WEAK COMPACTNESS IN CERTAIN STAR-SHIFT INVARIANT SUBSPACES

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ABSTRACT. He clasps the crag with crooked hands,
close to the sun in lonely lands,
ringed with the azure world he stands.

The wrinkled sea beneath him crawls,
He watches from his mountain walls,
and like a thunderbolt he falls.
(The Eagle, by Tennyson)

1. INTRODUCTION

Let $\mathbb{D} = \{z : |z| < 1\}$ be the unit disk and $\mathbb{T} := \partial\mathbb{D} = \{z : |z| = 1\}$ be the unit circle. For $0 < p < \infty$, $H^p := H^p(\mathbb{D})$ denotes the Hardy space of functions analytic in $\mathbb{D}$ and satisfying:

$$||f||_p^p := \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

It is well-known that, for $1 \leq p < \infty$, H^p can be identified with the closed subspace of functions in $L^p := L^p(\mathbb{T}, \frac{d\theta}{2\pi})$ whose negatively indexed Fourier coefficients are zero. Those functions on $\mathbb{T}$ are in fact nontangential boundary values of H^p-functions in the disk. Also, from

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standard theory we know that every nonzero H^p-function can be factored:

$$f = e^{i\alpha} BFS_\mu,$$

where $\alpha \in \mathbb{R}$ is a constant,

$$B(z) = z^m \prod_{n=1}^{\infty} \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a}_n z}$$

is a Blaschke product built on the zeros of f (m is nonnegative integer and, necessarily, $\sum_{n=1}^{\infty}(1-|a_n|) < \infty$),

$$F(z) = \exp\left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \ln|f(e^{i\theta})| d\theta \right\}$$

is an outer function ($|F| = |f|$ a.e. on $\mathbb{T}$) and

$$S_\mu(z) = \exp\left\{ -\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} d\mu(\theta) \right\}$$

is a singular inner function; $\mu \geq 0$ is singular with respect to Lebesgue measure. Note that H^2 is a Hilbert space; a subspace of $L^2(\mathbb{T})$ with the usual inner product

$$< f, g > = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \overline{g(e^{i\theta})} d\theta.$$

For a detailed account of Hardy spaces on the disk, we refer the reader to [Du], [Ga], [Ho] and [Ko] and references therein.

The following question, posed to the third author by Anders Lindquist and Tryphon Georgiou, has arisen in their work on nonlinear optimal control.

Question 1.1. Let B be an infinite Blaschke product and $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions in the space $K_B := H^2 \ominus BH^2$ – the orthogonal complement of BH^2 in H^2 – such that $\{\log|f_n|\}_{n=1}^{\infty}$ is bounded in $L^1(\mathbb{T})$. If $\{f_n\}_{n=1}^{\infty}$ converges to $f \not\equiv 0$ weakly in H^2, then does this imply that $\{\log|f_n|\}_{n=1}^{\infty}$ is weakly compact in $L^1(\mathbb{T})$?

We remark in passing that if the functions f_n are not required to belong to some special subspace of H^2, then the answer to Question 1.1 is firmly in the negative. Indeed, let S be the singular inner function given by

$$S(z) = \exp\left\{\frac{z+1}{z-1}\right\}$$

and for $r_n := 1 - 2^{-n}$, let $f_n(z) = S(r_n z)$. Then $\{f_n\}_{n=1}^{\infty}$ converges to S in H^2 (as $n \to \infty$), yet the measures $\{\log|f_n|d\theta\}_{n=1}^{\infty}$ converge weak-star to a point mass at 1 and so the family $\{\log|f_n|\}_{n=1}^{\infty}$ cannot be uniformly integrable on $\mathbb{T}$. Accordingly, $\{\log|f_n|\}_{n=1}^{\infty}$ is not weakly compact in $L^1(\mathbb{T})$; cf. [DS2], [Na] – Vitali's Theorem.

The spaces K_B $(= H^2 \ominus BH^2)$ are representatives of a wide class of subspaces in H^2 that are invariant with respect to the so-called backward shift opeartor T^* (cf. [CR], [DSS] and [Ni]). Indeed, if we let T denote the forward shift operator acting on H^2 $(T(f) := zf)$, then a celebrated theorem of A. Beurling states that any nontrivial closed subspace $\mathcal{M}$ of H^2 that is invariant with respect to T has the form

$$\mathcal{M} = IH^2,$$

where $I = BS_\mu$ is an inner function; B is a Blaschke product and S_μ is a singular inner function, and one or both of these factors may reduce to a constant. Hence, by duality, it immediately follows that $K_I := H^2 \ominus IH^2$ is invariant with respect to T^*, the adjoint of T. One easily calculates that

$$T^* f = \frac{f(z) - f(0)}{z},$$

which explains the terminology "backward shift" when we identify a function f in H^2 with a vector in ℓ^2 via its Taylor coefficients.

Returning to Question 1.1, note that the answer is a resounding "yes" if $B = B_N$ is a *finite* Blaschke product. Indeed, in that case,

$$K_{B_N} = \operatorname{Span}\left\{\frac{1}{1-\overline{a}_j z} : j = 1, 2, 3, ..., N\right\},$$

where $\{a_j\}_{j=1}^N$ are the zeros of B_N. Evidently, K_{B_N} is a finite dimensional space of rational functions with fixed poles $\{\frac{1}{\overline{a}_j} : j = 1, 2, 3, ..., N\}$. Note that if $f = FJ \in K_I$, where F is an outer function and J is an inner function, then F must be in K_I as well since

$$0 = \int \overline{FJ} IJh d\theta = \int \overline{F} Ih d\theta$$

for all h in H^2. Thus K_I is stable under division by inner factors. Now, with regard to Question 1.1, if we let F_n denote the outer part of f_n, then we obtain

$$F_n(z) = \frac{c_n \prod_{j=1}^{m_n}(z - b_j^{(n)})}{\prod_{j \in J_n}(1 - \overline{a}_j z)},$$

where $J_n \subseteq \{1, 2, ..., N\}$, and c_n and $\{b_j^{(n)} : 1 \leq j \leq m_n < |J_n|\}$ are constants; $|b_j^{(n)}| \geq 1$. Since $|F_n| = |f_n|$ a.e. on $\mathbb{T}$, and since K_{B_N} is stable under division by inner factors, we may assume that $\{F_n\}_{n=1}^\infty$ converges weakly to F in K_{B_N}; $|F| \geq |f|$ and so $F \not\equiv 0$. Thus, $\{c_n\}_{n=1}^\infty$ are bounded away from zero and weak compactness of $\{\log|f_n| = \log|F_n|\}_{n=1}^\infty$ follows from the uniform integrability of $\{\log|F_n|\}_{n=1}^\infty$; since $z \mapsto \log|z - b| \in L^1(\mathbb{T})$ for any b in $\mathbb{T}$. This example suggests that, in order for Question 1.1 to have an affirmative answer, it is perhaps necessary that functions in K_B do not allow a singular inner factor. We note here that some special examples of such subspaces were constructed by K. Dyakonov (in [Dy]). In section 2 we considerably extend this collection of known examples and describe a wide variety of such K_B-spaces; our methods are essentially different from those of K. Dyakonov. In section 3, we show that, for these spaces K_B, the answer to Question 1.1 is indeed "yes". Moreover, we further show that all weak-star limit points of measures $\{\log|f_n|d\theta\}_{n=1}^{\infty}$ in the weak-star topology of $M(\mathbb{T}) = C(\mathbb{T})^*$ are absolutely continuous with respect to Lebesgue measure on $\mathbb{T}$; a result that is reminiscent of the F. and M. Riesz theorem, though for the logarithms of the moduli. In section 4 we discuss simple examples of spaces K_B that have plentiful singular inner factors; this follows the work of K. Dyakonov in [Dy], Theorem 5 (i). In section 5, using Paley-Wiener theory, we construct an example of a K_B-space for which Question 1.1 has a negative answer. Some final remarks and further questions are contained in section 6.

2. T^*-Invariant Subspaces without Singular Inner Factors

In this section we develop a large class of infinite Blaschke products B for which the T^*-invariant subspaces $K_B := H^2 \ominus BH^2$ contain no functions with nontrivial singular inner factors. The examples we discuss here are somewhat complementary to the one given by [Dy], Theorem 5 (ii); where the zeros (of B) have just one accumulation point on $\mathbb{T}$ and are chosen (in a very special way) to approach the unit circle fast and tangentially from both sides of that accumulation point.

Theorem 2.1. *Let B be an infinite Blaschke product whose zeros cluster at only finitely many points $w_1, w_2, ..., w_m$ of the unit circle $\mathbb{T}$. Moreover, suppose that for each $j = 1, 2, ..., m$ there is a chord of the unit circle with one endpoint at w_j so that B has no zeros in one of the domains bounded by this chord and $\mathbb{T}$. Then $K_B := H^2 \ominus BH^2$ contains no functions with nontrivial singular inner factors.*

Proof (of Theorem 2.1). Assume, to the contrary, that K_B contains gS, where S is a nontrivial singular inner function and $0 \not\equiv g \in H^2$.

As we noted in section 1, K_B is stable with respect to division by inner factors and so we may assume that g is an outer function. As is well-known and immediately checked (cf. [CR]), the function

$$k(z,\zeta)=k_B(z,\zeta):=\frac{1}{2\pi}\left[\frac{1-\overline{B(z)}B(\zeta)}{1-\overline{z}\zeta}\right] \qquad (2.1.1)$$

is the reproducing kernel for K_B. Hence, writing any f in K_B as

$$f(z)=<f(\zeta),k(z,\zeta)>,$$

we conclude that (together with $\zeta\longmapsto k(z,\zeta)$) all functions f in K_B are analytically continuable to $\mathbb{C}\setminus\overline{(\{\frac{1}{\overline{b}_n}\}_{n=1}^{\infty})}$, where $\{b_n\}_{n=1}^{\infty}$ are the zeros of B. In particular, all such functions f are analytically continuable across $\mathbb{T}\setminus\{w_1,w_2,...,w_m\}$. Thus the singular measure of S consists only of atoms at some of the points w_j. Again, since we have the freedom to cancel inner factors, we may assume (applying a rotation if needed) that S has only one atom and in fact that atom is located at $w_1=1$. So we are reduced to the case that

$$S(z)=\exp\left\{c\left(\frac{z+1}{z-1}\right)\right\}, \qquad (2.1.2)$$

where c is some positive constant. Since $g\not\equiv 0$ is is an outer function, for $0\le r<1$, $\log|g(re^{it})|$ is a Poisson integral of the L^1-function $\log|g(e^{i\theta})|$ whose Fourier coefficients $\{c_n\}_{n=-\infty}^{\infty}$ satisfy $c_n\longrightarrow 0$ as $|n|\to\infty$. Thus,

$$\log|g(re^{i\theta})|=\sum_{n=-\infty}^{\infty}c_n r^{|n|}e^{in\theta}$$

and hence

$$\log|g(re^{i\theta})|=o\left(\frac{1}{1-r}\right), \qquad (2.1.3)$$

as $r \to 1$ uniformly in θ. From (2.1.2) and (2.1.3) it follows that, for $0 \leq x < 1$,

$$|S(x)g(x)| \leq A \exp\left\{-\frac{b}{1-x}\right\}, \tag{2.1.4}$$

for suitable positive constants A and b. Now (2.1.4) will lead us to a contradiction. In order to clarify the reasoning, from this point on we transplant the whole scenario to the upper half-plane $U = U_+ := \{w = x+iy : y > 0\}$ and send $w_1 = 1$ to ∞ via $\varphi(z) = \frac{i(1+z)}{1-z}$. The hypothesis of the theorem applied to the boundary point ∞ reads:

(2.1.5) – there is a ray L (in U) such that one of the open sectorial (or "wedge") domains bounded by L and $\mathbb{R}$ is free from zeros of $\mathcal{B} := B \circ \varphi^{-1}$.

Let t in $\mathbb{R}$ denote the point where L meets the real axis and let W be the zero-free wedge domain. By decreasing the angle of W (if necessary), we may assume that the zeros of B approach $\overline{W}$ nontangentially. Note that

$$\mathcal{S}(w) := S \circ \varphi^{-1}(w) = \exp\{icw\},$$

where $c > 0$, $\mathcal{G} := g \circ \varphi^{-1}$ ($\not\equiv 0$) is an outer function in the Hardy space $H^2(U)$ (cf. [Du] or [Ho]) and (2.1.4) becomes

$$|\mathcal{S}(iy)\mathcal{G}(iy)| \leq A \exp\{-by\}, \tag{2.1.6}$$

for $y > 0$; again, b is a positive constant. Furthermore, transplanting (2.1.3) into U and observing that $\mathcal{S}$ tends to zero exponentially along all vertical rays in U, it follows that (2.1.6) holds (perhaps with different positive constants A and b) along each vertical ray in U. In particular, we can find positive constants A_o and b_o such that

$$|\mathcal{S}(t+iy)\mathcal{G}(t+iy)| \leq A_o \exp\{-b_o y\}, \tag{2.1.7}$$

whenever $y \geq 0$. Since $\mathcal{SG}$ is orthogonal to $\mathcal{B}H^2$ in U, it follows from the F. and M. Riesz Theorem (in U) that there is f in $H^2(U)$ such that

$$\overline{\mathcal{SG}}\mathcal{B} = f$$

a.e. on $\mathbb{R}$; that is,

$$\mathcal{SG} = \mathcal{B}\overline{f} \tag{2.1.8}$$

a.e. on $\mathbb{R}$. Let F be a function in $H^2(U_-)$, $U_- := \{w = x+iy : y < 0\}$ is the lower half-plane, so that $F = \overline{f}$ a.e. on $\mathbb{R}$. The existence of such an F follows right away from the Paley-Wiener Theorem (cf. [Ho], Chapter 8, and [Br]) or by transplanting f back to $\mathbb{D}$, noting that H^2-functions are precisely those whose Taylor coefficients are square summable and conjugating that Taylor series on $\mathbb{T}$. Now rewrite (2.1.8) as

$$\mathcal{SG} = \mathcal{B}F \tag{2.1.9}$$

a.e. on $\mathbb{R}$. The Blaschke product $\mathcal{B}$ has a meromorphic continuation to U_- across all points of $\mathbb{R}$ that are not cluster points of the zeros of $\mathcal{B}$. The continued function has poles precisely at the points in U_- that are reflections of zeros of $\mathcal{B}$ through $\mathbb{R}$. Letting $W_- = \{\overline{w} : w \in W\}$, we then have:

$$G(w) = \begin{cases} \mathcal{S}(w)\mathcal{G}(w) & \text{if } w \in U \\ \mathcal{B}(w)F(w) & \text{if } w \in W_- \end{cases} \tag{2.1.10}$$

which is analytic across the common boundary of U and W_-; this analytic continuation follows from representation (2.1.1) and the sentence that follows it (in the proof of Theorem 2.1). Without loss of generality, we may assume that this common bondary is $\{x : -\infty < x < t\}$ ($\subseteq \mathbb{R}$). Thus, by (2.1.10), G is holomorphic in the wedge domain V that is bounded by $L_- := \{\overline{w} : w \in L\}$ and the vertical ray $\{t + iy : y > 0\}$; at t, the sides of V form an angle of less than π. Also, in view of (2.1.10) and (2.1.7), G decays exponentially along $\{t + iy : y > 0\}$. Also, along L_-, G has "subexponential" growth. That is,

$$\limsup_{\substack{|w|\to\infty \\ w\in L_-}} \frac{\log |G(w)|}{|w|} \leq 0, \qquad (2.1.11)$$

since, on L_-, $G = \mathcal{B}F$ (cf. (2.1.10)) and both F and the analytically continued Blaschke product $\mathcal{B}$ each separately satisfy such an estimate. Indeed, for F (an H^2-function in U_-) it follows at once from, for example, the Paley-Wiener theorem (cf. [Ho], Chapter 8, and [Br]) that the limit in (2.1.11) is in fact less than 0. As to the continued Blaschke product $\mathcal{B}$, the assertion is equivalent (in view of the reflection law: $\mathcal{B}(w)\overline{\mathcal{B}(\overline{w})} = 1$) to the following result, whose proof we defer for the moment.

Lemma 2.2. *Let $\mathcal{B}$ be a Blaschke product in U whose zeros $\{a_n\}_{n=1}^{\infty}$ tend to ∞. If W is a wedge domain in U such that $W \cap \{a_n\}_{n=1}^{\infty} = \emptyset$ and, indeed, $\{a_n\}_{n=1}^{\infty}$ has nontangential approach to $\overline{W}$ as $n \to \infty$, then*

$$\lim_{\substack{|w|\to\infty \\ w\in W}} \frac{\log |\mathcal{B}(w)|}{|w|} = 0.$$

There are much stronger versions of this lemma in the literature (due to L. Ahlfors and M. H. Heins) that make no restriction concerning a zero-free wedge domain W; we refer the reader to [Bo], Theorem 7.2.3. Returning to the proof of our theorem, we have a function G of exponential type in the wedge domain V, where V has opening less than

π and is bounded by the ray L_- and the vertical ray $\{t+iy : y>0\}$. The indicator funtion h ($h(\theta) := \limsup_{r\to\infty} \frac{\log|G(t+re^{i\theta})|}{r}$ – cf. [Bo], page 66) is negative on L_- and is nonpositive on $\{t+iy : y>0\}$. Therefore, by [Bo], Theorem 5.1.2, $h<0$ in V along all rays emanating from the vertex t of V. In particular, $h<0$ on $(-\infty,t]$ (in $\mathbb{R}$). But the latter is impossible since G, being the boundary values of a nonzero function in H^2, must satisfy

$$\int_{-\infty}^{t} \frac{\log|G(x)|}{1+x^2}dx > -\infty$$

(cf. [Du], [Ho] and [Ga]) and hence cannot have exponential decay as $x\to-\infty$.

To finish the proof of our theorem, it only remains for us to establish Lemma 2.2.

Proof (of Lemma 2.2). The argument proceeds more easily under the automorphism (of U) $\psi(w) := 1/\overline{w}$; we let $W^* = \psi(W)$. So we assume that $\mathcal{B}$ is a Blaschke product in U whose zeros $\{a_n\}_{n=1}^{\infty}$ tend to zero (as $n\to\infty$) such that $W^*\cap\{a_n\}_{n=1}^{\infty}=\emptyset$ and a_n approaches $\overline{W}^*$ nontangentially as $n\to\infty$. To establish this lemma we need to show that

$$\lim_{\substack{|w|\to 0\\ w\in W^*}} |w|\cdot\sum_{n=1}^{\infty}\log\left(\frac{|w-a_n|}{|w-\overline{a}_n|}\right)=0.$$

Now, by our assumption concerning nontangential approach, there is a constant $C>1$ such that

$$\frac{1}{C}\le\frac{|w-a_n|}{|w-\overline{a}_n|}\le 1,\ \text{ and }\ \frac{|w|}{|w-\overline{a}_n|}\le C$$

for all w in W^* and all n. And so we can find positive constants M_1 and M_2 such that for any natural number N and for any w in W^*,

$$\begin{aligned} 0 > |w| \cdot \sum_{n=1}^{\infty} \log\left(\frac{|w-a_n|}{|w-\overline{a}_n|}\right) &\geq M_1|w| \cdot \sum_{n=1}^{\infty}\left(\frac{|w-a_n|}{|w-\overline{a}_n|} - 1\right) \\ &= M_1|w| \cdot \sum_{n=1}^{\infty}\left(\frac{[|w-a_n| - |w-\overline{a}_n|]}{|w-\overline{a}_n|}\right) \\ &\geq M_2 \cdot \left(\sum_{n=1}^{N}[|w-a_n| - |w-\overline{a}_n|] - \sum_{n=N+1}^{\infty}|a_n - \overline{a}_n|\right). \end{aligned}$$

Since $\mathcal{B}$ is a Blaschke product, $\sum_{n=1}^{\infty}|a_n - \overline{a}_n|$ is convergent and so we can make $\sum_{n=N+1}^{\infty}|a_n - \overline{a}_n|$ as small as we like by choosing N sufficiently large. The result now follows from the simple observation that, for any fixed N,

$$\sum_{n=1}^{N}[|w-a_n| - |w-\overline{a}_n|] \longrightarrow 0$$

as $|w| \to 0$ (for z in W^*). $\square$

There is a different proof of Theorem 2.1 that involves an argument based on harmonic measure estimates. It is worth mentioning, and thus we sketch its outline in what follows.

Another Proof (of Theorem 2.1). We proceed as we did in the beginning of our first proof and arrive at the assumption that gS is in $K_B := H^2 \ominus BH^2$, where $g \not\equiv 0$ is an outer function and S is a singular inner function of the form

$$S(z) = \exp\left\{c\left(\frac{z+1}{z-1}\right)\right\};$$

c is some positive constant. Since the zeros of B have nontangential approach to $w_1 = 1$ on at least one side of 1, we can find r, $0 < r < 1$, and a point z_o in $\mathbb{D}$ ($z_o \neq 1 - r$, yet $|1 - z_o| = r$) such that the closure of $\Delta := \{z : |z - z_o| < r\}$ contains none of the poles of B (in $\mathbb{C} \setminus \overline{\mathbb{D}}$).

In fact, we may assume that these poles approach $\overline{\Delta}$ nontangentially. Note that gS has an analytic continuation to Δ and indeed, for z in $\Omega := \Delta \setminus \overline{\mathbb{D}}$, we have:

$$f(z)S(z) = < f(\zeta)S(\zeta), k(z,\zeta) >,$$

where $k(z,\zeta)$ is the reproducing kernel function for K_B. Using this integral representation, we obtain:

$$|f(z)S(z)| < C_1 \frac{|B(z)|}{(|z|-1)}$$

whenever $z \in \Omega$; C_1 is some positive constant. So, for such z,

i) $|f(z)| < C_1 \frac{|B(z)|}{(|z|-1)|S(z)|}$, and

ii) $|f(z)S_o(z)| < C_1 \frac{|B(z)|}{(|z|-1)|S_o(z)|}$,

where S_o is given by $S_o^2 = S$. By (i), (ii) and Lemma 2.2, we can now find another positive constant C_2 such that

iii) $|f(z)| \leq \frac{C_2}{(|z|-1)}$ and $|f(z)S_o(z)| \leq \frac{C_2}{(|z|-1)}$

for all z in Ω. For any s, $0 < s < r$, let $\alpha(s) = \{\theta \in [0, 2\pi) : |z_o - se^{i\theta}| \leq 1\}$ and let $\beta(s) = \{\theta \in [0, 2\pi) : |z_o - se^{i\theta}| > 1\}$. Now arclength measure on an arc of a circle in $\mathbb{D}$ is a Carleson measure, and the associated (Carleson) constant M can be chosen independent of the arc or the circle. Therefore,

$$\begin{aligned} \sup_{0<s<r} \frac{1}{2\pi} \int_{\alpha(s)} \log^+ |f(z_o + se^{i\theta})| d\theta &\leq \sup_{0<s<r} \frac{1}{2\pi} \int_{\alpha(s)} |f(z_o + se^{i\theta})| d\theta \\ &\leq M ||f||_{H^2} < \infty, \end{aligned}$$

and likewise with fS_o in place of f. And, by (iii),

$$\sup_{0<s<r} \frac{1}{2\pi} \int_{\beta(s)} \log^+ |f(z_o + se^{i\theta})| d\theta < \infty,$$

and likewise with fS_o in place of f. Therefore,

$$\sup_{0<s<r} \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(z_o + se^{i\theta})| d\theta < \infty,$$

and likewise with fS_o in place of f. Consequently, both f and fS_o are in the Nevanlinna class of Δ. But then S_o is in the Nevanlinna class of Δ, which we know is false! Our (second) proof is now complete. □

3. Weak Compactness in T^*-Invariant Subspaces Without Inner Factors

We first prove the "F. and M. Riesz Theorem" for logarithms of functions in K_B, where B is such that K_B contains no function with a nontrivial singular inner factor.

Theorem 3.1. *Let B be an infinite Blaschke product such that $K_B := H^2 \ominus BH^2$ contains no functions with nontrivial singular inner factors. For positive constants a and M, let $E(M,a) = \{f \in K_B : ||f||_{H^2} \leq M$ and $\int \log |f(e^{i\theta})| d\theta > -a\}$. Then the set of measures $\{\log |f(e^{i\theta})| d\theta : f \in E(M,a)\}$ is compact in the weak-star topology of $M(\mathbb{T}) := C(\mathbb{T})^*$.*

Corollary 3.2. *The conclusion of Theorem 3.1 holds for spaces K_B, where B satisfies the hypothesis of Theorem 2.1.*

Proof (of Theorem 3.1). Note that having a parameter a in the definition of $E(M,a)$ is important in order to prevent unboundedness of the negative variations of the measures $\{\log |f_n| d\theta\}_{n=1}^{\infty}$.

The hypothesis implies that the measures $\{\log |f(e^{i\theta})| d\theta : f \in E(M,a)\}$ have bounded total variations. Indeed, $|\log |f|| = \log^+ |f| + \log^- |f|$. Therefore, we have

$$\int \log^+ |f| d\theta \leq \int |f| d\theta \leq 2\pi M < \infty,$$

and hence

$$\int \log^- |f| d\theta = -\int \log |f| d\theta + \int \log^+ |f| d\theta \leq a + 2\pi M < \infty.$$

Thus, for any sequence $\{f_n\}_{n=1}^{\infty}$ in $E(M,a)$, choosing a subsequence if necessary, we may assume that $\{f_n\}_{n=1}^{\infty}$ converges to some f (in K_B) weakly in H^2 and that the measures $\{\log |f_n(e^{i\theta})d\theta\}$ converge weak-star to a bounded real measure on $\mathbb{T}$. Since K_B is stable with respect to dividing by inner factors, dividing out the Blaschke products of the functions f_n (by our hypothesis, they have no singular inner factors), we may further assume that (for each n) $0 \not\equiv f_n$ is an outer function. We then have a sequence of outer functions $\{f_n\}_{n=1}^{\infty}$ convergent to f uniformly on compact subsets of $\mathbb{D}$ to f; indeed, for each z in $\mathbb{D}$, letting $P_z(\theta)$ denote the Poisson kernel for evaluation at z, we have

$$\lim_{n\to\infty} \log |f_n(z)| = \lim_{n\to\infty} \int \log |f_n(e^{i\theta})| P_z(\theta) d\theta \geq -a \left(\frac{1+|z|}{1-|z|}\right) > -\infty.$$

And so, by Hurwitz's Theorem, f vanishes nowhere in $\mathbb{D}$. Since, by our hypothesis, f has no singular inner factor, it is an outer function. That is, $\log |f(z)|$ is the Poisson integral of the integrable function $\log |f(e^{i\theta})|$. Since, obviously, $\log |f(z)|$ is also the Poisson integral of the measure μ that is the weak-star limit of $\{\log |f_n(e^{i\theta})| d\theta\}_{n=1}^{\infty}$, we have: $d\mu = \log |f(e^{i\theta})| d\theta$. The proof is now complete.□

Although this theorem (i.e., Theorem 3.1) is not yet a positive answer to the L^1-weak compactness question, it turns out to be the main ingredient needed to establish L^1-weak compactness of the logarithms in the K_B spaces described in Theorem 2.1.

Theorem 3.3. *Let B be a Blaschke product satisfying the hypothesis of Theorem 2.1. Then Question 1.1 has the affirmative answer. More precisely, the set $\{\log|f(e^{i\theta})| : f \in E(M,a)\}$ is weakly compact in $L^1(\mathbb{T})$.*

Proof. We begin with some elementary lemmas.

Lemma 3.4. *Let E be a bounded subset of $L^1(\mathbb{T})$ with the property that, for any compact subset F of $\mathbb{T}\setminus\{w_1, w_2, ..., w_m\}$, there is a positive constant $M(F)$ such that*

$$\sup\{\int_F |g|^2 d\theta : g \in E\} \leq M(F) \qquad (3.4.1).$$

Then any measure on $\mathbb{T}$ that is the weak-star limit of a sequence $\{g_n d\theta : g_n \in E\}$ has the form $gd\theta + \mu$, where μ is an atomic measure supported on $\{w_1, w_2, ..., w_m\}$.

Proof (of Lemma 3.4). Let $d\nu$ be a measure on $\mathbb{T}$ that is the weak-star limit of a sequence $\{g_n d\theta : g_n \in E\}$. Since, restricted to any compact subset F of $\mathbb{T} \setminus \{w_1, w_2, ..., w_m\}$, $\{g_n\}_{n=1}^{\infty}$ is bounded in the L^2-norm, we could, by taking an increasing sequence $\{F_j\}_{n=1}^{\infty}$ of compact subsets of $\mathbb{T}\setminus\{w_1, w_2, ..., w_m\}$ ($\cup_{j=1}^{\infty} F_j = \mathbb{T}\setminus\{w_1, w_2, ..., w_m\}$) and applying Cantor's diagonalization process in extracting subsequences, assume that $\{g_n\}_{n=1}^{\infty}$ converges to g weakly in L^2 on all compact subsets of $\mathbb{T} \setminus \{w_1, w_2, ..., w_m\}$. Let V be an open subset of $\mathbb{T}$ such that $\{w_1, w_2, ..., w_m\} \subseteq V$ and let φ be a continuous function on $\mathbb{T}$ that vanishes on V. Then,

$$\int \varphi d\nu = \lim_{n\to\infty} \int_{\mathbb{T}} \varphi g_n d\theta = \lim_{n\to\infty} \int_{\mathbb{T}\setminus V} \varphi g_n d\theta = \int_{\mathbb{T}\setminus V} \varphi g d\theta.$$

It follows that $d\nu$ and $gd\theta$ coincide on $\mathbb{T} \setminus \{w_1, w_2, ..., w_m\}$.$\square$

Remark 3.5. Of course, replacing the exponent 2 in (3.4.1) by any $p > 1$, leads to the same conclusion.

Lemma 3.6. *Under the hypothesis of Lemma 3.4, assume further that*

i) *any limit measure of a weak-star convergent sequence* $\{g_n d\theta : g_n \in E\}$ *has vanishing atomic component, and*

ii) *all* g *in* E *are nonnegative.*

Then the family E *is uniformly integrable and hence (by* [DS1]*, Chap. 4, Section 8, Thm. 7) is compact in the weak topology of* $L^1(\mathbb{T})$*.*

Proof (of Lemma 3.6). As in the proof of Lemma 3.4, applying a Cantor diagonalization argument, we may assume that the sequence $\{g_n\}_{n=1}^{\infty}$ converges weakly in L^2 on compact subsets of $\mathbb{T}\backslash\{w_1, w_2, ..., w_m\}$. By passing to another subsequence (if necessary) we may further assume that the measures $\{g_n d\theta\}_{n=1}^{\infty}$ converge weak-star to ν, and by (i) and Lemma 3.4, $d\nu = g d\theta$, where $g \in L^1(\mathbb{T})$. Choose ψ in $L^\infty(\mathbb{T})$ and V an open subset of $\mathbb{T}$ such that $\{w_1, w_2, ..., w_m\} \subseteq V$. Then,

$$\int_{\mathbb{T}\backslash V} g_n \psi d\theta \longrightarrow \int_{\mathbb{T}\backslash V} g\psi d\theta \qquad (3.6.1).$$

And furthermore,

$$\left|\int_{\mathbb{T}} g_n \psi d\theta - \int_{\mathbb{T}} g\psi d\theta\right| \leq \left|\int_{\mathbb{T}\backslash V} (g_n - g)\psi d\theta\right| + ||\psi||_\infty \cdot \int_V (|g_n| + |g|) d\theta \ \ (3.6.2).$$

Letting n tend to ∞ and using assumption (ii) and (3.6.1), we see that

$$\limsup_{n\to\infty} \left|\int_{\mathbb{T}} (g_n - g)\psi d\theta\right| \leq ||\psi||_\infty \cdot \left(\int_V |g| d\theta + \limsup_{n\to\infty} \int_V g_n d\theta\right) \ \ (3.6.3).$$

Now the first term on the right hand side in (3.6.3) obviously tends to zero uniformly as $|V| \longrightarrow 0$; $|V|$ denotes the Lebesgue measure of V. The second term does as well, because of:

Lemma 3.7. $\limsup_{|V|\to 0}\{\int_V g d\theta : g \in E\} = 0.$

This is precisely the assertion of uniform integrability locally at the points $w_1, w_2, ..., w_m$.

Proof (of Lemma 3.7). Assume, to the contrary, that $\limsup_{|V|\to 0}\{\int_V g d\theta : g \in E\} > 0$. So, we can find $\varepsilon > 0$, a sequence of open sets $\{V_n\}_{n=1}^{\infty}$ in $\mathbb{T}$, where $|V_n| < 2^{-n}$ and $\{w_1, w_2, ..., w_m\} \subseteq V_n$ (for all n), and a sequence $\{g_n\}_{n=1}^{\infty} \subseteq E$ such that

$$\int_{V_n} g_n d\theta \geq \varepsilon, \tag{3.7.1}$$

for $n = 1, 2, 3, ...$. In fact, we may assume that $V_{n+1} \subseteq V_n$ for all n, because if this is not the case, then (for $n = 1, 2, 3, ...$) we replace V_n with $V_n' := \cup_{k=n}^{\infty} V_k$ and observe that:

a) $V_{n+1}' \subseteq V_n'$,
b) $|V_n'| < 2^{1-n}$, and
c) $\int_{V_n'} g_n d\theta \geq \varepsilon$ for all n.

Note that (c) holds by (3.7.1), and since $V_n \subseteq V_n'$ and $g_n \geq 0$ for all n. Moreover, passing to a subsequence, we may further assume (by (i)) that $\{g_n d\theta\}_{n=1}^{\infty}$ converges to $Gd\theta$ weak-star, and weakly in L^2 on any compact subset of $\mathbb{T} \setminus \{w_1, w_2, ..., w_n\}$; $G \in E$. Since, for any fixed natural number N, ∂V_N is a compact subset of $\mathbb{T} \setminus \{w_1, w_2, ..., w_m\}$, and since $g_n \geq 0$ for all n, we now have:

$$\varepsilon \leq \lim_{n\to\infty} \int_{V_N} g_n d\theta = \int_{V_N} G d\theta.$$

Letting N approach ∞, a contradiction to the integrability of G arises. So Lemma 3.7 must hold, and thus Lemma 3.6 is also established.□

Finally, to finish the proof of Theorem 3.3, let us note that in that context, the functions g_n in Lemmas 3.4, 3.6 and 3.7 are of the form $\log^- |f_n|$, where $f_n \in K_B \cap E(M, a)$. Since $\log^+ |x| \leq |x|$, $\log^+ |f_n|$ are bounded in $L^2(\mathbb{T})$ and hence are uniformly integrable on $\mathbb{T}$. So, in order to apply the L^1-weak precompactness criterion ([DS1], Chapter IV, Section 8, Theorem 7), we only need to study the uniform integrability of $\log^- |f_n|$; So, hypothesis (ii) of Lemma 3.6 is satisfied. Hypothesis (i) of Lemma 3.6 holds in view of Theorem 2.1. Lastly, the hypothesis of Lemma 3.4 holds simply since, as we noted earlier, any function f in K_B ($\supseteq K_B \cap E(M,a)$) is analytically continuable across $\mathbb{T}\backslash\{w_1, w_2, ..., w_m\}$, where the $w_j's$ are the only accumulation points of the zeros of the Blaschke product B. Thus, Lemma 3.7 holds for $|\log|f||$, where $f \in K_B \cap E(M,a)$, and the compactness in L^1 of $\{\log|f| : f \in K_B \cap E(M,a)\}$ follows.□

4. T^*-Invariant Subspaces that contain Singular Inner Factors

Let S_μ be a singular inner function. By Frostman's Theorem (cf. [Ga], Chapter II), for all α in $\mathbb{D}$, except for a set of logarithmic capacity zero, the inner function

$$b(z) := \frac{S_\mu(z) - \alpha}{1 - \overline{\alpha} S_\mu(z)} \tag{4.0}$$

is an infinite Blaschke product. The following is a result of K. Dyakonov (cf. [Dy], Theorem 5). We state the result and give a proof for completeness sake.

Theorem 4.1. *Let b be an infinite Blaschke product as in (4.0), and let B be the Blaschke product $B(z) := zb(z)$. Then $K_B := H^2 \ominus BH^2$ contains a function fS_μ, where f is an outer function for which there exists $M > 1$ such that $\frac{1}{M} \leq |f| \leq M$ on $\mathbb{D}$.*

Proof. Since $b(z) := \frac{S_\mu(z)-\alpha}{1-\overline{\alpha}S_\mu(z)}$, it follows that

$$S_\mu(z) = \frac{b(z) + \alpha}{1 + \overline{\alpha} b(z)}$$

and hence

$$(1+\overline{\alpha}b(z))S_\mu(z) = b(z)+\alpha.$$

Note that $(b(z)+\alpha) \in K_B$ and that $f := 1+\overline{\alpha}b$ is an outer function satisfying the conclusion of this theorem.□

Since K_B is stable under division by singular factors, we have an immediate corollary (to Theorem 4.1).

Corollary 4.2. *Under the hypothesis and terms of Theorem 4.1, if S_ν is a singular inner function and $\frac{S_\mu}{S_\nu} \in H^\infty$, then $fS_\nu \in K_B$.*

In fact, Theorem 4.1 helps us find a great variety of outer functions in K_B.

Theorem 4.3. *Let S_μ, B and f be as above and let $K_{S_\mu} = H^2 \ominus S_\mu H^2$. If $g \in K_{S_\mu}$, then $fg \in K_B$.*

Proof. By our assumption, $< g, S_\mu h >= 0$ for all h in H^2. Therefore, by the F. and M. Riesz Theorem, there exists h_o in H^2 such that $h_o(0) = 0$ and $\overline{g}S_\mu = h_o$ a.e. on $\mathbb{T}$; and hence, $g = S_\mu\overline{h_o}$ a.e. on $\mathbb{T}$. Thus

$$< gf, Bh >=< fS_\mu\overline{h_o}, Bh >=< fS_\mu, Bh_oh >= 0$$

for all h in H^∞. Since BH^∞ is dense in BH^2, the conclusion follows. □

Remark 4.4. Theorem 4.3 suggests a strategy for finding a K_B-space for which the answer to Question 1.1 is in the negative. To be precise, if f, B and S_μ are as above and we find a sequence of functions $\{g_n\}_{n=1}^\infty$ in $K_{S_\mu} \cap E(M,a)$ such that $\{\log|g_n|\}_{n=1}^\infty$ are not uniformly integrable on $\mathbb{T}$, then $\{\log|fg_n|\}_{n=1}^\infty$ are also not uniformly integrable on $\mathbb{T}$. Since

$\{fg_n\}_{n=1}^{\infty} \subseteq K_B$, the answer to Question 1.1 for this K_B is in the negative. We employ this strategy and construct such an example in the next section.

5. An Example of Failure of L^1-Weak Compactness for the Logarithms of K_B-Functions

As before, let S denote the singular inner function given by

$$S(z) = \exp\left\{\frac{z+1}{z-1}\right\},$$

and let α be some point in $\mathbb{D}$ such that

$$b(z) := \frac{S(z) - \alpha}{1 - \overline{\alpha} S(z)}$$

is an infinite Blaschke product. Define B to be the Blaschke product $B(z) = zb(z)$. In view of Remark 4.4, we look to construct a sequence of functions $\{g_n\}_{n=1}^{\infty}$ in K_S whose logarithms are not uniformly integrable on $\mathbb{T}$. It turns out to be much more convenient to perform this construction in the upper half-plane $U := \{w = x + iy : y > 0\}$. We carry over to this setting under the conformal mapping $\varphi(z) = \frac{i(1+z)}{1-z}$ and let

$$\mathcal{S}(w) := S \circ \varphi^{-1}(w) = \exp\{iw\};$$

the transplanted space $K_{\mathcal{S}}$ admits a very simple description in terms of the Fourier transform. Indeed, by the Paley-Wiener Theorem (cf. [Ho], Chapter 8, and [Br]), $H^2(U)$ is identified with Fourier transforms of L^2-functions supported on $[0, \infty)$. Hence, $\mathcal{S}H^2$ (on the Fourier side) is identified with the subset of $L^2(\mathbb{R})$ supported on $[1, \infty)$ (the Fourier transform of multiplication by an exponential "transforms" into translation). Thus, on the side of Fourier transforms, $K_{\mathcal{S}}$ becomes the subset of $L^2(\mathbb{R}^+)$ that is orthogonal to $L^2(1, \infty)$; that is, the subset of $L^2(\mathbb{R})$

that is supported on $[0,1]$. Recall that the subset of $L^2(\mathbb{R})$ with support in $[-a,a]$ consists of the Fourier transforms of the Paley-Wiener space of entire functions that are square-integrable on $\mathbb{R}$ and of type less than a. Thus, $K_{\mathcal{S}}$ is precisely the result of multiplying by $e^{\frac{i}{2}w}$ all the functions in the Paley-Wiener space on $[-\frac{1}{2},\frac{1}{2}]$. Furthermore, since we are only looking at real w and are interested in absolute values of H^2-functions on $\mathbb{R}$, the factor $e^{\frac{i}{2}w}$ plays no role at all, and so we can normalize the problem with respect to any convenient Paley-Wiener space; we may let $a=\pi$, the classical Paley-Wiener space. Thus the problem is now reduced to finding a sequence of functions $\{f_n\}_{n=1}^{\infty}$ in the classical Paley-Wiener space for which

$$\left\{\frac{|\log|f_n(x)||}{1+x^2}\right\}_{n=1}^{\infty}$$

are not uniformly integrable on $\mathbb{R}$. The latter is the subject of the following result.

Theorem 5.1. *In the Paley-Wiener space of exponential type less than or equal to some constant c and with norm $||f||:=||f||_{L^2(\mathbb{R})}$, there is a sequence $\{f_n\}_{n=1}^{\infty}$ such that*

i) $\{||f_n||\}_{n=1}^{\infty}$ *is bounded,*

ii) $\left\{L_n(x):=\frac{|\log|f_n(x)||}{1+x^2}\right\}_{n=1}^{\infty}$ *is also bounded in $L^1(\mathbb{R})$, and*

iii) *there exists a positive constant b such that*

$$\int_{|x|>n} L_n(x)dx \geq b$$

for all n.

Corollary 5.2. *For $H^2 := H^2(U)$ and for the inner function $\mathcal{S}(w) = \exp\{icw\}$ (in H^2) (c is some positive constant), there is a sequence of functions $\{g_n\}_{n=1}^{\infty}$ in $H^2 \ominus \mathcal{S}H^2$ which have bounded norms in H^2 and such that the family of measures $\{L_n(x)dx\}_{n=1}^{\infty}$ in $M(\hat{\mathbb{R}})$ is bounded and the weak-star limit of any subsequence has an atom at ∞ ($\hat{\mathbb{R}}$ is the one-point compactification of $\mathbb{R}$).*

Interpreting Corollary 5.2 in the context of the unit disk and in light of the comments at the beginning of this section, we now have:

Corollary 5.3. *For some positive constant c, let S denote the singular inner function*

$$S(z) = \exp\left\{c\left(\frac{z+1}{z-1}\right)\right\}.$$

Choose α in $\mathbb{D}$ so that

$$b(z) := \frac{S(z) - \alpha}{1 - \overline{\alpha}S(z)}$$

is an infinite Blaschke product, and let B be the Blaschke product given by $B(z) = zb(z)$. Then, for K_B, the answer to Question 1.1 is in the negative. More precisely, there is a sequence of functions $\{f_n\}_{n=1}^{\infty}$ in $K_B \cap E(M, a)$ (for some positive constants M and a) such that all weak-star limit points (in $M(\mathbb{T})$) of the sequence of measures $\{\log|f_n|d\theta\}_{n=1}^{\infty}$ contain an atom at 1.

Proof (of Theorem 5.1). We first note that it suffices to produce our sequence $\{f_n\}_{n=1}^{\infty}$ with properties (ii) and (iii), and satisfying

$$\text{(iv)} \qquad \sup\{|f_n(x)| : x \in \mathbb{R} \text{ and } n = 1, 2, 3, ...\} \leq 1$$

instead if (i). Indeed, suppose $\{f_n\}_{n=1}^{\infty}$ satisfies (ii), (iii) and (iv). Let $h(z) = \frac{sinz}{z}$ and let $F_n = hf_n$. Then $\{F_n\}_{n=1}^{\infty}$ is bounded in $L^2(\mathbb{R})$ and

F_n has type no greater than $c+1$, which makes no difference (we simply replace c by $c+1$). And furthermore, on $\mathbb{R}$ we have:

$$|\log|F_n|| = -\log|F_n| = -\log|f_n| - \log|h|.$$

So, for any $D \subseteq \mathbb{R}$,

$$\begin{aligned} (5.1.1) \qquad \int_D \frac{|\log|F_n(x)||}{1+x^2}dx &= -\int_D \frac{\log|f_n(x)|}{1+x^2}dx - \int_D \frac{\log|h(x)|}{1+x^2}dx \\ &=: A_1(D) + A_2(D). \end{aligned}$$

Since $A_2(D)$ is finite, we see, by (ii), that $\left\{\frac{|\log|F_n||}{1+x^2}\right\}_{n=1}^{\infty}$ is bounded in $L^1(\mathbb{R})$ by $M+C$, where M is a bound on $\{||L_n||\}_{n=1}^{\infty}$ and $C = \int_{\mathbb{R}} \frac{|\log|h||}{1+x^2}dx$. Whereas, choosing $D_n = \{|x| > n|\}$ we find that

$$(5.1.2) \qquad \int_{D_n} \frac{|\log|F_n(x)||}{1+x^2}dx = -\int_{D_n} \frac{\log|f_n(x)|}{1+x^2}dx - \int_{D_n} \frac{\log|h(x)|}{1+x^2}dx,$$

and the second term on the right-hand side goes to zero as $n \to \infty$. By (5.1.2) we see that, for large n, F_n satisfies (iii) (with $\frac{b}{2}$ in place of b).

To construct $\{f_n\}_{n=1}^{\infty}$ satisfying (ii), (iii) and (iv), we may assume that $c = 1$. Define f_n by

$$f_n(w) = \left(\frac{sin(\frac{w}{n})}{\frac{w}{n}}\right)^n,$$

for $n = 1, 2, 3, \ldots$. Clearly the functions f_n are entire of type 1 and are in $L^2(\mathbb{R})$. Also, $|f_n(x)| \leq 1$ on $\mathbb{R}$. Note that $\frac{\sin w}{w}$ is in the Nevanlinna class of the upper half-plane U (indeed, $(\frac{\sin w}{w})e^{iw} \in H^\infty(U)$), and so

$$\int_{\mathbb{R}} \frac{|\log|\frac{\sin(x)}{x}||}{1+x^2}dx < \infty.$$

In fact, since $\int_{-1}^{1} \frac{|\log|\frac{\sin(x)}{x}||}{x^2}dx < \infty$, we have:

$$\int_{\mathbb{R}} \frac{|\log|\frac{\sin(x)}{x}||}{x^2}dx < \infty.$$

Therefore, by a change of variables,

$$\int_{\mathbb{R}} \frac{|\log|f_n(x)||}{1+x^2}dx = \int_{\mathbb{R}} \frac{|\log|\frac{\sin(x)}{x}||}{\frac{1}{n^2}+x^2}dx \leq \int_{\mathbb{R}} \frac{|\log|\frac{\sin(x)}{x}||}{x^2}dx < \infty;$$

and so (ii) holds. Lastly we address (iii). On the interval $2n < x < \infty$ we have $|f_n(x)| \leq \left(\frac{n}{x}\right)^n < \frac{1}{2^n}$, and so

$$\int_{2n}^{\infty} \frac{|\log|f_n(x)||}{1+x^2}dx \geq (n\log 2)\cdot \int_{2n}^{\infty} \frac{dx}{1+x^2} \geq const > 0$$

independent of n. Theorem 5.1 (and Corollaries 5.2 and 5.3) are now established. □

6. Final Remarks

There are some natural questions that arise in response to results of this paper. One of the first has to do with Frostman's representation of certain Blaschke products (see (4.0)). Note that if a Blaschke product b can be represented as

$$b(z) := \frac{S_\mu(z)-\alpha}{1-\overline{\alpha}S_\mu(z)},$$

for some α in $\mathbb{D}$ and some singular inner function S_μ, then

$$S_\mu(z) = \frac{b(z)+\alpha}{1+\overline{\alpha}b(z)},$$

and so b omits the value $-\alpha$. Since this observation can be "reversed", we see that the Blaschke products b that can be represented in Frostman's way are precisely those which satisfy $b(\mathbb{D}) \neq \mathbb{D}$. The only known examples of K_B-spaces that contain functions with nontrivial singular inner factors are those for which B has a proper factor b such that $b(\mathbb{D}) \neq \mathbb{D}$. We modify the proof of [Dy], Theorem 5 (i) (or, Theorem 4.1 in this paper) to show that all such K_B-spaces have functions with nontrivial singular inner factors.

Theorem 6.1. *Let B be a Blaschke product that has a proper factor b such that $b(\mathbb{D}) \neq \mathbb{D}$. Then K_B contains functions with nontrivial singular inner factors.*

Proof. Since b is a proper factor of B, there is a nontrivial Blaschke product b_o such that $B = b_o b$. And since $b(\mathbb{D}) \neq \mathbb{D}$, there exists α in $\mathbb{D}$ and a nontrivial singular inner function S_μ such that

$$b(z) := \frac{S_\mu(z) - \alpha}{1 - \overline{\alpha} S_\mu(z)}.$$

Let $\{z_n\}_{n=1}^\infty$ be the zeros of b_o (in $\mathbb{D}$), and let $\mathcal{W}_{b_o}$ denote the collection of nonnegative, log-integrable L^1-weights w on $\mathbb{T}$ for which there is an absolutely summable sequence of real numbers $\{c_n\}_{n=1}^\infty$ (not necessarily all nonnegative) such that

$$w(z) = \sum_{n=1}^\infty c_n P_{z_n}(z);$$

$z \mapsto P_{z_n}(z)$ is the Poisson kernel on $\mathbb{T}$ for evaluation at z_n ($n = 1, 2, 3, ...$). If f is any function in H^2 and $h \in H^\infty$, then

$$\begin{aligned} \overline{fS}_\mu Bfh = |f|^2 \overline{S}_\mu Bh &= |f|^2 b_o h \frac{1 - \alpha \overline{S}_\mu}{1 - \overline{\alpha} S_\mu} \\ &= |f|^2 |1 - \overline{\alpha} S_\mu|^2 \frac{b_o h}{(1 - \overline{\alpha} S_\mu)^2}. \end{aligned}$$

Therefore, if f is chosen to be an outer function such that $z \mapsto |f(z)|^2|1 - \overline{\alpha} S_\mu(z)|^2 \in \mathcal{W}_{b_o}$, then BfH^∞ is dense in BH^2 and

$$\int \overline{fS}_\mu Bfh d\theta = 0$$

for all h in H^∞, and therefore $fS_\mu \in K_B$. Since b_o is nontrivial and

$$0 < 1 - |\alpha| \leq |1 - \overline{\alpha} S_\mu| \leq 1 + |\alpha|$$

on $\mathbb{D}$, we can find a variety (depending on the size of b_o) of outer functions f such that $z \mapsto |f(z)|^2|1-\overline{\alpha}S_\mu(z)|^2 \in \mathcal{W}_{b_o}$ and hence such that $fS_\mu \in K_B$. $\square$

Theorem 6.1 prompts the following question.

Question 6.2. If B is a Blaschke product and K_B contains functions with nontrivial singular inner factors, then does B necessarily have a factor b such that $b(\mathbb{D}) \neq \mathbb{D}$?

One theme of our study in this paper is the search for some understanding of the growth of functions in $K_B := H^2 \ominus BH^2$, when B is an infinite Blaschke product. We now briefly expand on an idea found in the proof of Theorem 6.1 that helps in this search. If B is any Blaschke product, we define $\mathcal{W}_B$ (as in the proof of Theorem 6.1) to be the set of nonnegative, log-integrable L^1-weights w on $\mathbb{T}$ for which there is an absolutely summable sequence of real numbers $\{c_n\}_{n=1}^\infty$ (not necessarily all nonnegative) such that

$$w(z) = \sum_{n=1}^{\infty} c_n P_{z_n}(z);$$

$\{z_n\}_{n=1}^\infty$ are the zeros of B (in $\mathbb{D}$) and $z \mapsto P_{z_n}(z)$ is the Poisson kernel on $\mathbb{T}$ for evaluation at z_n $(n = 1, 2, 3, \ldots)$.

Lemma 6.3. *Let B be a Blaschke product. If f is an outer function in H^2 and $z \mapsto |f(z)|^2 \in \mathcal{W}_B$, then $f \in K_B$*

Proof. We assume that f is an outer function in H^2 such that $z \mapsto |f(z)|^2 \in \mathcal{W}_B$. Then BfH^∞ is dense in BH^2 and, for any h in H^∞,

$$\int \overline{f} Bfh d\theta = \int Bh|f|^2 d\theta = \sum_{n=1}^{\infty} c_n B(z_n)h(z_n) = 0.$$

It follows that $f \in K_B$. $\square$

Lemma 6.3 provides a short and elementary proof of a result of B. A. Lotto and D. Sarason; cf. [LS], Lemma 13.4, or Theorem 6.4 below. The proof given in [LS] relies on some rather "heavy machinery" developed by H. S. Shapiro and A. L. Shields in [SS].

Theorem 6.4 (B. A. Lotto, D. Sarason)**.** . *Let φ be a nonnegative essentially unbounded measurable function on $\mathbb{T}$. Then there exists f in H^2 that is noncyclic for T^* such that $f\varphi \notin L^2$.*

Proof. Let $E = \{z \in \mathbb{T} : \varphi(z) > 1\}$ and find a nonnegative, Lebesgue integrable and essentially unbounded function ψ on $\mathbb{T}$ such that $\psi = 1$ on $\mathbb{T} \setminus E$ and $1 \le \psi \le \varphi$ on E. Then $\log(\psi)$ is integrable on $\mathbb{T}$ and so we can find an outer function g in H^2 such that $|g| = \psi$ a.e. on $\mathbb{T}$. Since $|g|$ is essentially unbounded on $\mathbb{T}$, there is a sequence of points $\{z_n\}_{n=1}^{\infty}$ in $\mathbb{D}$ such that $|z_n| \to 1$ and $|g(z_n)| \longrightarrow \infty$ (as $n \to \infty$). Passing to a subsequence if need be, we may assume that $\{z_n\}_{n=1}^{\infty}$ is a Blaschke sequence; let B be the corresponding Blaschke product. Since $|g(z_n)| \longrightarrow \infty$, as $n \to \infty$, we can find a *summable* sequence of positive constants $\{c_n\}_{n=1}^{\infty}$ such that

$$\sum_{n=1}^{\infty} c_n |g(z_n)|^2 = \infty.$$

Note that if we define w on $\mathbb{T}$ by

$$w(z) = \sum_{n=1}^{\infty} c_n P_{z_n}(z),$$

then $w \in \mathcal{W}_B$ and so, by Proposition 6.3, any outer function f in H^2 such that $|f|^2 = w$ on $\mathbb{T}$, is in K_B. Thus, for such an f, and by the subharmonicity of $|g|^2$ on $\mathbb{D}$,

$$\int |f|^2 |g|^2 d\theta \ge \sum_{n=1}^{\infty} c_n |g(z_n)|^2 = \infty;$$

and therefore $fg \notin L^2$. Since $|g| = \psi$ a.e. on $\mathbb{T}$ and by our construction of ψ, it follows immediately that $f\varphi \notin L^2$. $\square$

We now state a corollary to the proof of Theorem 6.4

Corollary 6.5. *Let B be an infinite Blaschke product, with zeros $\{z_n\}_{n=1}^{\infty}$ and let w_o be an accumulation point of $\{z_n\}_{n=1}^{\infty}$ on $\mathbb{T}$. Suppose $g \in H^2$ and*

$$\liminf_{\substack{z \to w_o \\ z \in \mathbb{D}}} |g(z)| = \infty.$$

Then there exists f in K_B such that $fg \notin L^2$.

The following question was also asked of us by A. Lindquist and T. Georgiou, and remains unsolved.

Question 6.6. Let B be an infinite Blaschke product, with zeros $\{z_n\}_{n=1}^{\infty}$ and let w_o be an accumulation point of $\{z_n\}_{n=1}^{\infty}$ on $\mathbb{T}$. Suppose $g \in L^2$ and g is essentially unbounded in every neighborhood of w_o. Then does there exist f in K_B such that $fg \notin H^2$? The same question can be raised more generally in the context of K_I, where I is an inner function and w_o is an essential singularity of I.

We end this section and the paper with a discussion of our work in section 5. In that section we show that the results of Theorems 3.1 and 3.3 fail if a specific K_B contains a nontrivial function of the form fS_μ, provided part of the singular measure μ is a point mass on $\mathbb{T}$. There is a fair amount of evidence that these theorems fail whenever K_B contains a function with a nontrivial singular inner factor, without requiring that the associated singular measure involves a point mass. Accordingly, Question 1.1 would have a negative answer in these spaces. However, we have not yet been able to prove this. A possible avenue of approach to this problem is as follows. Let B be a Blaschke product such that K_B contains functions with nontrivial singular inner factors and let $\{a_n\}_{n=1}^{\infty}$ be the zeros of B. As we noted earlier for finite Blaschke products, we have similarly for infinite Blaschke products B that

$$K_B = \text{Span}\left\{\frac{1}{1-\overline{a}_n z} : n = 1, 2, 3, ...\right\}.$$

Let fS_μ be a function in K_B, where S_μ is a notrivial singular inner function and, since K_B is stable under division by singular factors, f may be taken to be an outer function. Approximating fS_μ by a sequence of rational functions $\{q_N\}_{N=1}^{\infty}$, where $q_N \in K_{B_N}$ and B_N is the Blaschke product built on the zeros $\{a_n\}_{n=1}^{N}$, we then have:

$$f(z)S_\mu(z) = \lim_{N\to\infty} q_N(z) = \lim_{N\to\infty} \frac{c_N \prod_{j=1}^{m_N}(z - b_j^{(N)})}{\prod_{j\in J_N}(1-\overline{a}_j z)},$$

where $J_N \subseteq \{1, 2, ..., N\}$, $m_N < |J_N|$ and $\{b_j^{(N)}\}_{j=1}^{m_N}$ are the zeros of q_N. In order for such a sequence $\{q_N\}_{N=1}^{\infty}$ to violate Theorems 3.1 and 3.3, it suffices for the functions q_N to be outer. So the problem reduces to finding a way to push those zeros of q_N that are in $\mathbb{D}$ to the outside, while preserving the convergence. The obvious way would be to replace those zeros $b_j^{(N)}$ that are in $\mathbb{D}$ by $1/\overline{b}_j^{(N)}$, yet the convergence of the resulting rational functions to fS_μ becomes an issue. Without pushing the zeros of q_N outside $\mathbb{D}$, we may very well have Blaschke factors of q_N converging to S_μ, thus not yielding that the compactness of $\{\log|q_N|\}_{N=1}^{\infty}$ is being violated. This approach leads to a question that perhaps is of independent interest.

Question 6.7. Let B be an infinite Blaschke product and let F be a nonvanishing function in K_B. Is there a sequence $\{q_n\}_{n=1}^{\infty}$ of nonvanishing rational outer functions in K_B that converges to F?

Since we have been talking about pushing the zeros of functions in K_B outside of $\mathbb{D}$, the question naturally arises as to where these zeros might be. This question seems rather intractable, yet we pose it as follows.

Question 6.8. Let B be an infinite Blaschke product. What are the *zero sets* of $K_B := H^2 \ominus BH^2$? That is, for which subsets E of $\mathbb{D}$ does there exist $f \not\equiv 0$ in K_B such that $f(z) = 0$ for all z in E?

We have made modest progress in the direction of an answer to Question 6.8, which we present after the following lemma.

Lemma 6.9. *Let $\{u_k\}_{k=1}^n$ and $\{v_k\}_{k=1}^n$ be complex numbers, and let $\{M_k\}_{k=1}^n$ be real numbers such that $M_k \geq 1$ for $k = 1, 2, ..., n$. If*

$$|u_k| \leq M_k \text{ and } |v_k| \leq M_k,$$

for $k = 1, 2, ..., n$, then

$$|(u_1 \cdots u_n) - (v_1 \cdots v_n)| \leq (M_1 \cdot \cdots \cdot M_n) \cdot \sum_{k=1}^{n} |u_k - v_k|.$$

Proof. We proceed by induction. Clearly the inequality in this lemma holds for $n = 1$. Suppose that it holds for $n - 1$ factors. Then, by this induction hypothesis, and since both 1 and $|v_n|$ are less than or equal to M_n,

$$\begin{aligned} |(u_1\cdots u_n)-(v_1\cdots v_n)| &= |(u_1\cdots u_{n-1})u_n-(u_1\cdots u_{n-1})v_n+ \\ &\qquad (u_1\cdots u_{n-1})v_n-(v_1\cdots v_{n-1})v_n| \\ &\leq |u_1\cdots u_{n-1}||u_n-v_n|+ \\ &\qquad |(u_1\cdots u_{n-1})-(v_1\cdots v_{n-1})||v_n| \\ &\leq (M_1\cdots M_n)|u_n-v_n|+ \\ &\qquad (M_1\cdots M_n)\cdot\sum_{k=1}^{n-1}|u_k-v_k| \\ &= (M_1\cdots M_n)\cdot\sum_{k=1}^{n}|u_k-v_k|.\Box \end{aligned}$$

If B is an infinite Blaschke product, then, by taking linear combinations of elements of K_B, we see that any finite subset of $\mathbb{D}$ would then be a zero set for K_B. Our next and final result adds to this.

Theorem 6.10. *Let B be an infinite Blaschke product with zeros $\{a_n\}_{n=1}^{\infty}$ and let $\{c_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers, chosen so that*

$$\sum_{n=1}^{\infty}\frac{c_n}{1-|a_n|}<\infty.$$

If $\{w_n\}_{n=2}^{\infty}$ is a sequence in $\mathbb{D}$ such that $|a_n-w_n|\leq c_n$ (for $n=2,3,...$), then $\{w_n\}_{n=2}^{\infty}$ is a zero set for K_B.

Proof. By reducing to a factor of B (if necessary), we may assume that the zeros of B (i.e., $\{a_n\}_{n=1}^{\infty}$ – listed according to multiplicity) are distinct and $a_n\neq 0$ for all n.

For $n=2,3,...$ let

$$g_n(z)=\frac{1}{(1-\overline{a}_1 z)}\cdot[b_2(z)\cdots b_n(z)],$$

where

$$b_k(z) := \frac{|a_k|}{a_k} \cdot \frac{a_k - z}{1 - \overline{a}_k z} \qquad (k = 1, 2, 3, ...)$$

are the elementary factors of B. Then $\{g_n\}_{n=2}^{\infty} \subseteq K_B$, $|g_n(0)| = |a_2 a_3 \cdots a_n|$ and $||g_n||_{H^2} = \frac{1}{\sqrt{1-|a_1|^2}}$ for all n. So $\{g_n\}_{n=1}^{\infty}$ is uniformly bounded in H^2 and there exists $c > 0$ such that $|g_n(0)| \geq c$ (for all n). Let $\{c_n\}_{n=1}^{\infty}$ be a sequence of positive constants so that

$$\sum_{n=1}^{\infty} \frac{c_n}{1 - |a_n|} < \infty, \qquad (6.10.1)$$

and choose a sequence of points $\{w_n\}_{n=2}^{\infty}$ (in $\mathbb{D}$) such that

$$|a_n - w_n| \leq c_n, \qquad (6.10.2)$$

for all $n \geq 2$. For $n = 2, 3, ...$ let

$$G_n(z) = \frac{1}{(1 - \overline{a}_1 z)} \cdot [\beta_2(z) \cdots \beta_n(z)],$$

where

$$\beta_k(z) := \frac{|a_k|}{a_k} \cdot \frac{w_k - z}{1 - \overline{a}_k z} \qquad (k = 2, 3, ...).$$

Then, by a partial fractions argument, $G_n \in K_B$ $(n = 2, 3, ...)$. Moreover,

$$|b_k(z) - \beta_k(z)| \leq \frac{|a_k - w_k|}{|1 - \overline{a}_k z|} \leq \frac{c_k}{(1 - |a_k|)},$$

and so,

$$|\beta_k(z)| \leq 1 + \frac{c_k}{(1 - |a_k|)} \text{ for } n = 2, 3, ... $$

Therefore, by Lemma 6.9,

$$|G_n(z) - g_n(z)| \leq \left[\frac{M_2 \cdots M_n}{(1 - |a_1|)}\right] \cdot \sum_{k=2}^{n} \frac{c_k}{(1 - |a_k|)}, \qquad (6.10.3)$$

where $M_k := 1 + \frac{c_k}{(1-|a_k|)}$. By (6.10.3) and (6.10.1), $\{G_n\}_{n=1}^{\infty}$ is bounded in H^2, and since $|G_n(0)| = |w_1 \cdots w_n|$, (6.10.2) and (6.10.1) tell us that $\{|G_n(0)|\}_{n=2}^{\infty}$ is bounded away from zero. Hence,

$$G(z) := \frac{1}{(1 - \overline{a}_1 z)} \cdot \prod_{k=2}^{\infty} \beta_k(z)$$

converges in K_B, $G \not\equiv 0$, yet $G(w_k) = 0$ for $k = 2, 3, \ldots$.□

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