A NOTE ON A THEOREM OF J. GLOBEVNIK

DMITRY KHAVINSON

In this note we offer a short proof of Globevnik's theorem [Gl1, Gl2] for simply connected domains. Yet, this idea alone is insufficient to give the proof of his result in its full generality in finitely connected domains [Gl2].

Theorem 1. *Let* $\mathbf{D}$ *be the unit disk,* $\mathbf{T}$ *is the circle,* $f \in C(\mathbf{T})$*. If for any function* g *in the disk-algebra* $A := A(\mathbf{D})$ *such that* $f + g \neq 0$ *on* $\mathbf{T}$ $\Delta_{\mathbf{T}}\, arg(f + g) \geq 0$*, then* f *extends analytically to* $\mathbf{D}$*.*

Proof.

Assume that f does not extend analytically to $\mathbf{D}$, then $dist(f, A) = d > 0$, where the distance is understood in the $C(\mathbf{T})$-metric. Of course, then for all rational functions f_n uniformly approximating f on $\mathbf{T}$ we can assume that $dist(f_n, A) = d_n > d/2$. Let g_n be the best $A(\mathbf{D})$ approximation to f_n. As is known from the rudiments of the theory of extremal problems as in, e.g., [Kh], [Du] (Chapter 8), g_n is analytic across $\mathbf{T}$, and $|g_n - f_n| = d_n > d/2 > 0$ everywhere on $\mathbf{T}$. Moreover, since f_n is continuous on $\mathbf{T}$, there is an extremal function F_n in the dual problem of finding

$$max\, |\int_{\mathbf{T}} f_n h\, dz|$$

over all h in the unit ball of the Hardy space H^1.

F_n is also analytic across T and

$$F_n(f_n - g_n)dz = |F_n|\, d_n\, |dz| \geq 0 \qquad \text{on} \qquad \mathbf{T}. \qquad (1)$$

Since $\Delta_{\mathbf{T}}\, arg\, F_n \geq 0$ and $\Delta_{\mathbf{T}}\, arg\, dz = 2\pi$, (1) implies that

$$\Delta_{\mathbf{T}}\, arg\,(f_n - g_n) \leq -2\pi \qquad (2)$$

for all sufficiently large n.

The rest is standard. Assume that f_n approximate f , so that $|f - f_n| < \varepsilon$ on $\mathbf{T}$ and $|f - g_n| = |f_n - g_n - (f_n - f)| \geq d_n - \varepsilon > 0$ for all n. Then,

$$\Delta_{\mathbf{T}}\, arg\,(f - g_n) = \Delta_{\mathbf{T}}\, arg\,[(f_n - g_n)(1 - \frac{f_n - f}{f_n - g_n})] = \Delta_{\mathbf{T}}\, arg\,(f_n - g_n) \leq -2\pi,$$

giving a desired contradiction.

Remarks. (i) The proof could still be shortened if one notices that the function $f_n - g_n$ is *badly approximable*, cf. [Ga] (p.177). Then, Poreda's theorem yields (2). (ii) This particular argument does not directly extend to multiply connected domains, the focus of the Globevnik's paper [Gl2]. There, $\Delta\, arg\, dz = 2 - k$, k being the number of the boundary components, and the latter number is ≤ 0 for all $k \geq 2$.

The author gratefully acknowledges support by the National Science Foundation.

References

[Du] P. Duren, *Theory of H^p Spaces*, Academic Press, 1970.

[Ga] J. Garnett, *Bounded Analytic Functions*, Academic Press, 1983.

[Gl1] J. Globevnik, *Holomorphic extendibility and the argument principle*,to appear in *Complex Analysis and Dynamical Systems II*, Proceedings of a conference held in honor of Professor Lawrence Zalcman's sixtieth birthday in Naharia, Israel, June 9-12,2003, Contemp. Math. [http://arxiv.org/abs/math/math.CV/0403446].

[Gl2] J. Globevnik, *The argument principle and holomorphic extendibility*, J. d'Analyse, to appear [http://arxiv.org/abs/math/math.CVCV/0405409].

[Kh] S.Ya. Khavinson, *Two papers on extremal problems in complex analysis*, AMS Translations, ser. 2, 1986, v. 129, 1-114

Department of Mathematical Sciences, University of Arkansas, Fayetteville, Arkansas 72701

E-mail address: dmitry@uark.edu