Planar Elliptic Growth

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To our friend Björn Gustafsson

Abstract. The planar elliptic extension of the Laplacian growth is, after a proper parametrization, given in a form of a solution to the equation for area-preserving diffeomorphisms. The infinite set of conservation laws associated with such elliptic growth is interpreted in terms of potential theory, and the relations between two major forms of the elliptic growth are analyzed. The constants of integration for closed form solutions are identified as the singularities of the Schwarz function, which are located both inside and outside the moving contour. Well-posedness of the recovery of the elliptic operator governing the process from the continuum of interfaces parametrized by time is addressed and two examples of exact solutions of elliptic growth are presented.

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1. Introduction

Several moving boundary processes, such as solidification [1], electrodeposition [2], viscous fingering [3], and bacterial growth [4], to name a few, can be reduced, after some idealizations, to the Laplacian growth, which can be described as follows:

$$V(\xi) = \partial_n G_{D(t)}(\xi, a). \tag{1.1}$$

Here V is the normal component of the velocity of the boundary $\partial D(t)$ of the moving domain $D(t) \subset \mathbb{R}^d$, $\xi \in \partial D(t)$, t is time, ∂_n is the normal component

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of the gradient, and $G_{D(t)}(\xi, a)$ is the Green function of the domain $D(t)$ for the Laplace operator with a unit source located at the point $a \in D(t)$.

In two dimensions this equation can be rewritten as the area-preserving diffeomorphism identity

$$\Im\left(\bar{z}_t z_\phi\right) = 1, \tag{1.2}$$

where $z(t, \phi) := \partial D(t)$ is the moving boundary parameterized by $\phi \in [0, 2\pi]$ and conformal when analytically extended in the region $\Im\phi \leq 0$ [5, 6]. The equation (1.2) possesses many remarkable properties, among which, the most noticeable ones are the existence of an infinite set of conservation laws:

$$C_n = \int_{D(t)} z^n \, dx \, dy, \tag{1.3}$$

where n runs over all non-negative [7] (non-positive [8]) integers in the case of a finite (infinite) domain $D(t)$, and an impressive list of exact time-dependent closed form solutions [21]. For a beautiful interpretation of conserved quantities C_n as coefficients of the multi-pole expansion of the fictitious Newtonian potential created by matter uniformly occupying the domain $D(t)$ see, e.g., [21].

It was established in [22] that the interface dynamics described by (1.2) is equivalent to the dispersionless integrable 2D Toda hierarchy [23], constrained by the string equation. Remarkably, this hierarchy, being one of the richest existing integrable structures, describes an existing theory of 2D quantum gravity (see the comprehensive review [23] and references therein). The work [22] generated a splash of activity in apparently different mathematical and physical directions revealing profound connections between Laplacian growth and random matrices [24], the Whitham theory [25], and quadrature domains [26].

In this paper we present a natural extension of the Laplacian growth, where the Green function of $D(t)$ for the Laplace operator ∇^2 in the RHS of (1.1) is replaced by the Green function of a linear elliptic operator,

$$L = \nabla \cdot (\lambda(\mathbf{x})\nabla) - u(\mathbf{x}), \qquad \lambda(\mathbf{x}) > 0, \qquad \mathbf{x} \in \mathbb{R}^d. \tag{1.4}$$

Such a process, which is natural to be named *an elliptic growth*, is clearly much more common in physics than the Laplacian growth.

Consider, for instance, viscous fingering between viscous and inviscid fluids in the porous media governed by Darcy's law

$$\mathbf{v} = -\lambda \nabla p, \tag{1.5}$$

where λ is the filtration coefficient of the media and p is the pressure (equal to the Green function, $G_{D(t)}$, defined in (1.1) in most of the cases of interest for us). One can easily imagine a non-homogeneous media where the filtration coefficient λ is space-dependent. Such examples of elliptic growth, where the elliptic operator L has the form of the Laplace-Beltrami operator, $L = \nabla \cdot \lambda \nabla$, and λ is a prescribed function of $\mathbf{x}$, will be called an elliptic growth *of the Beltrami type*. It is clear that all moving boundary problems other than viscous fingering with a non-homogeneous kinetic coefficient λ fall into this category.

From a mathematical point of view this process is the Laplacian growth occurring on curved surfaces instead of the Euclidean plane. In this case the Laplace equation is naturally replaced by the Laplace-Beltrami equation, and λ (that can be a matrix instead of a scalar as it is in our case) is related to the metric tensor. There are several works addressing the Hele-Shaw problem on curved surfaces and we will mention below those few related to the integrable mathematical structure of elliptic growth.

Another major source of examples of elliptic growth is related to screening effects, when $u \neq 0$, while λ is constant in (1.4). The simplest example of this kind is an electrodeposition, where the field p is the electrostatic potential of the electrolyte. It is known that in reality electrolytes ions are always locally surrounded by a cloud of oppositely charged ions. This screening modifies the Laplace equation for the electrostatic potential by adding to the Laplace operator the negative screening term, $-u(x)$, which stands for the inverse square of the radius of the Debye-Hukkel screening in the classical plasma [27]. For the homogeneous screening u is a (positive) constant, so the operator L becomes the Helmholtz operator, while for the non-homogeneous case, when u is not a constant, L is a standard Schrödinger operator. Motivated by this example, we will call the moving boundary problem for $L = \nabla^2 - u$ an elliptic growth of *Schrödinger type*.

We show that these rather general types of elliptic growth still retain remarkable mathematical properties, similar to those possessed by the Laplacian growth. A mixed case with a non-constant λ and non-zero u also shares similar properties but is less representative in physics and can always be reduced to one of the two former types of elliptic growth by a simple transformation described later on in the article. For completeness we shall indicate another class of elliptic growth when the fluid density ρ changes in space while it is constant in time. This happens for instance when porosity (fraction of porous media accessible for fluid) is space-dependent. In this case the continuity equation for incompressible fluid in porous media has the form

$$\nabla(\rho(\mathbf{x})\lambda(\mathbf{x})\nabla p) = 0, \tag{1.6}$$

while (1.5) still holds. This case presents an additional extension of the elliptic growth related to potential theory with a non-uniform density, as will be shown below.

In prior works on elliptic growth an infinite number of conservation laws, regarded as extensions of (1.3), were identified in [21, 28]. Also an integrable example in 2D, which corresponds to a very special choice of the conductivity function, $\lambda(\mathbf{x})$, was explicitly constructed in [29]-[31]. The elliptic growth in these works was reduced to the well-known Calogero-Moser integrable system.

The present article contains several new results on elliptic growth and reviews the known conservation laws from a slightly novel perspective. It is organized as follows:

- In Section 2 we interpret (1.2) as the equation of the area-preserved diffeomorphism in 2D and analyze its connections with the Laplacian growth.

- Section 3 contains the definition of elliptic growth, the conservation laws for this process and recasts the latter in terms of the inverse non-Newtonian potential theory.

- In Section 4 we obtain the equation (1.2) for elliptic growth of the Beltrami type by introducing the function q, conjugate with respect to p defined in (1.4), which plays the role of a stream function for the incompressible fluid; furthermore, we obtain the Beltrami equation for the function $p + iq$.

- Section 5 is devoted to analyzing connections between the elliptic growth of the Beltrami and Schrödinger types, elucidating the difficulties of parametrization of the interface for the Schrödinger type.

- In Section 6 we reformulate the elliptic growth in terms of the Schwarz function of the moving interface.

- Section 7 addresses a well-posedness of a recovery problem for the operator L from the continuum of moving interfaces parameterized by time.

- In Section 8 we discuss Herglotz' theorem as the main device to generate exact solutions and present two examples of the exact closed form solutions of the elliptic growth. We also identify the constants of motion of these solutions as the singularities of the Schwarz function of the moving contour.

- Section 9 contains brief conclusions.

Due to the fact that one of us is labelled as a theoretical physicist and following the customs of the physics community the references do not appear in alphabetical order.

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1.1. List of notations and conventions

We collect below a few basic definitions and notations used throughout the text.

$\nabla^2 = \Delta, \qquad \nabla f = \operatorname{grad} f$;
$\nabla(\mathbf{U}) = \nabla \cdot \mathbf{U} = \operatorname{div} \mathbf{U}$, where $\mathbf{U}$ is a vector field;
$\mathcal{C}^\omega$ denotes the class of real analytic functions;
$\dot{h} = \frac{\partial h}{\partial t}$;
n stands for the outer unit normal to the moving boundary $\Gamma = \Gamma(t)$;
ℓ denotes the arc length on the boundary Γ;
$dA = \frac{1}{\pi} d\, Area = \frac{dx \wedge dy}{\pi}$;
an *analytic Jordan curve* means a smooth Jordan curve which admits a real analytic parametrization.

2. Area preserving diffeomorphisms

This section contains some immediate implications of the equation of area preserving diffeomorphisms related to the parametrization of an analytic Jordan curve.

Later on we shall see that this, apparently innocent, Jacobian identity plays an important role in the study of moving boundaries governed by elliptic growth.

2.1. Fourier expansion

Consider the equation

$$\Im(\overline{z_t} z_q) = 1, \tag{2.1}$$

where $t \in [0,T]$ is a non-negative variable (usually identified with time), while $q \in [0, 2\pi]$ is the parameter along the Jordan analytic curve C_t. Equation (2.1) can be interpreted as an area preserving property: that is the Jacobian of the transformation

$$(t,q) \mapsto (x,y), \quad \text{where} \quad z(t,q) = x + iy,$$

is equal to 1.

To be more precise, for a fixed t, we assume that the 2π-periodic real analytic map

$$z(t,\cdot) : [0,2\pi] \longrightarrow \mathbb{C}$$

is an embedding, and its range is denoted by C_t. We denote by $D(t)$ the interior of the Jordan curve C_t.

In view of the smoothness hypothesis imposed on $z(t,q)$ we can expand the function $z(t,q)$ in a Fourier series

$$z(t,q) = \sum_{-\infty}^{\infty} a_k(t) e^{ikq}. \tag{2.2}$$

We will assume that the dependence $t \mapsto z(t,q)$ is C^1. Also, we put $w = e^{iq}$, so that we can rewrite (2.2) as

$$z(t,w) = \sum_{-\infty}^{\infty} a_k(t) w^k.$$

By the analyticity assumption, there exists $\epsilon, 0 < \epsilon < 1$, so that the above Laurent series is convergent in the annulus $1 - \epsilon < |w| < 1 + \epsilon$.

Due to the real analyticity of the map z, the Fourier series for z and its derivatives are absolutely and uniformly convergent, whence

$$z_q(t,q) = i \sum_{-\infty}^{\infty} k a_k(t) e^{ikq},$$

and

$$\overline{z_t}(t,q) = \sum_{-\infty}^{\infty} \overline{\dot{a}_n(t)} e^{-inq}.$$

Thus, equation (2.1) becomes

$$1 = \Im(\overline{z_t} z_q) = \frac{1}{2} \sum_{k,n} (\overline{\dot{a}_n} k a_k + \dot{a}_k n \overline{a_n}) e^{i(k-n)q} =$$

$$\sum_{m=-\infty}^{\infty}[\frac{1}{2}\sum_{n=-\infty}^{\infty}((n+m)\overline{\dot{a}_n}a_{n+m}+n\dot{a}_{n+m}\overline{a_n})]e^{imq}.$$

By equating the coefficients we find

$$1=\frac{1}{2}\sum_{n=-\infty}^{\infty}(n\overline{\dot{a}_n}a_n+n\dot{a}_n\overline{a_n}),$$

and

$$0=\frac{1}{2}\sum_{n=-\infty}^{\infty}((n+m)\overline{\dot{a}_n}a_{n+m}+n\dot{a}_{n+m}\overline{a_n}),$$

whenever $m\neq 0$.

Since

$$\text{Area}(D(t))=\int_{C_t}\frac{\bar{z}dz}{2i}=\int_{|w|=1}\overline{z(t,q)}\frac{\partial z(t,w)}{\partial w}\frac{dw}{2i}=\int_0^{2\pi}\overline{z(t,q)}\frac{\partial z(t,q)}{\partial q}\frac{dq}{2i},$$

we derive the following remarkable identity.

Proposition 2.1. *Under the assumption* (2.1)*, the family of domains bounded by the curves* $z(t,q),\ 0\leq q\leq 2\pi$, *satisfy*

$$\frac{d\text{Area}(D(t))}{dt}=2\pi.$$

By regarding z now as a function of t and the complex variable $w=e^{iq}$, we obtain, along the curve C_t, the following:

$$\frac{\partial z}{\partial q}=\frac{\partial z}{\partial w}\frac{\partial w}{\partial q}=z_w iw.$$

Denote, for the sake of simplicity, $z'=z_w$. Then, the master equation (2.1) becomes

$$\Re(wz'\overline{\dot{z}})=1.$$

Since $z(t,\cdot)$ is analytic in the annulus $0<1-\epsilon<|w|<1+\epsilon$, the following result follows.

Proposition 2.2. *Under the above assumptions*

$$wz'(t,w)\dot{z}^\sharp(t,1/w)+\frac{1}{w}z'^\sharp(t,\frac{1}{w})\dot{z}(t,w)=2,\quad 1-\epsilon<|w|<1+\epsilon.$$

(As usual, for a complex analytic function $h(w)$, we denote by

$$h^\sharp(w)=\overline{h(\overline{w})},$$

obtained from h by conjugating its Taylor coefficients.)

2.2. Analytic parametrization.

The most studied case of the Laplacian growth process requires an additional analyticity assumption. We devote the present subsection to this scenario. Assume that for all $t \in [0, T]$ the negative Fourier coefficients vanish, i.e.,

$$a_k(t) = 0, \quad k < 0. \tag{2.3}$$

It is not difficult to see that this will hold the whole evolution, $t \in [0, T]$. The equation (2.3) simply means that for a fixed t the function $z(t, w)$ extends analytically to the unit disk $w \in \mathbb{D}$. Since $z(t, \cdot)$ is a homeomorphism from the boundary $\mathbb{T} = \partial\mathbb{D}$ to the curve $\partial D(t)$, the argument principle implies that

$$z(t, \cdot) : \mathbb{D} \longrightarrow D(t)$$

is a conformal mapping. Let $w = \Psi(t, z)$ denote the inverse conformal mapping. Since a linear transformation $z \mapsto \alpha z + \beta, |\alpha| = 1$, leaves the equation (2.1) invariant, we can assume without loss of generality that $z(t, 0) = 0$ and that $\rho(t) = z'(t, 0) > 0$. To distinguish this case from the general case considered in the previous subsection we shall denote $\partial D(t)$ by $\Gamma(t)$. The function $p(t, z) = \log |\Psi(t, z)|$ is, up to a constant factor, the Green function of the domain $D(t)$, with the source at $z = 0$. This means that $p(t, .)$ is the unique harmonic function in the punctured domain $D(t) \setminus \{0\}$ having zero boundary values on $\Gamma(t)$ and such that $p(t, z) - \log |z|$ is harmonic at $z = 0$.

Moreover, the harmonic conjugate function $\arg \Psi(t, z)$ is, up to an additive constant, equal to $q(z), \ z \in \Gamma(t)$.

In other words, for a fixed value of the parameter t, we have:

$$\nabla^2 p(t, \cdot) = 2\pi\delta(\cdot), \quad \text{in} \quad D(t),$$

$$p(t, \cdot)|_{\Gamma(t)} = 0,$$

and

$$(q_y(t, \cdot), -q_x(t, \cdot)) = (p_x(t, \cdot), p_y(t, \cdot)).$$

Consequently, the normal velocity of the boundary equals

$$V = \frac{\partial q(t, \cdot)}{\partial \ell} = \frac{\partial p(t, \cdot)}{\partial n}.$$

So, by the area conservation property, we have

$$V dz \wedge d\ell = dz \wedge dq.$$

These equations define a specific dynamics of planar boundaries known as Laplacian growth. For recent guides to the mathematics and physics behind Laplacian growth we refer to the volume [32] and the survey [26]. We will return to this case after discussing the geometry of the moving boundaries.

2.3. The Schwarz function

An important tool for studying the changing geometry of the moving boundaries is the Schwarz function [33], [45]. Up to the complex conjugation it is simply the (local) Schwarz reflection with respect to an analytic curve.

On the real analytic smooth boundary $\Gamma(t)$ of $D(t)$ we introduce the Schwarz function

$$\overline{z} = S(t, z),$$

where S is analytic in the variable z. The domain of definition for $S(t, .)$ is at least a tubular neighborhood of $\Gamma(t)$, although the function may possess analytic extensions to much larger sets. For instance, the Schwarz function of a disk centered at $z = a$ and of radius r is the rational function

$$S(z) = \overline{a} + \frac{r^2}{z-a}.$$

If a polynomial $P(z, \overline{z})$ vanishes on $\Gamma(t)$, then, necessarily, the associated Schwarz function satisfies the algebraic equation

$$P(z, S(t, z)) = 0, \quad z \in \Gamma(t).$$

Proposition 2.3. [34] *The normal velocity of the boundary satisfies*

$$V = \frac{S_t}{2i\sqrt{S_z}},$$

with the proper choice of the branch of the square root, so that $1/\sqrt{S_z} = dz/d\ell$ *along* $\Gamma(t)$.

Proof. By taking derivatives with respect to t we have

$$\overline{z_t} = S_t + S_z z_t.$$

When restricted to the boundary curve,

$$S_z = \frac{d\overline{z}}{dz}$$

is a complex number of modulus one.

Fix a single-valued branch of $\sqrt{S_z}$ along $\partial D(t)$. This is always possible since $1/\sqrt{S_z}$ equals to the unit tangent vector to $\partial D(t)$ and, hence, is single valued near $\partial D(t)$. Then the above equation becomes

$$\frac{-S_t}{\sqrt{S_z}} = \sqrt{S_z} z_t - \frac{\overline{z_t}}{\sqrt{S_z}},$$

or, equivalently,

$$\frac{S_t}{\sqrt{S_z}} = 2i\Im\frac{\overline{z_t}}{\sqrt{S_z}}.$$

Since

$$\frac{1}{\sqrt{S_z}} = \frac{dz}{d\ell}$$

is the unit tangent vector, then

$$\frac{S_t}{\sqrt{S_z}} = 2iV$$

□

Theorem 2.4. *There exists a multivalued analytic function $W(t, z)$, defined in a neighborhood of $\partial D(t)$, with the property*

$$S_t = \partial_z W,$$

and such that $\Re W$ is constant along $\partial D(t)$.

Proof. From the above computations we find

$$\overline{S_t} = -2i\overline{\sqrt{S_z}}V = \frac{-2i}{\sqrt{S_z}}V,$$

whence the vector S_t is collinear with the complex conjugate of the normal to $\Gamma(t)$. Moreover, rewriting the last equation in the form

$$\overline{S_t} = \overline{\partial W}$$

we infer

$$|S_t|n = \overline{S_t} = \overline{\partial W} = \nabla(\Re W),$$

where n is the normal to $\partial D(t)$. Hence, it follows that the boundary, $\partial D(t)$ is a level set of $\Re W$. □

2.4. Laplacian growth

In this subsection we merely illustrate few classical observations related to the consequences of the dynamics (2.1) under the analyticity assumption $a_k(t) = 0, \quad k < 0,\ t \in [0, T]$. That is, we assume again that the parametrization $z(t, \cdot)$ of the curve $\Gamma(t)$ analytically extends to the interior of the unit disk $\mathbb{D}$ and will use intensively the Schwarz function techniques.

By returning to the notations introduced in Section 2.2 we can identify the complex potential W with the multivalued function $\zeta(t, z) = p(t, z) + iq(t, z) = \log \Psi(t, z)$ and then study the analytic extension of the Schwarz function $S(t, z)$.

Theorem 2.5. [34] *For every t, there exists a tubular neighborhood U of $\Gamma(t)$, such that*

$$S_t(t, z) = 2\zeta_z(t, z), \quad z \in U. \tag{2.4}$$

Note that the function ζ is multivalued and analytic in the punctured domain $D(t) \setminus \{0\}$. Its derivative ζ_z is therefore meromorphic there with a simple pole at $z = 0$ and the residue equal to 1.

Proof. Indeed, according to Proposition 2.3, we have along $\Gamma(t)$:

$$\frac{S_t}{2i\sqrt{S_z}} = \frac{\partial p}{\partial n},$$

so

$$S_t = \frac{2i}{dz/d\ell}\frac{\partial p}{\partial n} = \frac{2i(\partial p/\partial n)d\ell}{dz} =$$
$$2\frac{(i\partial q/\partial \ell)d\ell}{dz} = 2\frac{\partial(p+iq)/\partial\ell)d\ell}{dz} = 2\frac{\partial \zeta}{\partial z}.$$

□

As simple as it looks, equation (2.4) has surprising consequences. In order to unveil them, we start with the known Plemelj-Privalov-Sokhotsky formula applied for a fixed t to the function $S(t,\cdot)$. Let

$$S_{\pm}(t,z) = \frac{1}{2\pi i}\int_{\Gamma(t)}\frac{\overline{\sigma}d\sigma}{\sigma - z}, \quad z \in D(t), \quad \text{respectively} \quad z \in \mathbb{C}\setminus\overline{D(t)}.$$

Then

$$S(t,z) = S_+(t,z) - S_-(t,z), \qquad z \in U, \tag{2.5}$$

where U denotes, as before, a neighborhood of $\Gamma(t)$. Similarly we decompose the function $\zeta_z(t,z)$ and find

$$(\zeta_z)_-(t,z) = \frac{-1}{z}.$$

From the uniqueness of the above decompositions and (2.4), we infer

$$S_-(t,z)_t = \frac{-2}{z},$$

or, for $z \notin D(t)$,

$$\frac{d}{dt}\frac{1}{2\pi i}\int_{\Gamma(t)}\frac{\overline{\sigma}d\sigma}{\sigma - z} = \frac{-2}{z},$$

that is

$$\frac{d}{dt}\int_{D(t)}\frac{dA(\sigma)}{\sigma - z} = \frac{-2}{z}.$$

By integrating against a polynomial f along the circle $|z| = R$, with R sufficiently large, we find the following general identity.

Proposition 2.6. *If the parametrization $z(t,w)$ of the boundary of the domain $D(t)$ extends analytically to the interior of the unit disk $\mathbb{D} := \{w : |w| < 1\}$, then for every polynomial $f(z)$ the following identity holds:*

$$\frac{d}{dt}\int_{D(t)} f(z)dA(z) = 2f(0).$$

Equivalently, $S_-(t,z) = \frac{-2t}{z} + h(z)$, where $h(z)$ is an analytic function in the neighborhood of $\mathbb{C}\setminus\overline{D(t)}$, vanishing at infinity, and independent of t. A simple application of Cauchy's formula now yields that there exists a complex valued measure μ supported on a compact set $K \subset D(t)$, independent of t (as proved above) and such that

$$S_-(t,w) = \frac{-2t}{z} - \frac{1}{\pi}\int_K\frac{d\mu(\sigma)}{\sigma - z}, \quad z \in \mathbb{C}\setminus\overline{D(t)}.$$

By repeating the above calculations we find

$$\int_{D(t)} \frac{dA(\sigma)}{\sigma - z} = \frac{-2t}{z} - \frac{1}{\pi}\int_K \frac{d\mu(\sigma)}{\sigma - z}$$

and, consequently, the following quadrature identity follows.

Corollary 2.7. *Under the same hypotheses as in Proposition* 2.6*, for every polynomial f we have*

$$\int_{D(t)} f(z)dA(z) = 2tf(0) + \frac{1}{\pi}\int_K f(z)d\mu(z). \tag{2.6}$$

Simple examples show that the measure μ is not unique. If one insists that the supporting set K is "minimal", and the representing measure μ is positive, then one can prove in most interesting cases the uniqueness of μ. The case of quadrature domains $D(t)$, corresponding by definition to a positive finite atomic measure μ, is by far the best understood from the constructive point of view. In this case the conformal mappings $z(t,w)$ are rational. Examples, a discussion of the alluded uniqueness and further details and references can be found in the collection of articles [35], cf. also [42], [45].

3. Elliptic growth

Guided by Laplacian growth as a prototype, we introduce in this section the elliptic growth phenomenon mentioned in the Introduction. It is surprising to see that many features of Laplacian growth persist and yet sharp differences occur. Let us start the formulation in arbitrary dimension d for a possibly multiply connected domain $D(t)$ in $\mathbb{R}^d$ with many sources, but later on we will focus on a homotopically trivial 2D case with a single source at the origin in more detail.

Consider a family $D(t)$ of bounded domains in $\mathbb{R}^d$ with smooth analytic boundaries. Moreover, dependence of $D(t)$ on t is assumed to be real analytic (in the sense of a chosen parametrization) as well.

Let G be an open set containing as relative compact subsets all $D(t), -1 < t < 1$, and let $\lambda : G \longrightarrow (0,\infty)$ be a real analytic function. We consider the elliptic (non-positive) differential operator

$$L = \nabla \cdot \lambda\nabla = \operatorname{div}(\lambda \operatorname{grad}).$$

As we noted in Section 1.1, when there is no danger for confusion we shall omit the dot in the notation, and write, for example, $\Delta = \nabla^2$.

The moving boundary problem with N sources s_k at $\mathbf{x}_k \in D(t)$ is the following:

Given $D(0)$, find domains $D(t)$ satisfying the system of equations:

$$L\,p = \sum_{k=1}^{N} s_k\delta(\mathbf{x} - \mathbf{x}_k) \quad \text{in} \quad D(t),$$

$$p|_{\partial D(t)} = 0,$$

$$V = \lambda\,\partial_n p \quad \text{on} \quad \partial D(t).$$

Note that the first two conditions simply assert that p is the linear combination of the Green functions for the operator L of the domain $D(t)$, with singularities at $\mathbf{x_k}$, while the third condition determines the dynamics of the moving boundary.

Theorem 3.1. *For every function $\psi \in C^2(G)$ satisfying $L\psi = 0$, we have*

$$\frac{d}{dt}\int_{D(t)} \psi \,\mathrm{dVol} = \sum_{k=1}^{N} s_k \psi(\mathbf{x}_k).$$

Here, dVol stands for Lebesgue measure on $D(t)$.

Since the constant function $\psi = \mathbf{1}$ is annihilated by the operator L, the above formula implies

$$\frac{d}{dt}\mathrm{Vol}\,D(t) = \sum_{k=1}^{N} s_k, \quad |t| < 1.$$

Thus, in this moving boundary process, the volume is still proportional to time.

Proof. Let $d\Gamma$ denote the surface element on each connected component of the boundary of $D(t)$. Then, we have:

$$\frac{d}{dt}\int_{D(t)} \psi d\mathrm{Vol} = \int_{\partial D(t)} \psi V\, d\Gamma = \int_{\partial D(t)} (\psi\lambda\partial_n p - p\lambda\partial_n\psi) d\Gamma =$$

$$\int_{\partial D(t)} (\psi\lambda\nabla p - p\lambda\nabla\psi)\cdot n\, d\Gamma = \int_{D(t)} \nabla\cdot(\psi\lambda\nabla p - p\lambda\nabla\psi) d\mathrm{Vol} =$$

$$\int_{D(t)} [\psi L p - pL\psi] d\mathrm{Vol} = \sum_{k=1}^{N} s_k\psi(\mathbf{x}_k).$$

□

Corollary 3.2. *In the case when the domains $D(t)$ are all homeomorphic to a ball and contain a single source $s_1 > 0$, the moments*

$$C(\psi) = \int \psi\, d\mathrm{Vol}, \quad L\,\psi = 0,$$

determine the domains $D(t)$ (locally in t).

Proof. Indeed, it is sufficient to consider a single moment $C(\mathbf{1})$. As remarked earlier,

$$\frac{d\mathrm{Vol}(D(t))}{dt} = \int_{\partial D(t)} V d\Gamma = s_1 > 0$$

and the corollary follows, after observing that the family $D(t)$ is increasing with respect to the ordering by inclusion. □

Remark. Theorem 3.1 and the Corollary extend word for word to more general elliptic operators

$$L = \operatorname{div}(\Lambda \operatorname{grad}) - u(\mathbf{x}),$$

where the matrix $\Lambda = (\lambda_{i,j}(\mathbf{x})_{i,j=1,2})$ is uniformly elliptic on the domain G and all the coefficients $\lambda_{i,j}, u(\mathbf{x})$ are assumed to be real analytic in G and $u \geq 0$. The only modification needed is that in the last boundary condition in (1.1) where one ought to require $V = \Lambda(\nabla p) \cdot n$. The existence of the Green function p for such operators is well known [36, 37]. The fact that $V > 0$ on $\partial D(t)$ then follows from the maximum principle and Hopf's lemma which hold for such operators cf. [36, 37, 43].

Assuming that the sources strengths $s_k(t)$ depend on time we then obtain another notable corollary of Theorem 3.1. The functionals $\int_{D(t)} \psi d\mu$ do not depend on $s_k(t)$, but only on the value of the integral $\int_0^t s(t)\, dt$ [38, 21, 42].

The functionals $\int_{D(t)} \psi d\mathrm{Vol}$ have a remarkable potential theoretic interpretation [21, 28]. Indeed, imagine that a domain $D(t)$ is occupied by matter with a unit density, which creates the potential Φ, governed by the Poisson's equation

$$L\Phi = \chi_{D(t)},$$

where χ_D is the characteristic function of the domain D. A solution of the last equation is

$$\Phi(\mathbf{x}) = \int_{D(t)} G_0(\mathbf{x}, \mathbf{y})\, d\mathrm{Vol}(\mathrm{y}),$$

where $G_0(\mathbf{x}, \mathbf{y})$ is the fundamental solution for the operator L. In important particular cases which are relevant for physical applications (for instance, for the Helmholtz operator $\Delta - 1$) it is possible to expand $G_0(\mathbf{x}, \mathbf{y})$ into the series

$$G_0(\mathbf{x}, \mathbf{y}) = \sum_n \tilde{\psi}_n(\mathbf{x}) \psi_n(\mathbf{y}), \quad x \notin D(t), y \in D(t),$$

where $\{\psi_n\}$ and $\{\tilde{\psi}_n\}$ are bases of the null space of L in $D(t)$ and its complement in $\mathbf{R}^d$ respectively. Then, assuming commutativity of summation and integration, we obtain

$$\Phi(\mathbf{x}) = \sum_n \tilde{\psi}_n(\mathbf{x}) \int_{D(t)} \psi_n(\mathbf{y})\, d\mathrm{Vol}(\mathrm{y}).$$

Therefore, we have obtained the functionals introduced in the Theorem 3.1 as the coefficients of the multi-pole expansion of the non-Newtonian potential given in a far field. We would like to add that the gradient of Φ is a generalization of the Cauchy transform for the domain $D(t)$ - the notion that was so useful in the Laplacian growth and the related field of quadrature domains.

In the case of elliptic growth with nonhomogeneous density mentioned in the Introduction, one should modify the formulation given in the beginning of this section by adding a positive space-dependent factor, ρ, namely,

$$\lambda \to \rho\lambda;$$

under these assumptions we still encounter an infinite set of conservation laws similar to the previous case, when $\rho = 1$, namely:

$$\frac{d}{dt}\int_{D(t)} \frac{\psi}{\rho} d\mathrm{Vol} = \sum_{k=1}^{N} q_k \psi(\mathbf{x}_k).$$

The elementary proof is not included here. From the point of view of potential theory this corresponds to the case of occupation of the domain $D(t)$ by non-uniform matter with density $1/\rho(\mathbf{x})$. Some aspects of this case in 2D were discussed in [39], also cf. [43]. It is clear that the inverse potential problem of recovery of $D(t)$ is considerably more difficult in this situation.

4. The conjugate function

From now on we return to a planar case with a single source of strength $s_1 = 2\pi$ and assume that all domains $D(t)$, $0 \le t \le T$, are *simply connected*. Define, using the notations from the previous section, a (multivalued) *conjugate function* $q \in C^\omega(D(t))$ by

$$q_y = \lambda p_x \tag{4.1}$$

$$q_x = -\lambda p_y. \tag{4.2}$$

Accordingly,

$$\nabla \cdot \frac{1}{\lambda}\nabla q = 0, \quad \text{in} \quad D(t).$$

In most computations below t is fixed. However, we stress that by its very definition, the function q depends on t also: $q(z, \overline{z}) = q(t; z, \overline{z})$. We hope that omitting t in the notations of q will not confuse the reader.

We have

$$V = \lambda \partial_n p = \partial_\ell\, q.$$

Let $z(t, \ell)$ be the parametrization of the contour $\Gamma(t) = \partial D(t)$ by an arc-length ℓ. Then a right angle rotation of the unit tangent vector gives

$$n = -iz_\ell,$$

hence the normal component of the boundary velocity is

$$V = z_t \cdot (-iz_\ell) = \Re(z_t(\overline{-iz_\ell})) = \Im(\overline{z_t} z_\ell) = \partial_\ell\, q,$$

where, as before, subscripts stand for partial derivatives.

Since λ is positive, then $\partial_\ell\, q > 0$ along $\Gamma(t)$, therefore $q(t, .)$ can equally well parameterize the boundary $\partial D(t) = \Gamma(t)$, in which case we can rewrite the above equation as

$$\Im(\overline{z_t} z_q) = 1, \quad \text{on} \quad \partial D(t).$$

Lemma 4.1. *The variation of q along the curve $\Gamma(t)$ is equal to 2π.*

Proof. Indeed

$$\text{var } q|_{\Gamma(t)} = \int_{\Gamma(t)} \frac{\partial q}{\partial \ell} d\ell = \int_{\Gamma(t)} \frac{\lambda \partial p}{\partial n} d\ell =$$

$$\int_{\Gamma(t)} \lambda \nabla p \cdot n d\ell = \int_{D(t)} \nabla(\lambda \nabla p) dArea = 2\pi.$$

□

Introducing the multivalued function

$$\zeta(z, \overline{z}) = p + iq,$$

we have

$$i\lambda \overline{\partial} p = \overline{\partial} q,$$

or, still,

$$\overline{\partial \zeta} = \frac{1+\lambda}{1-\lambda} \bar{\partial} \zeta, \tag{4.3}$$

which is a form of the Beltrami equation [44]. In terms of a new variable

$$\omega = \sqrt{\lambda} p + i \frac{q}{\sqrt{\lambda}},$$

the equation (4.3) takes the canonical Carleman form

$$\bar{\partial} \omega = (\bar{\partial} \log \sqrt{\lambda})\, \overline{\omega}. \tag{4.4}$$

An implicit solution of this equation is found from

$$\omega(z, \bar{z}) = F(z) \exp \int_{D(t)} \frac{\bar{\partial} \log(\sqrt{\lambda(\zeta, \bar{\zeta})})}{\zeta - z} \frac{\overline{\omega(\zeta, \bar{\zeta})}}{\omega(\zeta, \bar{\zeta})} \, dA(\zeta),$$

where $F(z)$ is analytic in $D(t)$ (cf. [44] for more details).

Thus the moving boundary problem of finding $D(t)$ can be reformulated as a Dirichlet boundary value problem:

Given the weight λ, find a function $\zeta(z, \bar{z})$ (or, $\omega(z, \bar{z})$) satisfying the Beltrami (or, Carleman) equation above and subject to the boundary condition $\Re\zeta = 0$ (or, $\Re\omega = 0$).

5. Elliptic growth of Schrödinger type

The previous section was devoted to the elliptic growth of the Beltrami type. In this section, we will consider the elliptic growth of Schrödinger type, which, as already mentioned in the introduction, is related to the theory of the Schrödinger operator. To be specific, we consider the problem:

Given a domain $D(0)$ find domains $D(t)$, satisfying the system of equations:

$$\begin{aligned} LP &= (\nabla^2 - u)P = \sum_{k=0}^{N} q_k \delta(\mathbf{x} - \mathbf{x}_k) \quad \text{in} \quad D(t), \\ P &= 0 \qquad \text{on } \partial D(t), \\ V &= \partial_n P \quad \text{on } \partial D(t). \end{aligned}$$

As has already been demonstrated, this problem has an infinite set of conservation laws which are time derivatives of integrals of null vectors of the operator L. These integrals have the potential-theoretic interpretation discussed in Section 3. Let us pose the following question. Does there exist in this case a function Q "conjugate" w.r.t. to P in the sense that P and Q are connected via some generalized Cauchy-Riemann equations? And, if such Q exists, can it be used as a parametrization of the moving contour $\partial D(t)$ similarly to how it was used in the case of the elliptic growth of the Beltrami type in the previous section?

The answer to the first question is 'yes', and to the second one - 'no'. One can see this from the generalized Cauchy-Riemann conditions which connect p and q in the Beltrami case. Namely,

$$\begin{aligned} \lambda \partial_x p &= \partial_y q, \\ \lambda \partial_y p &= -\partial_x q. \end{aligned}$$

These formulae suggest the substitution

$$p = \frac{P}{\sqrt{\lambda}},$$

$$q = \sqrt{\lambda} Q.$$

Thus, the new functions P and Q are connected via the system of linear euqations:

$$\partial_x P - P\partial_x(\log \sqrt{\lambda}) = \partial_y Q + \partial_y(\log \sqrt{\lambda}),$$
$$\partial_y P - P\partial_y(\log \sqrt{\lambda}) = -\partial_x Q - \partial_x(\log \sqrt{\lambda}).$$

Differentiating the first equation w.r.t. to x, the second one w.r.t. to y, and then adding them, one obtains

$$(\nabla^2 - u)P = 0,$$
$$(\nabla^2 - v)Q = 0,$$

where

$$u = \frac{\nabla^2(\lambda^{1/2})}{\lambda^{1/2}};$$
$$v = \frac{\nabla^2(\lambda^{-1/2})}{\lambda^{-1/2}}.$$

This simple transformation known as the "removal of the first derivative from linear differential equations of the second order" and, also, closely related to supersymmetry in physics, joints together the two major types of elliptic growth.

Now let us show that the function Q, unlike the function q cannot, in general, provide a parametrization of the contour. Indeed, it was shown in the previous section that q can serve as a parametrization since it is a monotonically increasing function of the arc-length along the contour. Since $q = \sqrt{\lambda}Q$, it is now clear that Q, generally speaking, is not monotone along the interface because of the space dependent factor $\lambda^{-1/2}$.

As one can easily see, $\sqrt{\lambda}$ solves the same Schrödinger equation as P, namely

$$(\nabla^2 - u)\sqrt{\lambda} = 0,$$

while the function $1/\sqrt{\lambda}$ solves the same Schrödinger equation as Q, namely

$$(\nabla^2 - v)\frac{1}{\sqrt{\lambda}} = 0.$$

6. An inverse problem

We address below the following natural question:

Is it possible to have the same "movie" $(t, \Gamma(t))$, $t \in [0, T]$ governed by the elliptic growth dynamics with different weights λ?

By studying a particular example we shall demonstrate that, indeed, such non-uniqueness may take place. To fix the ideas, assume that the elliptic growth dynamics, as specified above has the property that the conformal map $z(t, \cdot)$ onto a neighborhood U of $\Gamma(t)$:

$$z(t, \cdot) : \{w; 1 - \epsilon < |w| < 1 + \epsilon\} \longrightarrow U$$

extends analytically to the unit disk $|w| < 1$. Then, as we saw earlier, the equation (2.1) once again governs the whole evolution of $\Gamma(t)$ but this time the evolution of $\Gamma(t)$ is the Laplacian growth. In particular, the parameter along each boundary satisfies $q(t, z) = \Im \log w(t, z)$, that is $\nabla^2 q = 0$ at all points except the isolated singularity. On the other hand we have started with the assumption $\nabla\lambda^{-1}\nabla q = 0$. Hence

$$\lambda^{-1}\nabla^2 q - \lambda^{-2}(\nabla\lambda) \cdot (\nabla q) = 0,$$

so

$$(\nabla\lambda) \cdot (\nabla q) = 0.$$

The function $p(t, z)$ also, by its very definition, satisfies

$$(\nabla p) \cdot (\nabla q) = 0.$$

Since ∇p never vanishes, at least in a neighborhood of $\Gamma(t)$, a functional dependence

$$\lambda(z) = f(t, p(t, z))$$

must occur. This simple computation gives rise to the following counterexample.

Let us consider the simplest Laplacian growth process of expanding concentric discs

$$D(t) = D(0, \sqrt{2t}), \quad t > 0,$$

with associated functions

$$p(t, z) = \log \frac{|z|}{\sqrt{2t}}, \quad q(t, z) = \arg z.$$

Let $r = |z|$ and choose any positive, smooth radial function $\lambda(r)$, e.g., $\lambda(r) = \exp r$. Then, everywhere in $\mathbb{C} \setminus \{0\}$, we find

$$\nabla\lambda(r)^{-1}\nabla q = \lambda(r)^{-1}\Delta q - \lambda^{-2}\nabla\lambda \cdot \nabla q = 0.$$

Now we can solve the problem with a rotationally symmetric function $p(t, r)$:

$$\nabla\lambda(r)\nabla p = 2\pi\delta(0), \quad p(t, \sqrt{2t}) = 0,$$

so that p, q are L-conjugate, where $L = \nabla\lambda\nabla$. In particular, the normal velocity of the boundary is given by

$$\frac{\partial q}{\partial \ell} = \lambda \frac{\partial p}{\partial n},$$

thus it is independent of the choice of λ. That is, the same movie can be described as the Laplacian growth with $\lambda = 1$ and as elliptic growth with any other positive weight which is rotationally invariant.

Note that a point on the boundary $\Gamma(t) = \partial D(0, \sqrt{t})$ is parametrized as

$$z(t, q) = \sqrt{2t}e^{iq},$$

so that the associated conformal map is

$$z(t, w) = \sqrt{2t}w, \quad |w| = 1.$$

On the other side, if we choose an initial contour to be different from the level set of the function λ, the inverse problem may have a unique solution modulo an arbitrary function of λ. Obviously, the simplest way to verify if the “movie” in the previous example is the Laplacian growth, is to run the “movie” again but with a non-circular initial configuration. Then the ambiguity of the previous example should disappear. But this way of removing non-uniqueness will require two “movies”. Thus, it is probably correct to say that there a continuum of λ’s, which correspond to the same “movie”, and that there are two sources of non-uniqueness: (i) any smooth function of λ can replace the original function $\lambda(\mathbf{x})$ and (ii) any smooth function of p can be multiplied by the original $\lambda(\mathbf{x})$ without a change of the “movie”. This is an interesting feature that merits further investigation. Here, we have simply pointed out (by constructing an example) the non-uniqueness of elliptic operators L providing the same evolution (i.e., a “movie”).

7. Schwarz function in elliptic growth

It is instructive to state the moving boundary problem in terms of the Schwarz function $S(z)$. Start with the equation for the velocity of the moving boundary $V = \partial_\ell q$. By Proposition 2.3 its left hand side equals $S_t/(2i\sqrt{S_z})$, while the right hand side can be written as $\partial_\ell(\zeta - p)/i = \partial_\ell \zeta/i$ since $\partial_\ell p = 0$. (Here, as before $\zeta = p + iq$.) Then,

$$\zeta_\ell/i = (z_\ell \partial_z \zeta + \bar{z}_\ell \partial_{\bar{z}} \zeta)/i = (\partial_z \zeta + S_z \partial_{\bar{z}} \zeta)/i\sqrt{S}_z.$$

Hence, in virtue of the Beltrami equation (4.3) and the identity

$$\frac{S_t}{2i\sqrt{S_z}} = -i\zeta_\ell,$$

we obtain the following evolution law for the Schwarz function.

Proposition 7.1. *Under the above assumptions,*

$$S_t(t,z) = 2(\zeta_z + \frac{1-\lambda}{1+\lambda}\overline{\zeta_z} S_z) = 2\partial_z \zeta(z, S(t,z)) \tag{7.1}$$

for all z in a neighborhood U of $\Gamma(t)$.

Notice that the above formula is surprisingly similar to the evolution law of $S(t,z)$ in the Laplacian growth process - cf. Theorem 2.5.

7.1. Dynamics of singularities of the Schwarz function

The above proposition has interesting consequences. We consider first the dynamics of the singularities of the Schwarz function and associated universal quadrature formulas that are preserved during the elliptic growth. We have seen in the section devoted to the Laplacian growth that the poles, or more generally, the Cauchy integral density of the Schwarz functions of the moving boundaries are unchanged with the exception of the residue of the pole at $z = 0$ which depends linearly on time t. Naturally, we expect that the relations in the elliptic growth process are more complicated. The present subsection collects some observations along these lines.

Theorem 7.2. *Let $f(z)$ be a real analytic function defined in a neighborhood of the closed domains $\overline{D(t)}$ which are moving according to the elliptic growth law with an associated operator L. Then,*

$$\frac{d}{dt}\int_{D(t)} f dArea = 2\pi \tilde{f}(0), \tag{7.2}$$

where $\tilde{f}$ solves the elliptic Dirichlet problem:

$$Lu = 0 \quad \text{in } D(t), \qquad u|_{\Gamma(t)} = f.$$

Proof. Using the notations introduced in the earlier sections and previous computations, we infer

$$\frac{d}{dt}\int_{D(t)} f dArea = \int_{\Gamma(t)} fV d\ell = \int_{\Gamma(t)} f\lambda\partial_n p d\ell = 2\pi\tilde{f}(0).$$

□

If the boundaries $\Gamma(t)$ remain real analytic during the time of growth as is tacitly assumed throughout this note we can, as before, decompose the Schwarz function as follows:

$$S(t,z) = S_+(t,z) - S_-(t,z), \quad z \in \Gamma(t),$$

where

$$S_-(t,z) = -\int_K \frac{d\mu(t,\sigma)}{\sigma - z}, \quad z \notin K.$$

Here, K is a compact subset of $D(t)$ independent of t and μ is a complex valued measure smoothly depending on t. Then, assuming in addition that f is (complex) analytic in a neighborhood of $\overline{D(t)}$, we find

$$\frac{d}{dt}\int_{D(t)} f dArea = \frac{1}{2i}\frac{d}{dt}\int_{\Gamma(t)} f(z)S(z,t)dz =$$

$$\frac{1}{2i}\frac{d}{dt}\int_{\Gamma(t)} f(z)S_-(z,t)dz = \pi\frac{d}{dt}\int_K [\frac{1}{2\pi i}\int_{\Gamma(t)}\frac{f(z)dz}{z-\sigma}]d\mu(t,z) = \pi\int_K f(\sigma)\frac{d}{dt}\{d\mu(t,\sigma)\}.$$

This representation of the derivative of the average of an analytic function becomes interesting in several particular cases. In particular, as in the conservation law obtained by taking $f = 1$, cf. Corollary 2.7 and Corollary 3.2, it follows that

$$\int_K \frac{d}{dt}\{d\mu(t,\sigma)\} = 2.$$

Another application is given by the following.

Proposition 7.3. *Assume that for all times t the singularities of the Schwarz function contained in $D(t)$ are simple poles; i.e., we have in $D(t)$:*

$$S_-(t,z) = -\sum_{j=1}^{N}\frac{a_j(t)}{b_j(t) - z} + \text{ (analytic remainder)},$$

with $b_j(t) \in D(t)$ for all j and t. Then,

$$\sum_{j=1}^{N}[a_j'(t)f(b_j(t)) + a_j(t)b_j'(t)f'(b_j(t))] = 2\tilde{f}(0) \tag{7.3}$$

for every analytic function f.

Proof. According to the computations in the proof of Theorem 7.2, we have

$$\mu(t,\sigma) = \int \frac{d[\sum_{j=1}^{N} a_j(t)\delta_{b_j(t)}](\sigma)}{z-\sigma},$$

hence

$$\frac{d}{dt}\int_{D(t)} f dArea = \pi \sum_{j=1}^{N} [a_j'(t) f(b_j(t)) + a_j(t) b_j'(t) f'(b_j(t))]$$

and the statement follows from (7.2). □

Now, by choosing $2N$ linearly independent analytic functions f_k we can, based on (7.3), form a linear system of equations that at least in principle determines the dynamics of the poles b_j and residues a_j.

8. Herglotz theorem and generating closed-form solutions

The Herglotz theorem [40] establishes a fascinating one-to-one correspondence between the singularities of the Schwarz function of the contour and the singularities of conformal maps from a vicinity of the unit circle to a vicinity of the contour under consideration. Basically, it states that if a is a singular point of a conformal map $f(w)$ from the unit disk $\mathbb{D}$ to the domain D such that $\partial D = \{f(e^{iq}); q \in [0, 2\pi]\}$, then the Schwarz function of the curve ∂D has the singularity of the same kind at the point

$$b = f(1/\bar{a}). \tag{8.1}$$

Moreover, if an isolated singularity (i.e., a pole, an algebraic singularity, or a logarithmic singularity) at a appears in the function $f(w)$ with a coefficient A, then the corresponding singularity b is present in the Schwarz function with the coefficient$B$ determined from

$$\bar{B} = A\,(-\bar{a}^2 f'(1/\bar{a}))^m, \tag{8.2}$$

where m is the multiplicity of a pole if a is a pole; is a rational number if a is a an algebraic branch point; or, if a is a logarithmic singularity, m is equal to zero. Actually, the last two equations, which we call Herglotz' theorem, follow easily from the representation of the Schwarz function $S(z)$ in terms of the conformal map f [33],

$$S(z) = \bar{f} \circ (1/f^{-1})(z).$$

There is some evidence that Herglotz' theorem should be helpful in solving the elliptic growth problem in terms of $z = f(t, w)$ and generating exact solutions in the closed form. Here, we present a naive sketch of how we might expect generating of exact solutions for the elliptic growth should work.

First, find $\zeta = p+iq$ as a function of $z = x+iy$ and $\bar{z} = x-iy$, either by solving the Beltrami equations (4.3) or (4.4) for a given λ and a given initial domain $D(0)$, or by solving the Dirichlet problem in $D(0)$, thus finding the Green function p and consequently calculating the conjugate function q from the generalized Cauchy-Riemann equations (4.1), (4.2). Then, the equation $p(z, \bar{z}) = 0$ implicitly defines the Schwarz function of the moving boundary.

Second, either substitute $\zeta(z, S(z))$ into the equation (7.1) for the dynamics of the Schwarz function $S(z)$ (scenario A), or, "if one gets lucky", try to invert the solution $\zeta(z, \bar{z})$, thus obtaining the function $z(\zeta, \bar{\zeta})$ (scenario B).

Scenario A: As the third step, identify singularities of $S(z)$ (and their dynamics) that are already built in (7.1) through the function $\zeta(z, S(z))$ found at the previous step and, also, identify the singularities of $S(z)$ which are *not* the singularities of the RHS of (7.1). The latter are time-independent as one can see from (7.1) and represent constants of motion associated with the dynamics of the growth.

Scenario A. Fourth step. Using the formulae (8.1),(8.2) provided by the Herglotz theorem recover an *explicit* form for the moving boundary, $z = f(t, e^{iq})$, with the time dynamics of all parameters of f given implicitly by (8.1), (8.2).

Scenario B. Third step. Restrict $z(\zeta, \bar{\zeta})$ to the imaginary axis of ζ, that is the axis $p = 0$, thus defining the function $f(e^{iq}) = \zeta(iq, -iq)$, which, as was shown above, satisfies the equation (1.2) for area preserving diffeomorphisms.

Scenario B. Fourth step. Substitute $f(e^{iq})$ into (1.2) as an initial condition with time-dependent parameters and solve it. (The technique of integration is the same as that for the Laplacian growth [41], but the singularities of f are now lie both inside and outside the unit circle.) Alternatively, find the singularities of $S(z)$ that correspond to the singularities of $f(z)$ through (8.1),(8.2) and calculate their time dynamics using (7.1). In either case the Herglotz theorem plays the central role in finding exact solutions for elliptic growth.

While a more complete theory for solving (1.2) for the elliptic growth exactly will be published elsewhere, we will present below two examples of exact solutions in the case when the singularities of $f(w)$ are poles, both inside and outside the unit circle.

8.1. Example: two simple moving poles

Let us take

$$z(e^{iq}) = re^{iq} + \frac{A_1}{e^{iq} - a_1} + \frac{A_2}{e^{iq} - a_2} \tag{8.3}$$

as an initial condition for the equation (1.2), assuming $|a_1| < 1$ and $|a_2| > 1$. One can verify by a direct substitution that (8.3) is a solution of (1.2) with time dependent poles a_1, a_2, residues A_1, A_2 and the conformal radius r. The time dependence of these parameters is given by the equations

$$b_{1,2} = z(1/\bar{a}_{1,2}), \tag{8.4}$$

$$B_{1,2} = -\bar{A}_{1,2}\,\bar{a}_{1,2}^2\,z'(1/\bar{a}_{1,2}), \tag{8.5}$$

$$2t + C = r^2 - 2r\,\Re\frac{A_2}{a_2^2} - \frac{|A_1|^2}{(1-|a_1|^2)^2} + \frac{|A_2|^2}{(1-|a_2|^2)^2}, \tag{8.6}$$

where $b_{1,2}$, $B_{1,2}$, and C are constants of integration. Actually, the RHS of the last equation is the area of the domain $D(t)$ enclosed by the contour $z(t, e^{iq})$. When $A_2 = 0$ this is a standard Laplacian growth with a simple pole at $a_1(t)$. In this case, the contour is known to develop a finite time singularity by forming a cusp. When $A_1 = 0$, the process becomes a so-called inverse Laplacian growth, when a more viscous fluid displaces a less viscous one oppositely to a standard situation in which it is the other way around. The inverse Laplacian growth is stable and the shape rounds off during the evolution forming as a rule a circle as a long time limit. The case when both A_1 and A_2 are not equal to zero is a general case with a nontrivial dynamics caused by an interlay between the stabilization of a contour, due to the term in (8.3) with the pole a_2 that lies outside the unit circle, and destabilization, due to the term in (8.3) with the pole a_1 inside the unit circle.

It is interesting to note that the constants of integration in the LHS of (8.4)-(8.6) describe singularities of the Schwarz function of the moving contour, namely $b_{1,2}$ and $B_{1,2}$ are the simple poles and residues of $S(z)$ respectively. Note that $b_1 \notin D(t)$ while $b_2 \in D(t)$. Finally, $2t + C$ is the residue of the Schwarz function at the simple pole at the origin, which represents the area of $D(t)$. It only remains now to find the explicit expression of the function $\lambda(\mathbf{x})$ for this process described by (8.3).

8.2. Example: two multiple stationary poles

For the second example let us take, as the initial condition for the (2.1), a function

$$z(e^{iq}) = re^{iq} + ae^{i(1-n)q} + be^{i(1+n)q}, \tag{8.7}$$

that describes a contour with n-fold symmetry which represents a circle of radius r modulated by a monochromatic wave in such a way that exactly n waves of an amplitude $\sqrt{2(|a|^2 + |b|^2)}$ fit the circumference (at least when $|a|$ and $|b|$ are both small with respect to r).

One can substitute (8.7) into (2.1) and verify that (8.7) is a solution of the area preserving diffeomorphism (2.1) if the time dependent parameters a, b, and r obey the following algebraic equations:

$$A = a^{1/(1-n)}\,b^{1/(1+n)}, \tag{8.8}$$

$$\frac{B}{r} = (1+n)a^{1/(n-1)} + (1-n)Aa^{1/(1-n)}, \tag{8.9}$$

$$2t + C = r^2 - (n-1)\,|a|^2 + (n+1)\,|b|^2, \tag{8.10}$$

where A, B, and C are constants of integration.

The comments that can be made here about the dynamics of the contour described by (8.7) are very similar to those made above for the first example. When $b = 0$, the moving curve blows up in a finite time by forming cusps and thus ceases to exist after that. When $a = 0$ the process is the inverse (stable) Laplacian growth, it smoothes the curve to a circle in a long term asymptotics. When both $a \neq 0$ and $b \neq 0$, this is an intermediate case which can, for instance, stabilize the dynamics and prevent the finite time blow up with a proper choice of initial parameters.

Just as in the previous example, the constants of integration describe singularities of the Schwarz function, namely A and B are the $(1-n)^{th}$ and $(1+n)^{th}$ coefficients of the formal Laurent expansion of S(z), while $2t + C$ is the area of $D(t)$ enclosed by $z(t, e^{iq})$. Unfortunately, again, it is not clear at the moment what elliptic parameter λ is associated with this process. This question requires further investigation.

9. Conclusions

In this work we have shown that the elliptic growth processes, which present a natural generalization of the Laplacian growth, possess some remarkable mathematical properties strongly resembling those pertinent to the Laplacian growth. Specifically, there is an infinite set of conservation laws; these conservation laws can be interpreted in terms of potential theory; there exists a parametrization of the moving interface by the stream function of an associated fluid velocity vector field and, last but not least, there are several interesting accompanying features related to the singularities of the Schwarz functions of the moving boundaries. We expect the latter to be especially helpful in generating and illuminating particular closed form solutions of elliptic growth problems. We think that the next step in this direction should be a search for a class of multipliers λ, which will allow explicit solutions of the Beltrami equations and, therefore, will offer new closed form solutions of the elliptic growth phenomenon.

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