REMARKS ON THE BOHR PHENOMENON

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Abstract. Bohr's theorem ([10]) states that analytic functions bounded by 1 in the unit disk have power series $\sum a_n z^n$ such that $\sum |a_n||z|^n < 1$ in the disk of radius 1/3 (the so-called Bohr radius.) On the other hand, it is known that there is no such Bohr phenomenon in Hardy spaces with the usual norm, although it is possible to build equivalent norms for which a Bohr phenomenon does occur! In this paper, we consider Hardy space functions that vanish at the origin and obtain an exact positive Bohr radius. Also, following [4, 11], we discuss the growth and Bohr phenomena for series of the type $\sum |a_n|^p r^n$, $0 < p < 2$, that come from functions $f(z) = \sum a_n z^n$ in the Hardy spaces. We will then consider Bohr phenomena in more general normed spaces of analytic functions and show how renorming a space affects the Bohr radius. Finally, we extend our results to several variables and obtain as a consequence some general Schwarz-Pick type estimates for bounded analytic functions.

1. Introduction

Let $D = \{z : z \in \mathbb{C}, |z| < 1\}$ be the open unit disk in the complex plane and for $0 < q \leq \infty$, let $H^q := H^q(D)$ denote as usual the Hardy spaces in the disk (cf. [12, 13, 14, 16].) Bohr's theorem ([10]) says that for any $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in the unit ball of H^∞, $\sum |a_n| r^n \leq 1$ for any $r \leq \frac{1}{3}$, and this value $\frac{1}{3}$ is sharp. It is interesting to note that for any $0 < q < \infty$, there is no $0 < r < 1$ such that

$$\|f\|_{H^q} \leq 1 \Rightarrow \sum_{n=0}^{\infty} |a_n| r^n \leq 1.$$

One way of seeing this is to consider, following [7], the extremal functions in H^q maximizing the derivative at the origin when the value at the origin is prescribed: these are functions

$$f_c(z) = (c^{\frac{q}{2}} + \sqrt{1 - c^q}\, z)^{\frac{2}{q}}$$

of H^q norm 1 ($q \geq 1$) with $a_0 = f_c(0) = c$ and $a_1 = f_c'(0) = \frac{2}{q} c^{1-\frac{q}{2}} \sqrt{1 - c^q}$, for any $2^{-\frac{1}{q}} \leq c < 1$. Then for any fixed $0 < r < 1$, one can choose c close enough to 1 so that

$$|a_0| + |a_1| r > 1.$$

Since the H^q norm for $0 < q < 1$ is weaker than the H^1 norm, no such positive r exists for $0 < q < 1$ either. However, if one begins taking powers of the

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coefficients of the power series of f, one can ask whether a version of Bohr's theorem would hold in that case. For example, given p and q, does there exist $r > 0$ such that

$$\|f\|_{H^q} \leq 1 \Rightarrow \sum_{n=0}^{\infty} |a_n|^p r^n \leq 1?$$

For $1 \leq q \leq 2$ and $p = \frac{q}{q-1}$, the above is certainly true with $r = 1$, by the Hausdorff-Young inequality. For $q > 2$ and p again the conjugate exponent, there is no such r, again by [7, Corollary 6.3]. If $q = \infty$ and $0 < p < 1$, there is no such positive r either (cf also [4]): let $0 < \varepsilon < \frac{1}{2}$ and let $f_\varepsilon(z) = a + bz$ where $a = 1 - 2\varepsilon$ and $b = \varepsilon$. Then

$$\|f_\varepsilon\|_{H^\infty} = a + b < 1,$$

but $a^p + b^p r$ can be made bigger than 1, because $\frac{1-a^p}{b^p} \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. We can also consider the function

$$f(z) = \frac{c+z}{1+cz}$$

for some $0 < c < 1$, then

$$\sum_{n=0}^{\infty} |a_n|^p r^n = c^p + (1-c^2)^p \frac{r}{1-c^p r}.$$

This last sum is less than 1 when

$$r < \frac{1-c^p}{(1-c^2)^p}(1+c^p)$$

which goes to 0 as c goes to 1 ($0 < p < 1$.) On the other hand, for each $p \geq 1$ and $\|f\|_\infty \leq 1$, since

$$\sum_{n=0}^{\infty} |a_n|^p r^n \leq \sum_{n=0}^{\infty} |a_n| r^n$$

it is certainly true that there exists $R_p \geq \frac{1}{3}$ such that

$$\|f\|_{H^\infty} \leq 1 \Rightarrow \sum_{n=0}^{\infty} |a_n|^p r^n \leq 1$$

for $r \leq R_p$. It is an open problem to find the best possible R_p (called the Bohr radius) in this case.

Djakov and Ramanujan studied the growth of series of the type $\sum |a_n|^p r^n$ for bounded analytic functions in [11]. They obtained bounds on the Bohr radii R_p and extended their results to a several variables context. For analytic functions of several variables, there is no domain for which the Bohr radius is known exactly. However, Boas and Khavinson had previously obtained bounds on the Bohr radii for any complete Reinhardt domain in [9]. They noticed that the radii depend on the dimension of the space being considered and tend to zero as the dimension increases. In particular, there is no Bohr radius for holomorphic functions of infinitely many variables, contrary to what Bohr himself had probably envisioned (cf. [10].) Aizenberg, Aytuna, Djakov, and Tarkhanov, in a series of papers (cf.

[1, 2, 3, 5]) have studied Bohr phenomena in $\mathbb{C}^n$ and also for various bases different from that of monomials in spaces of analytic and harmonic functions equipped with a supremum-type norm. For further ramifications and extensions, see Boas' very nice survey [8]. Applications of Bohr phenomena to operator theory and other recent related references in that direction can be found in [20]. When the final version of this paper was being prepared, we became aware of a recent paper of Aizenberg, Grossman, and Korobeinik (cf. [4]) who obtain some further asymptotics for Bohr radii in the spirit of Djakov and Ramanujan. Our results in Section 2, although obtained independently, are quite similar in spirit to their discussion in that paper, although the discussion in [4] is restricted to bounded analytic functions, while our focus is on more general Hardy spaces.

We begin by studying series of the type $\sum |a_n|^p r^n$ (particularly when $0 < p < 2$) for analytic functions f in the unit ball of H^s, $s > 0$. We then consider Bohr-type phenomena in rather general normed spaces of analytic functions. As one of the more surprising consequences, we obtain essentially "free of charge" most general Schwarz-Pick estimates for derivatives of bounded analytic functions (cf. [18, 19].) Finally, we extend our results to functions of several variables.

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2. On absolute radial convergence in Hardy spaces

Fix $0 < p < \infty$ and $0 < r < 1$. For f analytic in the unit disc with Taylor expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$, we write

$$\|f\|_{p,r} = \left(\sum_{n=0}^{\infty} |a_n|^p r^n \right)^{1/p} .$$

Given such a function f we shall always write a_n for the n-th Taylor coefficient of f.

Our first result concerns the growth of $\|f\|_{p,r}$ as $r \to 1^-$ for Hardy space functions f.

Theorem 2.1. *Fix $0 < p < 2$ and suppose $f \in H^s$, for some $0 < s \leq \infty$. Then the following hold:*
(i) If $0 < s \leq 2$, then

$$\|f\|_{p,r} = o((1-r)^{-t}), \qquad \text{as } r \to 1^-,$$

where $t = 1/p + 1/s - 1$.
(ii) If $2 \leq s \leq \infty$, then

$$\|f\|_{p,r} = o((1-r)^{-t}), \qquad \text{as } r \to 1^-,$$

where $t = 1/p - 1/2$.

Remark. For $p = 1$ and $s = 2$ this result and the proof are due to Hardy (cf. [17]). Note that the case $p \geq 2$ is still open, besides the trivial cases covered by the Hausdorff-Young theorem as noted in the Introduction. Thus, extending Theorem 2.1 to treat the case $p > 2$ and $0 < s < p/(p-1)$ remains an open problem.

Proof. In the proof we write a_n for the n-th Taylor coefficient of f. First assume $1 < s \leq 2$. By the Hausdorff-Young inequality ([12], Theorem 6.1)

$$\sum_{n=0}^{\infty} |a_n|^{s'} \leq \|f\|_{H^s}^{s'}, \tag{1}$$

where s' is the conjugate exponent of s, i.e. $s' = s/(s-1)$. For any integer m we have

$$\sum_{n=0}^{\infty} |a_n|^p r^n \leq \sum_{n=0}^{m} |a_n|^p + \sum_{n=m+1}^{\infty} |a_n|^p r^n. \tag{2}$$

By Hölder's inequality and (1) we have

$$\begin{aligned}\sum_{n=m+1}^{\infty} |a_n|^p r^n &\leq \left(\sum_{n=m+1}^{\infty} |a_n|^{s'}\right)^{p/s'} \left(\sum_{n=m+1}^{\infty} r^{nq}\right)^{1/q} \\ &\leq \left(\sum_{n=m+1}^{\infty} |a_n|^{s'}\right)^{p/s'} (1-r^q)^{-1/q} \leq \|f - f_m\|_{H^s}^p (1-r^q)^{-1/q}\end{aligned}$$

where $f_m(z) = \sum_{n=0}^{m} a_n z^n$ and $q = s'/(s'-p)$ is the conjugate exponent to s'/p (note that $s'/p > 1$). Thus by (2)

$$\limsup_{r \to 1^-} (1-r)^{1/q} \|f\|_{p,r}^p \leq q^{-1/q} \|f - f_m\|_{H^s}^p, \tag{3}$$

which tends to 0 as $m \to \infty$. Since $1/qp = 1/p - 1/s' = 1/p + 1/s - 1$, this proves (i) for $1 < s \leq 2$.

For $s \geq 2$ we have $\|f\|_{H^2} \leq \|f\|_{H^s}$, and hence a calculation similar to the preceding one with $q = \frac{2}{2-p}$ shows that

$$\lim_{r \to 1^-} (1-r)^{1/q} \|f\|_{p,r}^p = 0,$$

which proves (ii) with $t = 1/qp$.

The remaining case, $0 < s \leq 1$, follows from the slightly more general proposition below, since in that case, $|a_n| = o(n^{1/s-1})$ (cf. [12], Theorem 6.4). ☐

Proposition 2.2. *Fix $\alpha > -1$. Suppose $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic in the unit disc and suppose $|a_n|^p = o(n^\alpha)$. Then*

$$\sum_{n \geq 0} |a_n|^p r^n = o((1-r)^{-1-\alpha}), \qquad \textit{as } r \to 1^-.$$

Proof. First we recall that

$$(1-z)^{-\beta} = \sum_{n=0}^{\infty} \frac{\Gamma(n+\beta)}{\Gamma(\beta) n!} z^n. \tag{4}$$

We claim that there is a constant C_β such that

$$\frac{\Gamma(n+\beta)}{n!} \geq C_\beta n^{\beta-1} \tag{5}$$

for $\beta > 0$ and $n = 1, 2, 3, \ldots$ For $n = 1$ this is obvious, so we may assume $n \geq 2$. By the convexity of $\log \Gamma$ on $(0, \infty)$ we have

$$\frac{\log \Gamma(n) - \log \Gamma(n-1)}{n - (n-1)} \leq \frac{\log \Gamma(n+\beta) - \log \Gamma(n)}{n + \beta - n}$$

which proves (5) with, say, $C_\beta = 2^{-\beta}$.

Now choose $\varepsilon > 0$ and then m so large that $|a_n|^p < \varepsilon n^\alpha$ for all $n \geq m$. Then

$$\sum_{n\geq 0} |a_n|^p r^n \leq \sum_{n=0}^{m-1} |a_n|^p + \varepsilon \sum_{n\geq m} n^\alpha r^n.$$

With $\beta = \alpha + 1$ in (4) and $C = C_\beta^{-1}$ in (5) we obtain

$$\sum_{n\geq m} n^\alpha r^n \leq C(1-r)^{-\alpha-1}.$$

Since $\alpha > -1$ we get

$$\lim_{r\to 1^-} (1-r)^{1+\alpha} \sum_{n\geq 0} |a_n|^p r^n \leq C\varepsilon$$

and the result follows. □

In connection with Theorem 2.1 it is natural to consider the uniform growth of the norms $\|f\|_{p,r}$ as the functions f vary in H^s. Formally, we consider

$$C(r, p, s) = \sup\{\|f\|_{p,r} : f \in H^s, \|f\|_{H^s} \leq 1\},$$

where, as before, $0 < p < 2$, $0 < s \leq \infty$ and $0 < r < 1$. A consequence of the following result is that the exponent in Theorem 2.1 cannot be improved in the case $s > 1/2$.

Theorem 2.3. *Let $0 < p < 2$. Then the following hold:*
(i)

$$\lim_{r\to 1^-} (1-r)^{1/p-1/2} C(r, p, 2) = (1-p/2)^{1/p-1/2}.$$

(ii) For $s \in (1, +\infty]$

$$\limsup_{r\to 1^-} (1-r)^t C(r, p, s) \leq (pt)^t,$$

where $t = 1/p + 1/s - 1$ for $1 < s \leq 2$ and $t = 1/p - 1/2$ for $s \geq 2$.
(iii) If $s \in (1/2, 2]$ and $t = 1/p + 1/s - 1$, then

$$\liminf_{r\to 1^-} (1-r)^t C(r, p, s) \geq \left(\frac{2s-1}{2^{s+1}s}\right)^{1/s} e^{-1/p}(1-e^{-2})^{1/p} > 0.$$

(iv) If $s \geq 2$ and $t = 1/p - 1/2$, then

$$\liminf_{r \to 1^-} (1-r)^t C(r,p,s) \geq (1-e^{-1})^{\frac{1}{p}} > 0.$$

Remark. The case $s = \infty$ in part (iv) of Theorem 2.3 gives a simplified and slightly sharper version of Theorems 2 and 3 in [11]; however, the main idea of using Kahane polynomials in the construction comes from the proof there.

We need the following lemma.

Lemma 2.4. *For $n = 0, 1, 2, \ldots,$ let K_n be the Fejér kernel,*

$$K_n(e^{it}) = \sum_{|k| \leq n} \left(1 - \frac{|k|}{n+1}\right) e^{it} = \frac{1}{n+1} \left(\frac{\sin \frac{(n+1)t}{2}}{\sin \frac{t}{2}} \right)^2.$$

For each $s \in (1/2, \infty)$ we have

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |K_n(e^{it})|^s dt \leq \frac{2s}{2s-1} (n+1)^{s-1}$$

for all n.

This follows immediately from the inequality

$$K_n(e^{it}) \leq \min\left(n+1, \frac{\pi^2}{(n+1)t^2}\right), \quad t \in (0, \pi).$$

We remark that estimates of $\sin \frac{(n+1)t}{2}$ from below, using sums of step functions, show that the corresponding statement is false for $s \leq 1/2$.

We now turn to the proof of Theorem 2.3

Proof. Part (ii) can be read out from the proof of Theorem 2.1. We omit the details. To prove (i) observe that by (ii) (or (iii)), it is sufficient to prove that

$$\liminf_{r \to 1^-} (1-r)^{1/p - 1/2} C(r,p,2) \geq (1 - p/2)^{1/p - 1/2}.$$

Fix an integer $n > 0$, $r \in (0,1)$ and let $a > 0$. Consider the polynomial $P_{n,r,a}(z) = \sum_{k=0}^n r^{ak} z^k = \frac{1 - r^{a(n+1)} z^{n+1}}{1 - r^a z}$ ($|z| < 1$), and let $Q_{n,r,a} = P_{n,r,a} / \|P_{n,r,a}\|_{H^2}$. A straighforward calculation yields

$$\|Q_{n,r,a}\|_{p,r} = \frac{\left(\sum_{k=0}^n r^{(ap+1)k} \right)^{1/p}}{\left(\sum_{j=0}^n r^{2aj} \right)^{1/2}} = \frac{\left(\frac{1 - r^{(ap+1)(n+1)}}{1 - r^{ap+1}} \right)^{1/p}}{\left(\frac{1 - r^{2a(n+1)}}{1 - r^{2a}} \right)^{1/2}}.$$

Hence

$$(1-r)^{1/p - 1/2} C(r,p,2) \geq \frac{(1-r)^{1/p}}{(1 - r^{ap+1})^{1/p}} \frac{(1 - r^{2a})^{1/2}}{(1-r)^{1/2}} \frac{\left(1 - r^{(ap+1)(n+1)}\right)^{1/p}}{\left(1 - r^{2a(n+1)}\right)^{1/2}}.$$

Now fix $\alpha > 0$ and let $(r, n) \to (1, \infty)$ so that $(1-r)(n+1) \to \alpha$. Then

$$\liminf_{r\to 1^-}(1-r)^{1/p-1/2}C(r,p,2) \geq \frac{(2a)^{1/2}}{(ap+1)^{1/p}}\frac{\left(1-e^{-\alpha(ap+1)}\right)^{1/p}}{\left(1-r^{-2a\alpha}\right)^{1/2}}.$$

Letting $\alpha \to \infty$ we get

$$\liminf_{r\to 1^-}(1-r)^{1/p-1/2}C(r,p,2) \geq \frac{(2a)^{1/2}}{(ap+1)^{1/p}}.$$

For the choice $a = 1/(2-p)$ we have $(2a)^{1/2}/(ap+1)^{1/p} = (1-2/p)^{1/p}$, which proves (i).

To prove (iii) put $P_n(z) = z^{2n+1}V_n(z)$, where V_n is the de la Vallée-Poussin kernel, defined by $V_n = 2K_{2n+1} - K_n$, and K_n is the Fejér kernel (cf. Lemma 2.4). By Minkowski's inequality and Lemma 2.4, we have

$$\|P_n\|_{H^s}^s \leq 2\frac{2s}{2s-1}(2n+2)^{s-1} + \frac{2s}{2s-1}(n+1)^{s-1} = C_s^s(n+1)^{s-1},$$

where $C_s = (2^{s+1}s/(2s-1))^{1/s}$. Note that the k-th Fourier coefficient of V_n is 1 for $|k| \leq n$, so that

$$\begin{aligned}\frac{\|P_n\|_{p,r}}{\|P_n\|_{H^s}} &\geq \frac{1}{\|P_n\|_{H^s}}\left(\sum_{|k|\leq n} r^{k+2n+1} + \text{a positive quantity}\right)^{1/p}\\ &> \frac{r^{(n+1)/p}}{\|P_n\|_{H^s}}\left(\frac{1-r^{2n+1}}{1-r}\right)^{1/p} \geq \frac{r^{(n+1)/p}}{C_s(n+1)^{1-1/s}}\left(\frac{1-r^{2n+1}}{1-r}\right)^{1/p}.\end{aligned}$$

From the above we get

$$(1-r)^{1/p+1/s-1}C(r,p,s) \geq \frac{r^{(n+1)/p}}{C_s((n+1)(1-r))^{1-1/s}}(1-r^{2n+1})^{1/p}.$$

Putting $r = 1 - 1/(n+1)$ and letting $r \to 1^-$ yields

$$\liminf_{r\to 1^-}(1-r)^{1/p+1/s-1}C(r,p,s) \geq C_s^{-1}e^{-1/p}(1-e^{-2})^{1/p} > 0.$$

It remains to prove (iv). Following [11], let $\varepsilon > 0$ and pick an integer n_ε and polynomials $P_n(z) = \sum_{k=0}^n a_{kn}z^k$ with $|a_{kn}| = 1$ such that

$$\sup_{z\in D}|P_n(z)| \leq (1+\varepsilon)\sqrt{n+1}$$

for all $n > n_\varepsilon$. Existence of such polynomials were proved by J.-P. Kahane (cf. for example [15]). This implies

$$C(r,p,\infty) \geq \frac{\|P_n\|_{p,r}}{(1+\varepsilon)\sqrt{n+1}} = \frac{1}{(1+\varepsilon)\sqrt{n+1}}\left(\frac{1-r^{n+1}}{1-r}\right)^{1/p}$$

for every $r \in (0,1)$ and n large enough. Put $r = 1 - 1/(n+1)$, let $r \to 1^-$ and then $\varepsilon \to 0^+$ to obtain

$$\liminf_{r\to 1^-}(1-r)^{1/p-1/2}C(r,p,\infty) \geq (1-1/e)^{1/p}.$$

□

Remark. It seems to be a hard task to find non-trivial estimates of $C(r,p,s)$ for $s \leq 1/2$. In fact, the "obvious" exponent might not even be the correct one (see the remark following Lemma 2.4).

It is interesting to note that if we consider functions whose first few Taylor coefficients vanish, we can sometimes obtain more accurate information.

Theorem 2.5. *Let $0 < p < 2$ and let $m \geq 1$ be an integer.*
(a) If $1 < s \leq 2$ and $f(z) = \sum_{n=m}^{\infty} a_n z^n \in H^s$ then

$$\|f\|_{p,r}^p \leq \frac{r^m}{(1-r^q)^{\frac{1}{q}}} \|f\|_{H^s}^p,$$

where $q = \frac{s}{s+p-ps}$. In particular, if $r^{qm} + r^q \leq 1$, then

$$\|f\|_{p,r}^p \leq \|f\|_{H^s}^p.$$

(b) If $f(z) = \sum_{n=m}^{\infty} a_n z^n \in H^s$ then

$$\|f\|_{p,r}^p \leq \frac{r^m}{(1-r^q)^{\frac{1}{q}}} \|f\|_{H^s}^p,$$

where $q = \frac{2}{2-p}$. In particular, if $r^{qm} + r^q \leq 1$, then

$$\|f\|_{p,r} \leq \|f\|_{H^s}.$$

(c) If $s = 2$ and $q = \frac{2}{2-p}$, then the inequality in (b) is sharp. In particular, if $f(z) = \sum_{n=m}^{\infty} a_n z^n \in H^2$ then

$$\|f\|_{p,r} \leq \|f\|_{H^2} \text{ for every } f \text{ in } H^2 \text{ if and only if } r^{qm} + r^q \leq 1.$$

(d) If $s \geq 2$ and $m = 1$, then

$$\|f\|_{p,r} \leq \|f\|_{H^s} \text{ for every } f \text{ in } H^s \text{ if and only if } r \leq 2^{-\frac{1}{q}}.$$

Remark. For the case $s = \infty$, similar results and related extremal problems can be found in [4, 8, 11].

Proof. The proof of part (a) is very similar to that of the case $1 < s \leq 2$ of part (i) in Theorem 2.1. Also note that

$$\sum_{n=m}^{\infty} |a_n|^p r^n \leq \|f\|_{H^s}^p$$

when $\frac{r^m}{(1-r^q)^{\frac{1}{q}}} \leq 1$, that is, when $r^{qm} + r^q \leq 1$.
(b) If $f(z) = \sum_{n=m}^{\infty} a_n z^n \in H^s$ then by Hölder's inequality,

$$\begin{aligned}\sum_{n=m}^{\infty} |a_n|^p r^n &\leq (\sum_{n=m}^{\infty} |a_n|^2)^{\frac{p}{2}} (\sum_{n=m}^{\infty} r^{qn})^{\frac{1}{q}} \\ &= \|f\|_{H^2}^p \frac{r^m}{(1-r^q)^{\frac{1}{q}}} \leq \|f\|_{H^s}^p \frac{r^m}{(1-r^q)^{\frac{1}{q}}}\end{aligned}$$

for $s \geq 2$.

(c) Notice that if $s = 2$ and c is any constant such that $|a_n|^2 = cr^{nq}$, we have equality in the Hölder application above. Let R be the unique solution in $(0, 1)$ to

$$R^{qm} + R^q = 1.$$

Let r be such that $R < r < 1$ and define

$$c = \frac{r^{mq}}{1 - r^q} > \frac{R^{mq}}{1 - R^q} = 1.$$

Define $a_n = \frac{1}{\sqrt{c}}\sqrt{r^{nq}}$. Then

$$\sum_{n=m}^{\infty} |a_n|^2 = \frac{1}{c} \sum_{n=m}^{\infty} r^{nq} = \frac{1}{c} \frac{r^{mq}}{1 - r^q} = 1.$$

Therefore

$$f(z) = \sum_{n=m}^{\infty} a_n z^n \in H^2$$

and $\|f\|_{H^2} = 1$. However,

$$\begin{aligned} \sum_{n=m}^{\infty} |a_n|^p r^n &= \|f\|_{H^2}^p \frac{r^m}{(1 - r^q)^{\frac{1}{q}}} \\ &= \frac{r^m}{(1 - r^q)^{\frac{1}{q}}} \\ &> 1. \end{aligned}$$

Therefore Bohr's inequality does not hold and the Bohr radius for this problem is exactly the solution to

$$R^{qm} + R^q = 1.$$

(d) If $s \geq 2$ and $m = 1$, we consider the function

$$\phi_a(z) = z \frac{a - z}{1 - az}$$

for $0 < a < 1$. Then $\|\phi_a(z)\|_{H^s} = 1$ and if we write $\phi_a(z) = \sum_{n=1}^{\infty} a_n z^n$, then

$$\sum_{n=1}^{\infty} |a_n|^p r^n = a^p r + r^2 \frac{(1 - a^2)^p}{1 - a^p r}.$$

Therefore

$$\sum_{n=1}^{\infty} |a_n|^p r^n > 1$$

if and only if

$$(1 - a^2)^p r^2 - a^{2p} r^2 + 2a^p r > 1.$$

If we choose $a = \frac{1}{\sqrt{2}}$, then the above inequality holds exactly when $r > 2^{-\frac{1}{q}}$. $\square$

Remark. The theorem still leaves unresolved the following question: if $s > 2$ and $m > 1$, is it true that

$$\|f\|_{p,r} \leq \|f\|_{H^s} \text{ if and only if } r^{qm} + r^q \leq 1?$$

Also, for the case $0 < s \leq 1$, it is possible, for example using the techniques of Proposition 2.2, to prove norm estimates similar to those in (a) and (b) of Theorem 2.5. However, we have been unable to control the constants involved (in the proof of Proposition 2.2 and Theorem 6.4 in [12].) Accordingly, we failed to obtain sharp estimates for these values of s.

We now turn to a discussion of Bohr's phenomenon in more general normed spaces of analytic functions.

3. Renorming

Let X be a Banach space of analytic functions in the disk. In the following discussion, we will assume that polynomials are dense (or weakly dense) in X, that the set of bounded point evaluations is the disk, and that if $f \in X$ with $\|f\|_X \leq 1$ is not constant then $|f(0)| < 1$. We say that a Bohr phenomenon holds for X if there exists $r > 0$ such that whenever $f(z) = \sum_{n=0}^{\infty} a_n z^n \in X$,

$$\|f\|_X \leq 1 \implies \sum_{n=0}^{\infty} |a_n| r^n \leq 1.$$

The largest such r is called the Bohr radius.

We have already noticed that there is no Bohr phenomenon for H^2. However, we can modify the definition of the norm of a function in $X = H^2$ slightly and force a Bohr phenomenon to occur: for $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H^2$, define

$$\|f\|_X = |a_0| + \sqrt{\sum_{n=1}^{\infty} |a_n|^2}.$$

This norm is equivalent to the usual H^2 norm. Notice that for $\|f\|_X \leq 1$, $|a_n| \leq 1 - |a_0|$. Hence, for such a function in the unit ball of X,

$$\sum_{n=0}^{\infty} |a_n| r^n \leq |a_0| + \sum_{n=1}^{\infty} (1 - |a_0|) r^n \leq 1$$

whenever $r \leq \frac{1}{2}$. Therefore X has a Bohr radius of at least $\frac{1}{2}$! This raises the question of characterizing the norms that give rise to a Bohr phenomenon. It turns out that the following necessary and sufficient condition must hold.

Theorem 3.1. *Let X be as above. Then Bohr's phenomenon holds in X if and only if*

$$\sup\{(\frac{|f^{(n)}(0)|}{n!(1 - |f(0)|)})^{\frac{1}{n}} : \|f\|_X \leq 1, |f(0)| < 1, n \geq 1\} = C < \infty.$$

Proof. Suppose a Bohr phenomenon holds in X and let $R > 0$ be the Bohr radius. Then for any function

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \in X$$

with $\|f\|_X \leq 1$ and for any integer $n \geq 1$,

$$|a_0| + |a_n| R^n \leq 1.$$

Therefore

$$\left(\frac{|f^{(n)}(0)|}{n!(1 - |f(0)|)}\right)^{\frac{1}{n}} \leq \frac{1}{R} < \infty.$$

On the other hand, suppose that for each

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \in X$$

such that $\|f\|_X \leq 1$, $f \neq 1$, we have

$$\left(\frac{|a_n|}{1 - |a_0|}\right)^{\frac{1}{n}} \leq C$$

for $n \geq 1$. Then

$$\begin{aligned}
\sum_{n=0}^{\infty} |a_n| r^n &\leq |a_0| + (1 - |a_0|) \sum_{n=1}^{\infty} C^n r^n \\
&= |a_0| + (1 - |a_0|) \frac{Cr}{1 - Cr} \\
&\leq 1
\end{aligned}$$

whenever $r \leq \frac{1}{2C}$. Therefore a Bohr phenomenon holds and in fact the Bohr radius is at least $\frac{1}{2C}$. □

Example 1. We have already seen that if $X = H^q$, $q \geq 1$, for $2^{-\frac{1}{q}} \leq c < 1$, the family of functions

$$f_c(z) = (c^{\frac{q}{2}} + \sqrt{1 - c^q} z^n)^{\frac{2}{q}}$$

discussed in the introduction satisfy (for $n = 1$!)

$$\sup_{2^{-1/q} \leq c < 1} \frac{|f_c'(0)|}{1 - |f_c(0)|} = \sup_{2^{-1/q} \leq c < 1} \frac{\frac{2}{q} c^{1 - \frac{q}{2}} \sqrt{1 - c^q}}{1 - c} = \infty.$$

Therefore, as we already know, there is no Bohr phenomenon for H^q $(1 \leq q < \infty)$ with the usual norm.

Example 2. Let X be the space of analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in the disk such that the following norm is finite:

$$\|f\|_X = |a_0| + \sqrt{\sum_{n=1}^{\infty} \frac{|a_n|^2}{n+1}}.$$

This is of course a norm equivalent to the usual A^2 (the Bergman space) norm. Recall that in the usual Bergman space, there is no Bohr phenomenon (since there is none even in H^2.) Notice that if $\|f\|_X \leq 1$, then $|a_0| + \frac{|a_n|}{\sqrt{n+1}} \leq 1$, so

$$(\frac{|a_n|}{1-|a_0|})^{\frac{1}{n}} \leq (n+1)^{\frac{1}{2n}}.$$

The left hand side attains the value on the right for the function $f(z) = |a_0| + (1-|a_0|)\sqrt{n+1}z^n$. The quantity $(n+1)^{\frac{1}{2n}}$ is decreasing and therefore

$$\sup\{(\frac{|f^{(n)}(0)|}{n!(1-|f(0)|)})^{\frac{1}{n}} : \|f\| \leq 1, |f(0)| < 1, n \geq 1\} = \sqrt{2}.$$

Hence according to the proof of the Theorem 3.1, we know that X has a Bohr radius of at least $\frac{1}{2\sqrt{2}}$.

Example 3. Since we know there is a positive Bohr radius for H^∞ and not for H^q $(0 < q < \infty)$ with the standard norms, it might be interesting to see what happens in the case of an intermediate space such as the BMO space (cf. [13].) Because the Bohr phenomenon is sensitive to different norms, the question of which norm to use becomes important. For example, if we define the BMO norm on H^1 modulo the constants and use the norm that arises from considering BMO modulo the constants as a subspace of the dual space of H^1, then there is a positive Bohr radius. In other words, suppose $f(z) = \sum a_n z^n \in$ BMO, then $\|f - f(0)\|_{BMO} = \|f\|_{BMO}$. Therefore without loss of generality, f vanishes at the origin. Applying f (as a linear functional on H^1) to z^n gives rise to a_n, so if $\|f\|_{BMO} \leq 1$, as a consequence $|a_n| \leq 1$. In that case,

$$\sum_{n=1}^{\infty} |a_n| r^n \leq \sum_{n=1}^{\infty} r^n = \frac{r}{1-r} \leq 1$$

whenever $r \leq \frac{1}{2}$. However, this comes as no surprise, since if we consider H^q functions that vanish at the origin, there is always a Bohr radius of at least $\frac{1}{2}$ in exactly the same manner. What happens if we factor in constants, still thinking of BMO normed as a dual space to H^1?

Question. *For $f \in BMOA$, let $\|f\|_{BMO} := \|f\|_{(H^1)^*}$. Is it true that*

$$\sup\{(\frac{|f'(0)|}{(1-|f(0)|)}) : \|f\|_{BMO} \leq 1, |f(0)| < 1\} < \infty?$$

However, suppose we define the norm as follows: for

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \in \text{ BMOA},$$

let

$$\|f\| := |f(0)| + \|f - f(0)\|_{BMO(T)}$$

where T is the unit circle and $\|g\|_{BMO(T)}$ is the usual Garsia BMO norm (cf. [13].) Then for any function $g \in$ BMO, since $\|g\|_{L^1(T)} \leq \|g\|_{BMO(T)}$ (cf. [13, pp. 224-225]), if $\|f\| \leq 1$ and $n \geq 1$,

$$|a_n| \leq \|f - f(0)\|_{L^1} \leq \|f - f(0)\|_{BMO(T)} \leq 1 - |f(0)|.$$

Therefore

$$\sum_{n=0}^{\infty} |a_n| r^n \leq |f(0)| + (1 - |f(0)|)\frac{r}{1-r} \leq 1$$

whenever $r \leq \frac{1}{2}$. In fact, if X is any normed space of analytic functions such that the norm on X dominates the L^1 norm, this same argument shows that defining a new norm

$$\|f\|_{new} := |f(0)| + \|f - f(0)\|_{old}$$

will always force a positive Bohr radius of at least $\frac{1}{2}$.

It is worth noticing that Bohr's phenomenon holds for all points in the disk. Let's suppose that X is a space whose norm behaves "well" with respect to translations and dilations, namely, satisfies the following condition:

> **(*)** suppose f is a function in X such that $\|f\|_X \leq 1$. Then for any $z_0 \in \mathbf{D}$ and radius $r > 0$ such that the disk $D(z_0, r)$ centered at z_0 of radius r is contained in the unit disk,
>
> $$\|f(z_0 + rz)\|_X \leq 1.$$

Remark. Of course, the natural assumption that the norm in X is lower semi-continuous with respect to pointwise convergence immediately implies that if X satisfies **(*)**, it is a subspace of H^∞, simply by letting $r \to 0$ in **(*)**.

If such a space X satisfies the Bohr phenomenon with Bohr radius R, fix any z_0 in the disk and take the Taylor expansion of a function $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$ about z_0 in the disk of radius $1 - |z_0|$. We obtain

$$\sum_{n=0}^{\infty} |\frac{f^{(n)}(z_0)}{n!}||z - z_0|^n \leq 1$$

for $|z-z_0| \leq (1-|z_0|)R$. To see this, we simply apply a linear change of variables, mapping the unit disk to the disk centered at z_0 of radius $1 - |z_0|$. This allows us to put the above criterion in invariant form.

Theorem 3.2. *(Invariant criterion) Let X be a Banach space of analytic functions on the disk as in the previous theorem satisfying condition (*). Then Bohr's phenomenon holds in X if and only if for every $z_0 \in \mathbf{D}$,*

$$\sup\{(\frac{|f^{(n)}(z_0)|(1 - |z_0|)^n}{n!(1 - |f(z_0)|)})^{\frac{1}{n}} : \|f\|_X \leq 1, |f(z_0)| < 1, n \geq 1\} = C < \infty$$

where the constant C depends only on the Bohr radius and not on z_0.

Using Stirling's formula, we obtain the following.

Corollary 3.3. *Let X be a Banach space of analytic functions as in Theorem 3.2 with Bohr radius R_X. Then for any function f in the unit ball of X,*

$$\sup_{z\in D}\limsup_{n\to\infty}\frac{|f^{(n)}(z)|^{\frac{1}{n}}(1-|z|)}{n}\leq\frac{e}{R_X}.$$

Applying Theorem 3.2 to $X=H^\infty$, we obtain the following.

Corollary 3.4. *Let $f\in H^\infty$ be such that $\|f\|_\infty<1$. Then for each integer $n\geq 1$,*

$$\sup_{z\in\mathbf{D}}\frac{|f^{(n)}(z)|(1-|z|^2)^n}{1-|f(z)|^2}\leq 6^n n!<\infty.$$

This result is contained in a recent paper [18]. In [18], it is obtained via different methods using a chain of composition operators on Bloch spaces. However, sharper results were known earlier - see Remark (i) below.

Proof. Notice that

$$\frac{|f^{(n)}(z)|(1-|z|^2)^n}{(1-|f(z)|^2)}\leq\frac{|f^{(n)}(z)|(1-|z|)^n}{n!(1-|f(z)|)}2^n n!$$

By Theorem 3.2 and the proof of Theorem 3.1, and since the Bohr radius for H^∞ is $1/3$, this last expression is less than $6^n n!$. □

Remark. (i) Instead of using Theorem 3.2, by composing f with a Mobius transformation, we could directly extend F. Wiener's estimate for the coefficients of bounded functions (cf. [10]) into the invariant form to obtain

$$\sup_{z\in\mathbf{D}}\frac{|f^{(n)}(z)|(1-|z|)^n}{1-|f(z)|^2}\leq n!,$$

which is sharper but still not sharp for a fixed z in $\mathbf{D}-\{0\}$ (cf. [7] for the discussion at $z=0$.) This leads to a slightly better constant $2^n n!$. The sharp constant, $2^{n-1}n!$, was obtained by classical methods in Ruscheweyh (cf [21].) Further extensions and generalizations can be found in [6, 22]. However, the "abstract nonsense" approach allows one to obtain at no cost similar estimates in several variables (cf. Section 4) in situations where the estimates of the Taylor coefficients are more involved.

(ii) As was noted before, for all practical purposes, **(*)** forces the space X to be inside H^∞. Yet for the norms equivalent to the standard norms in Hardy or Bergman spaces that force Bohr's phenomenon to take place (such as some of the norms considered in this section), some version of **(*)** holds with constants that depend on the radius r (distance from the point z to the unit circle.) In addition, there are often known simple growth estimates for functions in that space that also depend on r. In that case, one can carry out the above scheme and obtain pointwise estimates similar to Corollary 3.4 for the derivatives. We shall omit the details.

4. Functions of several variables

The above scheme applies to a several variables context as well. We will use the standard multivariate notations as in [9]: we write an n-variable power series $\sum_\alpha c_\alpha z^\alpha$ where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ is an n-tuple of non-negative integers, $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$, $\alpha! = \alpha_1!\alpha_2!...\alpha_n!$, $z = (z_1, z_2, ..., z_n)$ is an n-tuple of complex numbers, and $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} ... z_n^{\alpha_n}$. We consider analytic functions defined on the unit polydisk $D^n = \{z : \max_{1\le j\le n} |z_j| < 1\}$. We will denote by D^α the derivative

$$\frac{\partial^{|\alpha|}}{(\partial z_1)^{\alpha_1} ... (\partial z_n)^{\alpha_n}}.$$

The following theorem is a several variable analogue of Theorem 2.5, b). (Extensions of most of the other results in Section 2 could be carried out in a manner similar to that of [11], and we omit them.)

Theorem 4.1. *Let* $0 < p < 2$ *and* $q = \frac{2}{2-p}$. *Let* $f(z) = \sum_\alpha c_\alpha z^\alpha \in H^2(D^n)$ *be such that*

$$\|f\|_{H^2(D^n)} := \sum_{|\alpha|=0}^{\infty} |c_\alpha|^2 \le 1$$

and $c_0 = 0$. *Then*

$$\sum_\alpha |c_\alpha|^p |z^\alpha| < 1$$

for any $z \in R_n D^n$, *where* $R_n \le (\frac{1}{2n})^{\frac{1}{q}}$.

Proof. Let $z = Rw$ for $w \in D^n$. Then

$$\begin{aligned}
\sum_\alpha |c_\alpha|^p |z^\alpha| &= \sum_\alpha |c_\alpha|^p |(Rw)^\alpha| \\
&= \sum_\alpha |c_\alpha|^p |w^\alpha| R^{|\alpha|} \\
&\le (\sum_\alpha (|c_\alpha|^p |w^\alpha|)^{\frac{2}{p}})^{\frac{p}{2}} (\sum_\alpha |R^{|\alpha| q})^{\frac{1}{q}} \\
&\quad \text{(by Hölder's inequality)} \\
&= (\sum_\alpha (|c_\alpha|^2 |w^\alpha|^{\frac{2}{p}}))^{\frac{p}{2}} (\sum_{k=1}^{\infty} \sum_{|\alpha|=k} R^{kq})^{\frac{1}{q}} \\
&\le (\sum_\alpha |c_\alpha|^2)(\sum_{k=1}^{\infty} (R^q)^k n^k)^{\frac{1}{q}} \\
&\le 1 \qquad (\text{when } R \le (\frac{1}{2n})^{\frac{1}{q}}).
\end{aligned}$$

∎

In a similar manner, we can extend Theorem 3.1 to several variables.

Theorem 4.2. *Let X be a Banach space of analytic functions from D^n into $\mathbb{C}$ such that polynomials are dense in X, the set of bounded point evaluations is D^n, and if $f \in X$ with $\|f\|_X \leq 1$ is not constant then $|f(0)| < 1$. Then Bohr's phenomenon holds in X if and only if*

$$\sup\{\frac{|(D^\alpha f)(0)|}{\alpha!(1-|f(0)|)})^{\frac{1}{|\alpha|}} : \|f\|_X \leq 1, |f(0)| < 1, \alpha \in \mathbb{N}^n, \alpha \neq 0\} < \infty.$$

We leave it to the reader to restate Theorem 4.2 in a point-invariant form similarly to Theorem 3.2.

In particular, if we are interested in bounded functions on the polydisk, we have a several variable analogue of Corollary 3.4.

Corollary 4.3. *Let f be an analytic function from D^n to D. Then for each multi-index α,*

$$\sup_{z \in D^n} \frac{|(D^\alpha f)(z)|(1-|z_1|^2)^{\alpha_1}(1-|z_2|^2)^{\alpha_2}...(1-|z_n|^2)^{\alpha_n}}{1-|f(z)|^2} < \infty. \tag{6}$$

Notice that applying (6) coordinate-wise, one can easily extend this result to mappings from the polydisk into itself. Moreover, since as is shown in [9], Bohr's radius for any complete Reinhardt domain G is positive, Corollary 4.3 immediately extends to all such domains and accordingly, for example to holomorphic mappings of the unit ball into itself, a recent result of MacCluer, Stroethoff, and Zhao (cf. [19].)

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