omega \tag{1.5}$$

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where, say, $f \in C(\partial\Omega)$, would be given by

$$u(y) = \int_{\partial\Omega} f(x)P(x,y)dS_x. \tag{1.6}$$

Now, in general we do not know the Green function (nor the Poisson kernel), but we know its singular part since

$$g(x,y) = E(y-x) + h(x,y) \tag{1.7}$$

where E is the "fundamental singularity" of the Laplace operator, or "Newtonian kernel"

$$E(x) = c_n \cdot \begin{cases} |x|^{2-n} & (n \geq 3) \\ \log|x| & (n = 2) \end{cases} \tag{1.8}$$

(with c_n a normalization factor which depends on the dimension n) and $h(x,y)$ is a harmonic function of $x \in \Omega$ for each y (and, in fact, symmetric in x and y). Thus, comparing (1.6), (1.2) and (1.7) it is natural to look for a formula representing u in the form

$$u(y) = \int_{\partial\Omega} \varphi(x)(\partial/\partial N_x)E(y-x)dS_x \tag{1.9}$$

(a so-called double layer potential), with "dipole (or doublet) density" φ on $\partial\Omega$; cf. [K]. Of course we cannot expect that the harmonic function (1.9) will have the desired boundary values f on $\partial\Omega$ if we simply choose φ as (some constant times) f. Rather, we must introduce an operator J (z denoting a point of $\partial\Omega$) by

$$(J\varphi)(z) := \lim_{\substack{y\to z \\ y\in\Omega}} \int_{\partial\Omega} \varphi(x)(\partial/\partial N_x)E(y-x)dS_x \tag{1.10}$$

and show that $J\varphi = f$ is solvable. More precisely (and this is the essence of Fredholm's solution to the Dirichlet problem): with suitable regularity hypotheses and choice of Banach space X of functions on $\partial\Omega$, J operating on X is equal to $I/2 + K$, where I is the identity and K a compact operator on X. Hence, by Fredholm-Riesz theory its *surjectivity* (which implies the solvability of Dirichlet's problem for data f in X) is a consequence of its *injectivity.* The latter is relatively easy to check (for details see [K]). Thus, in the end, this program justifies the intuitive idea to replace the integral operator with kernel $\partial g/\partial N$ by that with kernel $\partial E/\partial N$, and a perturbation argument.

The compact operator K such that J is given by $I/2+K$ can be represented as an integral operator

$$(K\varphi)(z) := \int_{\partial\Omega} \varphi(x)(\partial/\partial N_x)E(z-x)dS_x, \qquad z \in \partial\Omega. \tag{1.11}$$

The inverse problem referred to in the title concerns the injectivity of this operator or, equivalently, whether or not $1/2$ is an eigenvalue of the operator J. In two dimensions, this question turns out to be equivalent to a certain matching problem for analytic functions. One of our main results, Theorem 3.19, establishes injectivity of K for a special class of domains in $\mathbb{R}^2$.

For convenience, we shall rescale the double layer potential so that the kernel of K corresponds to the space of fixed points rather than the space of eigenfunctions with eigenvalue $1/2$.

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1.2. **The two-dimensional case.** To make more precise statements, we now have to specify our smoothness assumptions, and the spaces of functions with which we work, as well as pay attention to normalization constants. We henceforth confine ourselves to the two-dimensional case and rescale the operator as mentioned above. The *double layer potential of the dipole density* F (on the boundary Γ of a simply connected planar domain Ω) is the harmonic function

$$u(z) = \frac{1}{\pi}\int_{\Gamma} F(\zeta)(\partial/\partial N_\zeta)(\log|z-\zeta|)ds_\zeta \tag{1.12}$$

where ds denotes arclength on Γ. Since our main interest here does not concern regularity questions we assume henceforth that Γ *is an analytic Jordan curve* (without cusp singularities) and, for some fixed $\sigma > 0$, F is in *the space* $\Lambda_\sigma(\Gamma)$ *of Hölder continuous (of order* σ*) complex-valued functions on* Γ. Then (1.12) defines a harmonic function u in Ω (and another one for z in the exterior domain bounded by Γ).

To solve the Dirichlet problem (for the interior domain) is to find, for given f in $\Lambda_\sigma(\Gamma)$ (say), a corresponding $F \in \Lambda_\sigma(\Gamma)$ such that u, defined in Ω by (1.12), has the boundary values f on Γ. It is easy to check that for F real-valued,

$$u(z) = \mathrm{Re}\left[\frac{1}{\pi i}\int_{\Gamma} \frac{F(\zeta)}{\zeta - z}\, d\zeta\right]. \tag{1.13}$$

Now, the Cauchy integral

$$\frac{1}{2\pi i}\int_{\Gamma} \frac{F(\zeta)}{\zeta - z}\, d\zeta,$$

where F is any complex valued function in $\Lambda_\sigma(\Gamma)$, defines a pair of analytic functions: one, denoted f_i, is defined and holomorphic in Ω and the other, denoted f_e, in $\Omega' := \widehat{\mathbb{C}} \setminus \overline{\Omega}$, where $\widehat{\mathbb{C}}$ denotes the complex plane (compactified with the point at infinity). We denote by $A_\sigma(\Omega)$ the Banach space of σ-Hölder continuous functions on $\overline{\Omega}$, that are analytic in Ω; and by $A_\sigma(\Omega')$ the corresponding space on Ω', which satisfy moreover the requirement that functions in this space vanish at ∞. Then, as is well known from the theory of singular integrals (see e.g. [D] or [M]),

$$f_i \in A_\sigma(\Omega), \qquad f_e \in A_\sigma(\Omega'). \tag{1.14}$$

For convenience henceforth (since σ remains fixed) we denote simply

$$A := A_\sigma(\Omega), \qquad A' := A_\sigma(\Omega'). \tag{1.15}$$

We shall also use the notations $A(\Gamma)$, $A'(\Gamma)$ to denote the restrictions to Γ of the spaces A, A'.

It is well known [M], that

$$f_i(\zeta) - f_e(\zeta) = F(\zeta), \qquad \zeta \in \Gamma. \tag{1.16}$$

1.3. **The "Hilbert operator" H.** Following Kerzman and Stein [**?**], we define the *Hilbert operator* H for the domain Ω as the map taking $F \in \Lambda_\sigma(\Gamma)$ to the boundary values of its Cauchy integral (from inside) on Γ, in other words $HF = f_i|_\Gamma$. Thus, H is a continuous linear map from $\Lambda_\sigma(\Gamma)$ to $\Lambda_\sigma(\Gamma)$, and it is *idempotent*: $H^2 = H$. Its kernel is $A'(\Gamma)$ and its range $A(\Gamma)$.

Remark 1.1. It is possible, and indeed desirable, to consider the analogous operator (still denoted H) in other spaces than the Hölder space, e.g. the Hilbert space $L^2(\Gamma; ds)$ but then certain complications may arise, which we will discuss later.

Now, in terms of the Hilbert operator H we can easily represent the operator corresponding to J in (1.10) on $\Lambda_\sigma(\Gamma)$. Indeed, denoting by ΠF the boundary values from Ω of the double layer potential with doublet density F, we have already noted (this is just a rewriting of (1.13)) that for real-valued F:

$$\begin{aligned} \Pi F &= 2\mathrm{Re}\, HF \\ &= HF + CHF, \end{aligned} \tag{1.17}$$

where C denotes complex conjugation.

Thus, for F, G real we have

$$\begin{aligned} \Pi(F+iG) &= HF + CHF + i(HG + CHG) \\ &= H(F+iG) + CH(F-iG) \\ &= (H + CHC)(F+iG) \end{aligned}$$

so we have

$$\begin{aligned} \Pi &= H + CHC \\ &= H + \widetilde{H}, \end{aligned} \tag{1.18}$$

where

$$\widetilde{H} := CHC \tag{1.19}$$

is an idempotent operator that projects $\Lambda_\sigma(\Gamma)$ on its subspace $CA(\Gamma)$.

We can now state the question which is the principal concern of this paper:

For which choices of Γ *does the operator* Π (which of course depends on Γ as do H, etc. although we suppress this in the notation) *admit a non-trivial fixed point?*

Note that $\Pi F = F$ means that, with the special choice of boundary data F, the double layer potential with this same density gives the solution to Dirichlet's problem. For instance, if Γ is a circle, then for all F in a space of codimension one, indeed for any F with mean value zero, (1.12) solves the Dirichlet problem for boundary values F. This is all well known, and easy to check. Now, we have the following:

Theorem 1.20. Π *admits a non-trivial fixed point if and only if there exist non-constant* $f \in A(\Omega)$, $g \in A(\Omega')$ *with* $f = \bar{g}$ *on* Γ. *Moreover, if there exist such* f *and* g, *then both* f *and* g *themselves are fixed points of* Π.

Proof. (i) Suppose F is a non-constant fixed point. Then

$$HF + \widetilde{H}F = F. \tag{1.21}$$

We consider now two cases:

(a) $\widetilde{H}F = 0$. In this case $H\overline{F} = 0$, so $\overline{F} \in A'(\Gamma)$. Also, from (1.21) we have $HF = F$, showing $F \in A(\Gamma)$. Thus $A(\Gamma)$ and $CA'(\Gamma)$ have a non-trivial common element, as was to be shown.

(b) $\widetilde{H}F \neq 0$. In this case, applying H to (1.21) gives $H\widetilde{H}F = 0$. Hence $\widetilde{H}F \in A'(\Gamma)$, and $\widetilde{H}F$ is not constant. On the other hand, $\widetilde{H}F = CH\overline{F}$ is the complex conjugate of an element of $A(\Gamma)$, so we conclude that $A'(\Gamma) \cap CA(\Gamma)$ contains a non-constant function. This concludes the proof of the "only if" assertion in Theorem 1.20.

(ii) In the other direction, suppose $f \in A(\Gamma)$, $g \in A'(\Gamma)$ and $f = \bar{g}$. We shall show $\Pi f = f$ and $\Pi g = g$. Indeed,

$$\Pi f = (H + CHC)f = Hf + CHg = f,$$

and

$$\Pi g = (H + CHC)g = Hg + CHf = g,$$

concluding the proof. □

We remark that the proof of Theorem 1.20 above also shows that if F is a non-trivial fixed point of Π, then either (a) $H\overline{F} = 0$, in which case $F \in A(\Gamma)$ and $\overline{F} \in A'(\Gamma)$, or (b) $H\overline{F}$, which belongs to $A(\Gamma)$, is also a non-trivial fixed point and $CH\overline{F} = \widetilde{H}F$ is in $A'(\Gamma)$.

Observe that the set of fixed points of Π is a linear space. Theorem 1.20 has the following remarkable consequence (which was also proved, using different methods, in [**?**]).

Corollary 1.22. *If the space of fixed points of* Π *contains a non-constant element, it is infinite dimensional.*

Proof. We shall make use of the elementary fact that $\Lambda_\sigma(\Gamma)$ is a subalgebra of $C(\Gamma)$. Suppose F is a non-constant fixed point of Π. According to the proof of Theorem 1.20 (and the subsequent remark) either

$$\text{(a)} \quad F \in A(\Gamma) \qquad \text{and} \qquad \overline{F} \in A'(\Gamma),$$

or

$$\text{(b)} \quad CH\overline{F} \;\text{ is in } A'(\Gamma) \text{ and non-constant.}$$

In case (a), all the powers $F, F^2, F^3, \dots$ are in $A(\Gamma)$ and linearly independent over $\mathbb{C}$, while the corresponding conjugates $\overline{F}, \overline{F}^2, \dots$ are in $A'(\Gamma)$. Therefore (by Theorem 1.20) $F, F^2, F^3, \dots$ (as well as their complex conjugates) all are fixed points of Π.

In case (b), the functions $(CH\overline{F})^n$ $(n = 1, 2, \dots)$ are all in $A'(\Gamma)$ and linearly independent. Moreover, each of these is in the range of CH, i.e. is the complex conjugate of an element of $A(\Gamma)$, namely $(H\overline{F})^n$. Hence all $(H\overline{F})^n$ $(n = 1, 2, \dots)$ (as well as their complex conjugates) are fixed points of Π.

Summing up, we have shown that if F is a non-trivial fixed point of Π, then either all powers F^n $(n = 1, 2, \dots)$, or $(H\overline{F})^n$ $(n = 1, 2, \dots)$ are also non-trivial fixed points of Π . This implies the corollary. $\square$

Thus, the problem of fixed points of Π is equivalent to a purely function-theoretic one which we henceforth shall call "*the matching problem*": to find non-constant $f \in A(\Omega)$, $g \in A'(\Omega)$ which "match" on Γ in the sense that they are complex conjugates of one another there.

Remark 1.2. That f is a fixed point of Π is the same as saying, f is an eigenvector of Π corresponding to the eigenvalue $\lambda = 1$. Thus, whenever 1 is an eigenvalue it is of infinite multiplicity! No other value of λ can be an eigenvalue of Π of infinite multiplicity because, with some regularity of Γ, as shown by Fredholm, $\Pi = I + 2K$ with K compact and $K \neq 0$ (given by the integral operator (1.11)), so the only possibility for an eigenvalue of K to be of infinite multiplicity is if it is 0.

We remark also that, apart from the case where Ω is a disk already noted, the set of fixed points of Π never has finite codimension. This was shown in [S, Theorem 7.6].

Henceforth, we will no longer explicitly refer to potentials nor the operator Π, just the matching problem.

Moreover, from a function-theoretic point of view, the Hölder continuity assumptions, so useful in potential theory, are somewhat unnatural, so we shall henceforth consider our question in the following form:

Matching problem. Determine for which Jordan domains Ω there exist non-constant functions f, g such that

(i) f is holomorphic in Ω and extends continously to $\Gamma = \partial\Omega$;

(ii) g is holomorphic in $\Omega' := \widehat{\mathbb{C}} \setminus \overline{\Omega}$, $g(\infty) = 0$ and g extends continuously to Γ;

(iii) $f(\zeta) = \overline{g(\zeta)}, \qquad \zeta \in \Gamma.$

Remark 1.3. We shall often impose further restrictions, e.g. that Γ is *analytic*. A very interesting question which, however, we shall not consider in this paper, is: *How much (if any) regularity of* Γ *is forced by the existence of a non-trivial matching pair?*

To appreciate the role played by regularity hypotheses, observe that the proof of Corollary 1.22 required that the underlying space of functions F on Γ be stable under multiplication. Now, the notions of double layer potential, Hilbert operator, etc. extend perfectly well to the setting of densities F that need not be continuous on Γ, but merely (say) in $L^2(\Gamma; ds)$ (and the corresponding space of holomorphic functions are then the *Hardy spaces* $H^2(\Omega)$, $H^2(\Omega'))$. But since L^2 is not an algebra the proof we gave that, if 1 is an eigenvalue of Π it has infinite multiplicity, no longer works in this setting, even though the *assertion* remains meaningful. We do not know if it is true.

2. Some domains for which the matching problem has non-trivial solutions

Thus far, the only known domains allowing non-trivial solutions for the matching problem are *lemniscates*. These lemniscate solutions (more precisely, the following theorem) were discovered by Mark Melnikov, who communicated the result to one of the present authors, and generously consented to its inclusion in our paper.

Theorem 2.1. (M. Melnikov, unpublished) *Let $R(z)$ be a rational function such that $\Gamma := \{z \in \mathbb{C} : |R(z)| = c\}$, where $c > 0$, is a Jordan curve. Suppose moreover that*

(i) *R has no poles in Ω (the interior domain bounded by Γ);*

(ii) *R has no zeroes in Ω' (the exterior domain);*

(iii) *$R(\infty) = \infty$.*

Then, the matching problem for Γ has non-trivial solutions.

Proof. The pair R, c^2/R satisfy the matching requirements. $\square$

We call these the *lemniscate solutions* to the matching problem. We do not know any other solutions.

Example 2.2. Let P be a polynomial of degree ≥ 1. Then, for sufficiently large c, $\Gamma := \{z \in \mathbb{C} : |P(z)| = c\}$ is a Jordan curve containing all zeroes of P. Hence the pair P, c^2/P solve the matching problem for Γ. (It is easy to construct also other, non-polynomial, lemniscate solutions).

3. Some domains for which the matching problem admits only the trivial solution

3.1. **Preliminary material.**

3.1.1. *The Schwarz function.* Our construction of domains that do not allow non-trivial solutions of the matching problem requires some rather delicate preliminary results concerning *anticonformal reflection* with respect to algebraic curves. These results are all known from earlier work of Avci, Ebenfelt, Gustafsson etc. (These, and other relevant references, may be found in [S]). Since they are often embedded in more general considerations concerning so-called quadrature domains, and afflicted with complications due to multiple connectivity, etc. we give here a simple, and mostly self-contained account of the results we shall need. First, let

$$\Gamma = \{(x, y) \in \mathbb{R}^2 : P(x, y) = 0\}, \tag{3.1}$$

where P is a polynomial with real coefficients, which we also express by $P \in \mathbb{R}[x, y]$, assumed irreducible in the ring $\mathbb{C}[x, y]$. If we write $x = (z + \bar{z})/2$, $y = (z - \bar{z})/2i$, the equation in (3.1) takes the form

$$Q(z, \bar{z}) = 0, \tag{3.2}$$

where

$$Q(z, w) := P\big((z + w)/2,\ (z - w)/2i\big). \tag{3.3}$$

Thus, $Q \in \mathbb{C}[z, w]$ and it is easy to see that Q is irreducible in this ring. Thus

$$V := \{(z, w) \in \mathbb{C}^2 : Q(z, w) = 0\}$$

is an irreducible one-dimensional complex algebraic variety, and Γ can be identified as the intersection of V with the "real plane" $\{w = \bar{z}\}$.

In a neighborhood of a non-singular point (x_0, y_0) of Γ we can (writing $x + iy = z,\ \ x_0 + iy_0 = z_0$) represent Γ by the equation $\bar{z} = S(z)$ where S is the *Schwarz function* of Γ (cf. [S]). From (3.2)

$$Q\big(z, S(z)\big) = 0. \tag{3.4}$$

This holds initially for complex z near z_0, and by analytic continuation, identically. Thus, S is an algebraic function of z.

Let us write

$$Q(z, w) = a_0(z) + a_1(z)w + \cdots + a_n(z)w^n, \tag{3.5}$$

where the polynomial $a_n(z)$ does not vanish identically. Considered as a polynomial in w, (3.5) has a discriminant $D(z)$, a polynomial whose zeroes give all possible branch points of S (i.e. a finite subset of $\mathbb{C}$ which contains all points "above" which, in one or more sheets of its Riemann surface, S has a branch point). Let us denote the totality of these last-mentioned points by B. Then, to each $z \in \mathbb{C} \setminus B$ there are precisely n distinct values which S takes at z, upon analytic continuation to z along a suitable path. (One of these values may be ∞, in case $a_n(z) = 0$).

It is often geometrically convenient to work with the antianalytic function

$$R(z) := \overline{S(z)} \tag{3.6}$$

which we call the (multi-valued) *anticonformal reflection* (ACR) associated with Γ.

For $z \in \mathbb{C} \setminus B$ we denote by $\{S(z)\}$ the unordered set (with n elements) comprising the values S takes on at z, and likewise for the symbol $\{R(z)\}$. We shall need the following known result. For the convenience of the reader, we supply a proof.

Reciprocity Theorem. *If $z_1, z_2 \in \mathbb{C} \setminus B$, then $z_1 \in \{R(z_2)\}$ if and only if $z_2 \in \{R(z_1)\}$.*

Proof. For any z, w in $\mathbb{C} \setminus B$,

$$w \in \big\{R(z)\big\} \Longleftrightarrow \bar{w} \in \big\{S(z)\big\} \Longleftrightarrow Q(z, \bar{w}) = 0 \qquad \text{(by (3.4))}$$

and by the same token,

$$z \in \big\{R(w)\big\} \Longleftrightarrow Q(w, \overline{z}) = 0.$$

So, the result follows from the known fact that,

$$Q(z, \bar{w}) = \overline{Q(w, \bar{z})}, \qquad z, w \in \mathbb{C} \tag{3.7}$$

or, what is equivalent,

$$Q(z, w) = \overline{Q(\bar{w}, \bar{z})}, \qquad z, w \in \mathbb{C}, \tag{3.8}$$

which reflects the fact that $Q(z, w)$ is constructed via (3.3) from a polynomial $P(x, y)$ with real coefficients.

To prove (3.8), simply note that $Q(z, \bar{z}) = P(x, y)$, where $z = x + iy$, and, in particular, $Q(z, \bar{z})$ is real, i.e.

$$Q(z, \bar{z}) = \overline{Q(z, \bar{z})}, \qquad z \in \mathbb{C}. \tag{3.9}$$

If we write $Q^{\#}(z, w)$ for the (holomorphic) polynomial $\overline{Q(\bar{z}, \bar{w})}$, then (3.9) can be written

$$Q(z, \bar{z}) = Q^{\#}(\bar{z}, z), \qquad z \in \mathbb{C}, \tag{3.10}$$

which implies, since both sides are analytic functions of z and $\bar{z}$, that

$$Q(z, w) = Q^{\#}(w, z), \qquad z, w \in \mathbb{C}. \tag{3.11}$$

The latter is equivalent to (3.8).

□

3.1.2. *Quadrature domains.* For our purposes in this paper, we shall only require some properties of a simple subclass of what are called "quadrature domains". For this subclass, which we call here *R-domains*, the properties needed can be developed very simply without outside references, and we shall do so in this section.

Definition. *An R-domain is a simply connected plane domain which is the image of the unit disk $\mathbb{D}$ under the conformal mapping by a rational function which is univalent in a neighborhood of $\overline{\mathbb{D}}$.*

Note that our definition implies that the boundary Γ of $\Omega := \varphi(\mathbb{D})$ (where φ is the rational conformal map) is an algebraic Jordan curve without singular points. What is of interest for us here is certain remarkable properties of the anticonformal reflection (abbreviated ACR) w.r.t. Γ.

Lemma 3.12. *Let Ω be an R-domain and R the anticonformal reflection function of its boundary Γ. Then:*

(i) *If $z \in \widehat{\mathbb{C}} \setminus (\overline{\Omega} \cup B)$, $\{R(z)\} \subset \Omega$*

(ii) *If $z \in (\Gamma \setminus B)$, then one point of $\{R(z)\}$ (namely z itself) lies on Γ, and the remaining points lie in Ω.*

In words, if we let n denote the degree of the rational mapping φ (by this we mean, writing $\varphi = p/q$ where p and q are polynomials without common zeroes, that n is the larger of $\deg p$ and $\deg q$), then (i) says: all n anticonformal image points of any point outside $\overline{\Omega} \cup B$ lie in Ω. Needless to say, this is a very special and remarkable property, not shared by all Jordan domains with algebraic boundary.

Proof of Lemma 3.12. The Schwarz function S of Γ satisfies

$$S\big(\varphi(\zeta)\big) = \overline{\varphi(\zeta)}, \qquad |\zeta| = 1. \tag{3.13}$$

Defining

$$\varphi^{\#}(\zeta) := \overline{\varphi(\bar{\zeta})}, \qquad \zeta \in \mathbb{C} \tag{3.14}$$

(so that $\varphi^{\#}$ is again a rational function of degree n), (3.13) becomes

$$S\big(\varphi(\zeta)\big) = \varphi^{\#}(1/\zeta) \tag{3.15}$$

initially for $|\zeta| = 1$ and, by analytic continuation globally. Or, in other words, S is represented parametrically by

$$t = \varphi(\zeta), \qquad S(t) = \varphi^{\#}(1/\zeta). \tag{3.16}$$

In terms of the ACR, $R(z) := \overline{S(z)}$, (3.15) becomes

$$R\big(\varphi(\zeta)\big) = \varphi\left(1/\bar{\zeta}\right), \tag{3.17}$$

or parametrically

$$t = \varphi(\zeta), \qquad R(t) = \varphi\left(1/\bar{\zeta}\right). \tag{3.18}$$

Let us fix a point $t \in \widehat{\mathbb{C}} \setminus (\overline{\Omega} \cup B)$, and compute $\{R(t)\}$. Solving $\varphi(\zeta) = t$ gives n roots $\zeta_1, \ldots, \zeta_n$ in $\widehat{\mathbb{C}}$ and all ζ_j satisfy $|\zeta_j| > 1$, since for $|\zeta| \leq 1$ we have $\varphi(\zeta) \in \overline{\Omega}$. Therefore the reflected points $\{1/\bar{\zeta}_j : j = 1, \ldots, n\}$ all lie in the open unit disk $\mathbb{D}$, which implies that the collection of points $\{\varphi(1/\bar{\zeta}_j) : j = 1, 2, \ldots, n\}$ all lie in Ω. By (3.18), this proves assertion (i) of the lemma.

Next, consider $t \in \Gamma$. The roots of $\varphi(\zeta) = t$ consist of one point, say ζ_1, on the unit circle and $n-1$ others necessarily outside $\overline{\mathbb{D}}$, call them $\zeta_2, \ldots, \zeta_n$. Then the elements of $\{R(t)\}$ are $\{\varphi(1/\bar{\zeta}_j) : j = 1, 2, \ldots, n\}$ and the first of these lies on Γ, the remaining ones in Ω. This proves (ii) of the lemma. ☐

3.2. **Main result.** The main result in this section is the following.

Theorem 3.19. *If Ω is an R-domain of degree ≥ 2, the matching problem has no non-trivial solution.*

Proof. Let $S(z)$ denote the Schwarz function of $\Gamma = \partial\Omega$ and, as usual, $R = \overline{S}$ the ACR associated to Γ. It follows from (3.16) that the direct analytic continuation of S from Γ to Ω (i.e. without leaving Ω) is meromorphic in Ω. Therefore the analytic continuation of S from Γ to $\Omega' := \widehat{\mathbb{C}} \setminus \overline{\Omega}$ must encounter a branch point, otherwise S would be meromorphic in all of $\mathbb{C}$ and this implies, by a theorem of P. Davis, that Γ is a circle (i.e. an R-domain of degree 1, contrary to hypothesis).

Therefore, there is a smooth Jordan arc α starting from a point $\zeta_0 \in \Gamma$ and returning to ζ_0, with $\alpha \setminus \{\zeta_0\} \subset \Omega'$ (where Ω', we recall, denotes $\widehat{\mathbb{C}} \setminus \overline{\Omega}$) such that, by analytic continuation of S around α it returns to ζ_0 as a different branch, which locally (near ζ_0) we shall denote by S_1. By Lemma 3.12 (ii), $\overline{S_1(\zeta_0)} \in \Omega$, or, expressing matters henceforth in terms of R: the (anti-)analytic continuation of R from ζ_0 to ζ_0 along α leads to a new branch R_1 at ζ_0, with $R_1(\zeta_0) \in \Omega$.

Now, suppose the matching problem has a solution

$$f \in A(\Gamma), \qquad g \in A'(\Gamma);$$

recall that these functions are holomorphic in Ω, Ω' respectively with $g(\infty) = 0$ and $\overline{f(z)} = g(z)$ for $z \in \Gamma$. Thus,

$$\overline{f\left(R(\zeta)\right)} = g(\zeta) \qquad \zeta \in \Gamma \tag{3.20}$$

where R denotes the "principal" branch of the ACR near Γ, i.e. $R(\zeta) = \zeta$ on Γ.

If we let ζ describe α starting from ζ_0, (3.20) makes sense and continues to hold, because $R(\zeta) \in \Omega$, by Lemma 3.12 (i), where f is defined and holomorphic, and $\overline{f(R(z))}$ is holomorphic as long as $R(z)$ (which itself is anti-holomorphic) lies in the domain of holomorphy of f.

Now, let ζ, after reaching ζ_0 along α, continue to move around Γ (making small detours into Ω' around any possible branch points of the branch R_1). Then (3.20) continues to make sense (with R now replaced by R_1) and be valid. As ζ traverses Γ, $R_1(\zeta)$ remains "trapped" in Ω and traverses an arc β, starting from $R_1(\zeta_0) \in \Omega$, which is a compact subset of Ω. This implies, in view of (3.20):

$$\max_{\zeta \in \Gamma} |g(\zeta)| = \max_{z \in \beta} |f(z)|.$$

But $|g(\zeta)| = |f(\zeta)|$ on Γ, and so

$$\max_{\zeta \in \Gamma} |f(\zeta)| = \max_{\zeta \in \beta} |f(z)|.$$

Hence the maximum modulus theorem implies f is constant. This proves the theorem. $\square$

A similar argument applies to some other domains, for example:

Theorem 3.21. *If Γ is a non-trivial ellipse (i.e. not a circle), the matching problem has only the trivial solution.*

Proof. The argument is essentially the same as for Theorem 3.19, except for interchanging the roles played by Ω and Ω'. (Indeed, now Ω' is a "quadrature domain", in the sense that it is a rational conformal map of $\{|\zeta| > 1\}$, and the Schwarz function of Γ is meromorphically extendable to Ω'.) We only sketch the details. S has branch points at the foci of the ellipse, hence we can continue $R = \overline{S}$ around a path α from some point $\zeta \in \Gamma$ to ζ_0 with $\alpha \setminus \{\zeta_0\} \subset \Omega$, such that R changes branch upon return to ζ_0. Reasoning like that used earlier shows that $R(z) \in \Omega'$ for $z \in \alpha \setminus \{\zeta_0\}$ and the new branch R_1 satisfies $R_1(\zeta_0) \in \Omega'$. As ζ now traverses Γ, $R_1(\zeta)$ remains "trapped" in Ω'. (For detailed calculations, see [E]). A contradiction to the supposition of existence of a non-trivial matching pair is reached as before. $\square$

4. Further remarks

We are aware that our results are extremely limited, and do not come anywhere near deciding when the "matching problem" has non-trivial solutions, even for domains bounded by algebraic curves. The following remark pinpoints the delicacy of the problem.

Consider first a polynomial lemniscate: it is defined by $\{|p(z)| = c\}$ for some $c > 0$, where p is a polynomial whose leading coefficient we may assume is 1. Thus, if

$$p(z) = (z - z_1) \dots (z - z_n)$$

the lemniscate is defined by

$$\prod_{j=1}^{n} (z - z_j)(\bar{z} - \bar{z}_j) - c^2 = 0$$

or, writing $z = x + iy$:

$$P(x, y) = 0$$

where

$$P(x, y) = \prod_{j=1}^{n} (x^2 + y^2 + a_j x + b_j y + c_j) - c^2,$$

and a_j, b_j, c_j are real constants (depending on z_j).

Hence,

$$P(x, y) = (x^2 + y^2)^n + \dots \text{ (terms of degree } < 2n). \tag{4.1}$$

Now, on the other hand, it is easy to check that *every R-domain is bounded by a curve* $\{P = 0\}$ *where* P *has the form* (4.1) *for some* n.

Thus, among domains bounded by curves $\{P = 0\}$ where P has the form (4.1) for some $n \geq 2$, there exist domains for which the matching problem admits non-trivial solutions, as well as ones for which this is not the case. This shows that the issue cannot be decided just by examining the highest degree terms that appear in P.

Our next remark concerns a result that follows by comparison of Theorems 2.1 and 3.19. We formulate it as:

Theorem 4.2. *Let* P *be a monic polynomial, and suppose the (lemniscate) domain* $\Omega := \{z : |P(z)| < 1\}$ *is a Jordan domain, which also is a quadrature domain. Then* Ω *is a disk.*

It seems of some interest to give a simple self-contained proof of this result, which we proceed to do:

Suppose the quadrature domain Ω has order m, so S, the Schwarz function of its boundary, has a branch equal to $\bar{z}$ on $\Gamma = \partial\Omega$, and extendible meromorphically throughout Ω with m poles (for background, see [S]). Now, on Γ, $|P(z)|^2 = 1$, i.e. $1 = P(z)\overline{P(z)} = P(z)P^{\#}(S(z))$. By analytic continuation, this holds for all z in Ω. This implies $P^{\#}(S(z))$ has $k = \deg P$ poles in Ω. But $P^{\#}(S(z))$ has km poles (counting always with multiplicities), so $m = 1$, which implies that Ω is a disk. This fact is well known, but for the convenience of the reader we give here the argument. If $m = 1$ then $S(z)$ has a simple pole at, say, $a \in \Omega$. Thus, the function $Q(z) := (z-a)(S(z)-\bar{a})$ has a holomorphic extension to a neighborhood of $\overline{\Omega}$. Moreover, on Γ we have $Q(z) = (z-a)(\bar{z}-\bar{a}) = |z-a|^2$ and, in particular, $Q(z)$ is real-valued on Γ . It follows that $Q(z)$ is constant in Ω and, hence, $|z-a|^2$ is constant on Γ, which implies that Ω is a disk.

In conclusion, let us briefly take note of some natural extensions of the "matching problem":

(i) *Extension to $n > 2$ variables.* The operator we called Π is, in essence, representable as the two-dimensional case of the operator $I + 2K$, where K is the integral operator mapping a function φ on Γ to u, where

$$u(y) := \int_{\Gamma} \varphi(x)(\partial/\partial N_x)E(y-x)dS_x, \qquad y \in \Gamma$$

and E is the Newtonian kernel. Our "matching problem" was equivalent to whether this operator (when $n = 2$) is *injective* (in one or another space of functions on Γ). This problem makes good sense in $\mathbb{R}^n$ for $n \geq 3$ also, but we have no results except for the easily verified fact that for the sphere in $\mathbb{R}^n$, with $n > 2$, the operator is indeed injective unlike the case $n = 2$. For in higher dimensions the kernel of the operator for a sphere is a constant times the Newtonian kernel, so a nontrivial nullspace would imply existence of a function whose single layer potential vanishes on the sphere and hence by the maximum principle throughout the space, an obvious contradiction.

To see that the kernel $k(x,y) := (\partial/\partial N_x)E(y-x)$ of the operator K is equal to a constant times $E(y-x)$ when $n \geq 3$ and Γ is the unit sphere, we recall that

$$E(y-x) = c_n \frac{1}{|y-x|^{n-2}} \tag{4.2}$$

and so, when Γ is the unit sphere in $\mathbb{R}^n$,

$$k(x,y) = -(n-2)c_n \frac{(y-x)\cdot x}{|y-x|^n}. \tag{4.3}$$

Now, for x, y on the unit sphere, we have

$$\begin{aligned} |y-x|^2 &= (y-x)\cdot(y-x) \\ &= 1 - x\cdot y - y\cdot x + 1 \\ &= 2(1 - x\cdot y) \\ &= 2(x\cdot x - x\cdot y) \\ &= -2(y-x)\cdot x \end{aligned} \tag{4.4}$$

and, hence,

$$k(x,y) = -\frac{n-2}{2}E(y-x), \tag{4.5}$$

as claimed.

(ii) *Generalized matching.* In the case $n = 2$ consider the following problem: Ω is a given Jordan domain with smooth boundary Γ, and we seek two functions f_1, f_2 holomorphic respectively in Ω and $\Omega' := \widehat{\mathbb{C}} \setminus \overline{\Omega}$, with $f_2(\infty) = 0$ and such that the following relations hold on Γ, where $f_j = u_j + iv_j$ $(j = 1, 2)$ with u_j, v_j real-valued:

$$(*) \qquad \begin{array}{lllllllll} A_1v_1 & + & B_1v_1 & + & A_2u_1 & + & B_2v_2 & = \varphi \\ C_1v_1 & + & D_1v_1 & + & C_2u_2 & + & D_2v_2 & = \psi \end{array}$$

where $A_j, \dots, D_j$ $(j = 1, 2)$ are given real constants and φ, ψ given real functions on Γ. Observe that our "matching problem" is the special case where this system reduces to $u_1 - u_2 = 0$, $v_1 + v_2 = 0$. For varying choices of the parameters, the system $(*)$ becomes the model for a wide range of problems in accoustics, electrostatics, gravitational attraction, diffraction theory, etc., see [C] for details and references. Still further generalizations are to allow A_j etc. to be functions on Γ. Also, we can study multivariable analogs where u_1, v_2 are replaced by U, $\partial U/\partial N$ (U being harmonic in Ω) and u_2, v_2 by V, $\partial V/\partial N$ where V is harmonic in the exterior domain with $V(\infty) = 0$. Few of these problems are completely solved.

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