EXTREMAL DOMAINS FOR SELF-COMMUTATORS IN THE BERGMAN SPACE

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Abstract. In [10], the authors have shown that Putnam's inequality for the norm of self-commutators can be improved by a factor of $\frac{1}{2}$ for Toeplitz operators with analytic symbol φ acting on the Bergman space $A^2(\Omega)$. This improved upper bound is sharp when $\varphi(\Omega)$ is a disk. In this paper we show that disks are the only domains for which the upper bound is attained.

1. Introduction

Let H be a complex Hilbert Space with inner product $\langle\cdot,\cdot\rangle : H \times H \to \mathbb{C}$ and T be a bounded linear operator with adjoint T^*. Assume $[T^*, T] := T^*T - TT^* \geq 0$, i.e. T is hyponormal, then Putnam's inequality states

$$\|[T^*, T]\| \leq \frac{Area(sp(T))}{\pi},$$

where $sp(T)$ denotes the spectrum of T (cf. [2]).

In [8], it was proved that when H is the Hardy space, $H^2(\Omega)$, or the Smirnov space, $E^2(\Omega)$ (cf. [4, p.2 and p.173]), where Ω is a domain bounded by finitely many rectifiable curves, this inequality is sharp. Indeed, if we take $T = T_\varphi$ to be the Toeplitz operator with symbol φ analytic in a neighborhood of $\overline{\Omega}$, then by the Spectral Mapping Theorem (cf. [13, p.263]) $sp(T_\varphi) = \overline{\varphi(\Omega)}$, and the following lower bound holds:

$$\|[T_\varphi^*, T_\varphi]\| \geq \frac{4Area^2(\varphi(\Omega))}{\|\varphi'\|^2_{E^2(\Omega)} \cdot P(\Omega)}, \tag{1.1}$$

where $P(\Omega)$ denotes the perimeter of Ω.

Since $[T_\varphi^*, T_\varphi]$ is a positive operator, an interesting consequence follows from (1.1) by setting $\varphi(z) = z$, so that $\|\varphi'\|^2_{E^2(\Omega)} = \|1\|^2_{E^2(\Omega)} = P(\Omega)$, and combining (1.1) with Putnam's inequality, we obtain

$$P^2(\Omega) \geq 4\pi Area(\Omega), \tag{1.2}$$

which is the classical isoperimetric inequality. The equality in (1.2) holds if and only if Ω is a disk.

We are interested in exploring similar questions in a Bergman space setting. Recall that the Bergman space $A^2(\mathbb{D})$ is defined by:

$$A^2(\mathbb{D}) := \{f \in Hol(\mathbb{D}) : \int_{\mathbb{D}} |f(z)|^2 dA(z) < \infty\},$$

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where dA denotes the area measure on $\mathbb{D}$. $A^2(\Omega)$ is defined accordingly (cf. [5]). The orthogonal projection from $L^2(\mathbb{D}, dA) = L^2(\mathbb{D})$ onto $A^2(\mathbb{D})$ is called the Bergman projection and has integral representation

$$Pf(z) = \int_{\mathbb{D}} \frac{f(\omega)}{(1-\bar{\omega}z)^2} dA(\omega).$$

Recently, Bell, Ferguson, and Lundberg in [3] obtained a different lower bound for $[T_\varphi^*, T_\varphi]$ when T_φ acts on the Bergman space $A^2(\Omega)$. This lower bound turned out to be connected with the torsional rigidity of Ω (cf. [11, p.2]). Intuitively, if we imagine a cylindrical object with cross-section Ω, then the torsional rigidity quantifies the resistance to twisting. There are several equivalent definitions (cf. [11, pp.87-89]). The one used in [3] and [10] is the following:

Definition. If Ω is a simply connected domain, the torsional rigidity $\rho = \rho(\Omega)$ is

$$\rho = 2\int_\Omega \nu,$$

where ν is the unique solution to the Dirichlet problem

$$\begin{cases} \Delta\nu & = -2 \\ \nu|_{\partial\Omega} & = \ 0 \end{cases}.$$

In [3] it is shown that for T_z acting on $A^2(\Omega)$,

$$\|[T_z^*, T_z]\| \geq \frac{\rho(\Omega)}{Area(\Omega)}.$$

The authors also conjectured that in the Bergman space setting Putnam's inequality could be improved by a factor of $\frac{1}{2}$. This conjecture was recently proven by Olsen and Reguera in [10]. Combined with the lower bound given by Bell, Ferguson, and Lundberg, this yields a new proof of the St. Venant inequality

$$\rho(\Omega) \leq \frac{Area(\Omega)^2}{2\pi}.$$

In this note we show that in the Bergman space setting disks are the only extremal domains for which $\|T_z^*, T_z\|$ achieves it's maximal bound of $\frac{Area(\Omega)}{2\pi}$. More precisely, in section 2, we present a simple argument illustrating that the upper bound in Putnam's inequality can only be attained when Ω is a disk in any Hilbert space, while in the Bergman space setting $\|T_\varphi^*, T_\varphi\| = \frac{Area(\varphi(\Omega))}{2\pi} = \frac{1}{2}$ when $\varphi(\Omega) = \mathbb{D}$, the unit disk. In section 3, we give a sketch of Olsen and Reguera's proof of the improved upper bound in the Bergman setting. This is needed for our argument in section 4 where we show that the upper bound for $\|T_\varphi^*, T_\varphi\|$ is achieved if and only if $\varphi(\Omega)$ is a disk. This gives another proof of the well known fact that St. Venant's inequality becomes equality only for disks.

2. Non-sharpness of Putnam's Inequality

In this section, we illustrate why Putnam's inequality is not sharp in a Bergman space setting. We start with the following elementary Lemma found in [5, p.13].

Lemma 1. *Suppose $\omega = \varphi(z)$ maps a domain D conformally onto a domain Ω. Then the linear map $T(f) = g$ defined by*

$$g(z) = f(\varphi(z))\varphi'(z)$$

defines an isometry of $A^2(\Omega)$ *onto* $A^2(D)$.

Proof. That T is an isometry is clear from the fact that

$$\int_\Omega |f(w)|^2 dA(\omega) = \int_D |f(\varphi(z))|^2 |\varphi'(z)|^2 dA(z),$$

where $|\varphi'(z)|^2$ is the Jacobian of the conformal map φ.

To see that T is onto, let $g \in A^2(D)$ and let $z = \psi(w)$ be the inverse mapping. Then $f(\omega) = g(\psi(\omega))\psi'(\omega)$ is in $A^2(\Omega)$ and $T(f) = g$ since

$$T(f) = f(\varphi(z))\varphi'(z) = g(\psi(\varphi(z))\varphi'(\psi(\omega))\psi'(\omega),$$

and we can write $\psi'(\omega) = \frac{1}{\varphi'(\psi(\omega))}$, which is well defined on D because $\varphi'|_D \neq 0$. So $T(f) = g$ and T is onto as claimed. $\square$

The following statement is now straightforward.

Theorem 2. *Suppose* Ω *is a bounded Jordan domain and* $\varphi : \mathbb{D} \to \Omega$ *is a conformal mapping. Then*

$$\|[T_\varphi^*, T_\varphi]\|_{A^2(\mathbb{D})\to A^2(\mathbb{D})} = \|[T_z^*, T_z]\|_{A^2(\Omega)\to A^2(\Omega)}.$$

Proof. We start with the following straightforward calculation (cf. [2]). If we take $A_1^2(\mathbb{D})$ to be the unit ball of $A^2(\mathbb{D})$, we have that

$$\begin{aligned}
\|[T_\varphi^*, T_\varphi]\| &= \sup_{f\in A_1^2(\mathbb{D})} \langle [T_\varphi^*, T_\varphi]f, f\rangle \\
&= \sup_{f\in A_1^2(\mathbb{D})} \left(\|T_\varphi f\|^2_{A^2(\mathbb{D})} - \|T_\varphi^* f\|^2_{A^2(\mathbb{D})}\right) \\
&= \sup_{f\in A_1^2(\mathbb{D})} \left(\|\varphi f\|^2_{A^2(\mathbb{D})} - \|P(\bar\varphi f)\|^2_{A^2(\mathbb{D})}\right) \\
&= \sup_{f\in A_1^2(\mathbb{D})} \left(\|\varphi f\|^2_{L^2(\mathbb{D})} - \|P(\bar\varphi f)\|^2_{A^2(\mathbb{D})}\right).
\end{aligned}$$

Thus we have that

$$\begin{aligned}
\|[T_\varphi^*, T_\varphi]\| &= \sup_{g\in A_1^2(\mathbb{D})} \left(\|\varphi f\|^2_{L^2(\mathbb{D})} - \|P(\bar\varphi g)\|^2_{A^2(\mathbb{D})}\right) \\
&= \sup_{g\in A_1^2(\mathbb{D})} \{ \inf_{f\in A^2(\mathbb{D})} \{\|\bar\varphi g - f\|^2_{L^2(\mathbb{D})}\}\}.
\end{aligned}$$

Fixing $f, g \in A^2(\mathbb{D})$, with $g \in A_1^2(\mathbb{D})$, and letting $\psi = \varphi^{-1}$, we see that

$$\begin{aligned}
\|\bar\varphi g - f\|^2_{L^2(\mathbb{D})} &= \int_{\mathbb{D}} |\bar\varphi g - f|^2 dA \\
&= \int_\Omega |\bar z g(\psi(\omega)) - f(\psi(\omega))|^2 |\psi'(\omega)|^2 dA(\omega) \\
&= \int_\Omega |\bar z g(\psi(\omega))\psi'(\omega) - f(\psi(\omega))\psi'(\omega)|^2 dA(\omega).
\end{aligned}$$

By Lemma 1, $T(f) = f(\psi(\omega))\psi'(\omega)$ is a surjective isometry from $A^2(\mathbb{D})$ onto $A^2(\Omega)$. So, we have that

$$\sup_{g\in A_1^2(\mathbb{D})} \{ \inf_{f\in A^2(\mathbb{D})} \{\|\bar\varphi g - f\|^2_{L^2(\mathbb{D})}\}\} = \sup_{g\in A_1^2(\Omega)} \{ \inf_{f\in A^2(\Omega)} \{\|\bar z g - f\|^2_{L^2(\Omega)}\}\},$$

and the proof is complete. $\square$

This leads to the following interesting observation.

Theorem 3. *Let φ and Ω be as in Theorem 2. Then $\|[T_\varphi^*, T_\varphi]\|$ can only achieve the upper bound stated in Putnam's inequality (cf.* [12]*) if $\varphi(\mathbb{D})$ is a disk.*

Proof. We argue as follows. Let $A_1^2(\Omega)$ be the unit ball in $A^2(\Omega)$. By Theorem 2, we have that

$$\|[T_\varphi^*, T_\varphi]\|_{A^2(\mathbb{D})\to A^2(\mathbb{D})} = \|[T_z^*, T_z]\|_{A^2(\Omega)\to A^2(\Omega)} = \sup_{g\in A_1^2(\Omega)}\{\inf_{f\in A^2(\Omega)}\{\|\bar{z}g-f\|^2_{A^2(\Omega)}\}\}.$$

Fix $g \in A_1^2(\Omega)$, we have

$$\inf_{f\in A^2(\Omega)} \|\bar{z}g - f\|^2_{L^2(\Omega)} = \inf_{f\in A^2(\Omega)} \int_\Omega |\bar{z}g - f|^2 dA$$

$$\leq \inf_{h:\, gh\in A^2(\Omega)} \int_\Omega |\bar{z} - f|^2 |g|^2 dA \leq \inf_{h:\, gh\in A^2(\Omega)} \|\bar{z} - h\|^2_\infty$$

since $g \in A_1^2(\Omega)$. Further, since the polynomials $\mathcal{P}$ are dense in $H^\infty(\Omega)$ for any bounded Jordan domain Ω, and since for all $g \in A_1^2(\Omega)$, and all $p \in \mathcal{P}$, we have that $gp \in A^2(\Omega)$, we obtain from the last inequality that

$$\inf_{f\in A^2(\Omega)} \|\bar{z}g - f\|^2_{L^2(\Omega)} \leq \inf_{h\in R(\overline{\Omega})} \|\bar{z} - h\|^2_{L^\infty(\Omega)}$$

where $R(\overline{\Omega})$ is the uniform closure of the algebra of rational functions in Ω with poles outside $\overline{\Omega}$. In [1], Alexander proved that

$$\inf_{f\in R(\overline{\Omega})} \|\bar{z} - f\|_{L^\infty(\Omega)} \leq \sqrt{\frac{Area(\Omega)}{\pi}},$$

and further that equality is achieved if, and only if, Ω is a disk (cf. [2, 6]). The theorem now immediately follows. $\square$

Remark. If we take H to be any Hilbert space and T to be any subnormal operator with a rationally cyclic vector, then there is a positive finite Borel measure μ on $sp(T)$ such that T is unitarily equivalent to multiplication by z on $R^2(sp(T), \mu)$ which is the closure of $R(sp(T))$ in $L^2(sp(T), \mu)$ (cf. [2]). From this, repeating the above argument word for word, we obtain that if

$$\|[T^*, T]\| = \frac{Area(sp(T))}{\pi},$$

then $sp(T)$ must be a disk. The case when T does not have a rationally cyclic vector follows from the above case as in [2], so that the above theorem extends to all Hilbert spaces and any subnormal operator T.

The following example shows that the converse fails, and in particular fails for Bergman spaces.

Example 4. Let $\varphi(z) = z^k$ for some $k \in \mathbb{N}$, and let $T_\varphi : A^2(\mathbb{D}) \to A^2(\mathbb{D})$, and recall that $P : L^2(\mathbb{D}) \to A^2(\mathbb{D})$ is the orthogonal projection of $L^2(\mathbb{D})$ onto $A^2(\mathbb{D})$. As we showed in Theorem 2,

$$\|[T_\varphi^*, T_\varphi]\| = \sup_{g\in A_1^2(\mathbb{D})} \left(\|\varphi g\|^2_{L^2(\mathbb{D})} - \|P(\bar{\varphi}g)\|^2_{A^2(\mathbb{D})}\right).$$

Let $\psi_n(z) = (\frac{n+1}{\pi})^{\frac{1}{2}} z^n$, where $n = 0, 1, 2, \ldots$. The collection $\{\psi_n(z)\}_{n=0}^{\infty}$ forms an orthonormal basis for $A^2(\mathbb{D})$ (cf. [5, p. 11]). For $g \in A_1^2(\mathbb{D})$, we may write

$$g(z) = \sum_{n=0}^{\infty} \hat{g}(n)\psi_n(z),$$

where $\hat{g}(n) := \langle g, \psi_n\rangle$ and $\sum_{n=0}^{\infty} |\hat{g}(n)|^2 = 1$. Since we have an orthonormal basis at hand, we may calculate $P(\bar{\varphi}g)$ explicitly.

$$P(\bar{z}^k g) = \sum_{n=0}^{\infty} \langle \bar{z}^k g, \psi_n\rangle \psi_n.$$

Calculating $\langle \bar{z}^k g, \psi_n\rangle$, we find that

$$\langle \bar{z}^k g, \psi_n\rangle = \langle \bar{z}^k \sum_{m=0}^{\infty} \hat{g}(m)\psi_m, \psi_n\rangle,$$

where

$$\begin{aligned}\langle \bar{z}^k \hat{g}(m)\psi_m, \psi_n\rangle &= \int_{\mathbb{D}} \frac{\sqrt{(m+1)(n+1)}}{\pi} \hat{g}(m) z^m \bar{z}^{n+k} dA \\ &= \frac{\sqrt{(m+1)(n+1)}}{\pi} \hat{g}(m) \frac{2\pi}{m+n+k+2} \delta_{m,n+k}.\end{aligned}$$

Where $\delta_{i,j}$ is the Kronecker symbol. Thus,

$$\langle \bar{z}^k g, \psi_n\rangle = \left(\frac{n+1}{n+k+1}\right)^{\frac{1}{2}} \hat{g}(n+k),$$

and so we obtain that

$$\|P(\bar{z}^k g)\|^2_{A^2(\mathbb{D})} = \sum_{n=0}^{\infty} \frac{n+1}{n+k+1} |\hat{g}(n+k)|^2. \tag{2.1}$$

Similarly, when we calculate $\|z^k g\|^2_{A^2(\mathbb{D})}$, we find that

$$\begin{aligned}\langle z^k \hat{g}(m)\psi_m, \psi_n\rangle &= \int_{\mathbb{D}} \frac{\sqrt{(m+1)(n+1)}}{\pi} \hat{g}(m) z^{m+k} \bar{z}^{n} dA \\ &= \frac{\sqrt{(m+1)(n+1)}}{\pi} \hat{g}(m) \frac{2\pi}{m+n+k+2} \delta_{m+k,n}.\end{aligned}$$

Thus

$$\langle z^k g, \psi_n\rangle = \begin{cases} \sqrt{\frac{n-k+1}{n+1}} \hat{g}(n-k) & n \geq k \\ 0 & n < k \end{cases}.$$

Hence,

$$\|z^k g\|^2_{L^2(\mathbb{D})} = \sum_{n=k}^{\infty} \frac{n-k+1}{n+1} |\hat{g}(n-k)|^2. \tag{2.2}$$

Combining (2.1) and (2.2), we obtain that

$$\|[T_\varphi^*, T_\varphi]\| = \sup_{g \in A_1^2(\mathbb{D})} \{\sum_{n=k}^{\infty} \frac{n-k+1}{n+1} |\hat{g}(n-k)|^2 - \sum_{n=0}^{\infty} \frac{n+1}{n+k+1} |\hat{g}(n+k)|^2\}$$

$$\sup_{g \in A_1^2(\mathbb{D})} \{\sum_{n=0}^{k-1} \frac{n+1}{n+k+1} |\hat{g}(n)|^2 + \sum_{n=k}^{\infty} (\frac{n+1}{n+k+1} - \frac{n-k+1}{n+1}) |\hat{g}(n)|^2\}$$

$$\leq \sup_{g\in A^2_1(\mathbb{D})} \{\sum_{n=0}^{k-1} \frac{n+1}{n+k+1}|\hat{g}(n)|^2 + \sum_{n=k}^{\infty} \frac{k}{n+k+1}|\hat{g}(n)|^2\}$$

since

$$\frac{n+1}{n+k+1} - \frac{n-k+1}{n+1} \leq \frac{n+1}{n+k+1} - \frac{n-k+1}{n+k+1} = \frac{k}{n+k+1}, \qquad k \geq 0.$$

Further, since $\frac{n+1}{n+k+1} \leq \frac{k}{2k}$ for $0 \leq n \leq k-1$, we obtain that

$$\|[T^*_\varphi, T_\varphi]\| \leq \sup_{g\in A^2_1(\mathbb{D})} \frac{k}{2k} \sum_{n=0}^{\infty} |\hat{g}(n)|^2 = \frac{1}{2}.$$

This upper bound is achieved if we take $g = \psi_{k-1}$, so that $\|[T^*_\varphi, T_\varphi]\| = \frac{1}{2}$, whenever $\varphi(z) = z^k$ for any $k \in \mathbb{N}$. Thus, we see that the converse to Theorem 3 fails.

This calculation, independently done by T. Ferguson, leads to the conjecture, following Bell et. al. that in the Bergman space setting, Putnam's inequality can be improved by a factor of $\frac{1}{2}$. This conjecture was recently proven by Olsen and Reguera in [10]. In the following section we give a sketch of their proof, which will be needed in §4.

3. Sketch of the Olsen and Reguera proof

In their paper, Olsen and Reguera worked with the Hankel operator on $A^2(\mathbb{D})$ with symbol $\varphi \in L^2(\mathbb{D})$ defined by

$$H_\varphi(f) := (I-P)(\varphi f), \qquad f \in A^2(\mathbb{D}).$$

They then proved the following theorem.

Theorem 5. *Let $\varphi \in A^2(\mathbb{D})$ be in the Dirichlet space $\mathscr{D}$, i.e $\varphi' \in A^2(\mathbb{D})$. Then*

$$\|H_{\bar{\varphi}}\| \leq \frac{1}{\sqrt{2}}\|\varphi'\|_{A^2(\mathbb{D})}.$$

Proof. For the reader's convenience, we give here a sketch of their proof. For full details, cf. [10, §2]. For $f \in A^2(\mathbb{D})$,we write $f(z) = \sum_{n\geq 0} a_n z^n$, and without loss of generality we assume that $\|f\|_{A^2(\mathbb{D})} = 1$, and set $\varphi(z) = \sum_{k\geq 1} c_k z^k$ (we can also assume without loss of generality that $\varphi(0) = 0$). The basic strategy is to calculate $H_{\bar{\varphi}} f$ in terms of these Taylor coefficients and obtain the desired norm estimate by working directly with the coefficients. Crucial to our purposes is the fact that the only inequality used in [10] is the arithmetic-geometric inequality $ab \leq \frac{a^2+b^2}{2}$.

First, by computing $P(\bar{\varphi} z^n)$ for each n, we find that

$$H_{\bar{\varphi}} f = \bar{\varphi}(z) f(z) - P(\overline{\varphi} f)(z)$$

$$= \sum_{l\geq 0}\sum_{n\geq 0} \overline{c_l} a_n \overline{z}^l z^n - \sum_{n\geq 1}\sum_{k=0}^{n-1} \frac{k+1}{n+1} a_n \overline{c_{n-k}} z^k. \tag{3.1}$$

Then, after rewriting the above expression to take advantage of the orthogonality, we let $z = re^{i\theta}$ and integrate the modulus squared with respect to $\frac{d\theta}{\pi}$. This yields that $\|H_{\bar{\varphi}} f\|^2_{A^2(\mathbb{D})}$ is equal to

$$2\sum_{k\geq 1} r^{2k} |\sum_{n\geq 0} a_n \overline{c_{n+k}} r^{2n}|^2 + 2\sum_{k\geq 0} r^{2k} |\sum_{n\geq k+1} a_n \overline{c_{n-k}} (r^{2(n-k)} - \frac{k+1}{n+1})|^2. \tag{3.2}$$

This expression is once again rewritten and then integrated with respect to rdr. If we set

$$(I) := \sum_{n,m\geq 1, k\geq 0} \frac{a_n\overline{a_m}c_{k+m}\overline{c_{k+n}}}{n+m+k+1},$$
$$(II) := \sum_{k\geq 0}\sum_{n,m\geq k+1} \frac{a_n\overline{a_m}c_{m-k}\overline{c_{n-k}}(m-k)(n-k)}{(n+1)(m+1)(n+m-k+1)},$$

then we obtain that

$$\|H_{\bar\varphi}f\|^2_{A^2(\mathbb{D})} = (I) + (II).$$

Relabeling the indices slightly, and setting $a_n = b_{n+1}(n+1)$, we find that

$$(I) = \sum_{n,m\geq 1, k\geq 0} b_n\overline{b_m}c_{k+m}\overline{c_{k+n}}\frac{nm}{n+m+k},$$
$$(II) = \sum_{n,m,k\geq 1} b_{n+k}\overline{b_{m+k}}c_m\overline{c_n}\frac{mn}{n+m+k}.$$

Using the symmetry in m and n we may interpret each term as being half that of its real part so that the inequality $2Re(ab) \leq |a|^2 + |b|^2$ may be applied to each term of the above expressions, and this is the only place where inequalities occur, which yields

$$(I) \leq \sum_{n,m\geq 1, k\geq 0} \left(|b_nc_{k+m}|^2 + |b_mc_{k+n}|^2\right)\frac{nm}{2(n+m+k)} = \sum_{n,m\geq 1,k\geq 0} |b_nc_{k+m}|^2\frac{nm}{n+m+k} =: (I_*),$$
$$(II) \leq \sum_{n,m,k\geq 1} \left(|b_{n+k}c_m|^2 + |b_{m+k}c_n|^2\right)\frac{nm}{2(n+m+k)} = \sum_{n,m,k\geq 1} |b_{n+k}c_m|^2\frac{mn}{m+n+k} =: (II_*).$$

By changing the order of summation properly with the goal of isolating unique pairs of indices, we arrive at the expression

$$(I_*) + (II_*) = \sum_{n,m\geq 1} |b_n|^2|c_m|^2\frac{nm}{2}.$$

Finally, replacing $a_n = b_{n+1}(n+1)$, we now see that the right hand side exactly equals

$$\frac{1}{2}\sum_{n,m\geq 0} |b_n|^2|c_m|^2 mn = \frac{1}{2}\left(\sum_{n\geq 0}\frac{|a_n|^2}{n+1}\right)\left(\sum_{m\geq 1}|c_m|^2 m\right) = \frac{1}{2}\|f\|^2_{A^2(\mathbb{D})}\|\varphi'\|^2_{A^2(\mathbb{D})}.$$

which was to be shown. □

Remark 6. From here, the inequality

$$\|[T_\varphi^*, T_\varphi]\| \leq \frac{\|\varphi'\|^2_{A^2(\Omega)}}{2} \tag{3.3}$$

is seen as a corollary by showing that if ψ is the conformal map from Ω to $\mathbb{D}$, then

$$\|[T_\varphi^*, T_\varphi]\|_{A^2(\Omega)\to A^2(\Omega)} = \|H_{\bar\varphi}\|^2_{A^2(\Omega)\to L^2(\Omega)} = \|H_{\overline{\varphi\circ\psi}}\|^2_{A^2(\mathbb{D})\to L^2(\mathbb{D})},$$

and thus we can apply Theorem 5 and the result follows. Again, refer to [10] for more details. Taking $\varphi(z) = z$, and combining (3.3) with the result of Bell, Ferguson, and Lundberg, one arrives at a proof of the sharp St. Venant inequality

$$\rho(\Omega) \leq \frac{Area^2(\Omega)}{2\pi}. \tag{3.4}$$

It should be noted that when $\varphi = z^k$ many of the terms in (3.2) become zero resulting in the value we found in Example 4 of $\frac{1}{2}$ rather than the Olsen-Reguera upper bound of $\frac{k}{2}$.

4. Unique extremality of the Disk

We now show that from the proof of Theorem 5, we may deduce that equality is obtained in (3.3) and only if $\varphi(\Omega)$ is a disk. This will come as a corollary to the following Theorem.

Theorem 7. *Suppose $\varphi(z)$ is analytic in $\mathbb{D}$ such that $\varphi(z) \in \mathscr{D}$, the Dirichlet space. Further suppose that*

$$\|[T_\varphi^*, T_\varphi]\|_{A^2(\mathbb{D})\to A^2(\mathbb{D})} = \frac{\|\varphi'\|^2_{A^2(\Omega)}}{2}.$$

Then $\varphi(\mathbb{D})$ is a disk.

Proof. Since $\varphi \in \mathscr{D}$, $H_{\overline{\varphi}}$ is compact (cf.[14, §7.4, p.145]), and so attains its norm on $A_1^2(\mathbb{D})$. Recall from the proof of Theorem 2 that

$$\begin{aligned}\|[T_\varphi^*, T_\varphi]\|_{A^2(\mathbb{D})\to A^2(\mathbb{D})} &= \left(\sup_{f\in A_1^2(\mathbb{D})} \|\varphi f\|^2_{L^2(\mathbb{D})} - \|P(\bar{\varphi} f)\|^2_{A^2(\mathbb{D})}\right) \\ &= \|H_{\bar{\varphi}}\|^2_{A^2(\mathbb{D})\to L^2(\mathbb{D})}.\end{aligned}$$

We now examine the proof of Theorem 5 to find exactly when equality may happen. Recall that if $f \in A_1^2(\mathbb{D})$, then

$$\|H_{\bar{\varphi}} f\|^2_{A^2(\mathbb{D})} = (I) + (II) \le (I_*) + (II_*) = \frac{1}{2}\|f\|^2_{A^2(\mathbb{D})}\|\varphi'\|^2_{A^2(\mathbb{D})},$$

where (I), (II), (I_*), and (II_*) are as in Theorem 5. The only inequality at work here is $2Re(ab) \le |a|^2 + |b|^2$, where equality is achieved if, and only if, $a = \bar{b}$. Thus we find that equality is achieved if $(I) = (I_*)$ and $(II) = (II_*)$, which will only happen if the following infinite system of equations is satisfied:

$$b_i c_{j+k} = b_j c_{i+k} \qquad i, j \ge 1, k \ge 0, \tag{4.1}$$

$$b_{i+k} c_j = b_{j+k} c_k \qquad i, j, k \ge 1, \tag{4.2}$$

where $\varphi(z) = \sum_{k\ge1} c_k z^k$ is given and $f(z) = \sum_{n\ge1} n b_n z^{n-1}$ is an extremal function in $A_1^2(\mathbb{D})$ such that the above equations are satisfied.

It is clear that if $c_k = 0$ for all but a single k, that is if $\varphi(z) = cz^k$ then the above equations can be satisfied by a non-zero $f \in A_1^2(\mathbb{D})$. In fact, we know from Example 4 that if we take $f = \psi_{k-1}$, then (4.1) and (4.2) will be trivially satisfied. As we remarked above, in this case the formula (3.2) is oversimplified, so that the resulting norm is $\frac{c^2}{2}$ instead of our expected upper bound of $\frac{c^2 k}{2}$ It is also clear that the above equations are satisfied when $\varphi(z) = \sum_{k\ge1} r^k z^k$ for some $r < 1$. Here, the extremal $f = \frac{1}{\|\varphi\|_{A^2(\mathbb{D})}} \sum_{k\ge0} r^k z^k$. In both cases $\varphi(\mathbb{D})$ is a disk.

We will now show that for all other φ, (4.1) and (4.2) only hold for $f \equiv 0$. We will do this by looking at two cases.

First suppose that $\varphi(z)$ has at least two non-zero Taylor coefficients, c_m, c_n, with $m < n$, and at least one zero coefficient c_{k_0} such that $k_0 > n$. This encompasses all Taylor series which do not have an infinite non-zero tail. Without loss of generality we can assume that $k_0 = n+1$ by taking c_{k_0} to be the first zero coefficient after at least two non-zero coefficients. We now assume that we have found an $f \in A_1^2(\mathbb{D})$ whose Taylor coefficients satisfy (4.1) and (4.2). By (4.2), we have that

$$b_{n+k}c_m = b_{m+k}c_n \qquad k \geq 1, \tag{4.3}$$

$$b_{n+k+1}c_m = b_{m+k}c_{n+k+1} \qquad k \geq 1. \tag{4.4}$$

Hence, we can conclude that $b_j = 0$ for all $j \geq n+2$ by (4.4), which implies that $b_{m+k} = 0$ for all $k \geq 2$ by (4.3). We now let $i = m+1$, $j = m$ and choose k such that $m + k = n$. Then by (4.1) we have that

$$b_{m+1}c_{m+k} = b_{m+1}c_n = b_m c_{m+1+k} = b_m c_{n+1} = 0,$$

which, shows that $b_{m+1} = 0$.

Now choosing $i < m+1$, $j = m+1$ and choosing k such that $m+1+k = n$, then by (4.1) we have that

$$b_i c_{m+1+k} = b_i c_n = b_{m+1}c_{n+k} = 0.$$

Hence, we have that in fact $b_i = 0$ for all $i \geq 1$, which means that $f \equiv 0$.

Suppose now instead, that $\varphi(z)$ is such that its Taylor series does have an infinite non-zero tail, but the coefficients do not exhibit a geometric progression. This means that we can find three non-zero coefficients, c_m, c_{m+1}, and c_{m+2} such that

$$\frac{c_m}{c_{m+1}} \neq \frac{c_{m+1}}{c_{m+2}}. \tag{4.5}$$

By (4.2), we have that

$$b_{m+k}c_{m+1} = b_{m+1+k}c_m \qquad k \geq 1, \tag{4.6}$$

$$b_{m+1+k}c_{m+2} = b_{m+2+k}c_{m+1} \qquad k \geq 1. \tag{4.7}$$

In particular, choosing $k = 2$ in (4.6) and $k = 1$ in (4.7) we have that

$$b_{m+2}c_{m+1} = b_{m+3}c_m,$$

and

$$b_{m+2}c_{m+2} = b_{m+3}c_{m+1},$$

which by (4.5) means that $b_{m+2} = b_{m+3} = 0$. In fact, the same argument shows that $b_j = 0$ for all $j \geq m+2$. But then of course, by (4.6) we immediately get that $b_j = 0$ for all $j \geq m+1$. Now once again simply let $i < m+1$, $j = m+1$, and $k = 1$, and then by (4.1) we once again have that $b_i = 0$ for all $i \geq 1$, and so $f \equiv 0$. $\square$

Our result now follows as a corollary.

Corollary 8. $\|[T_z^*, T_z]\|_{A^2(\Omega)\to A^2(\Omega)} = \frac{Area(\Omega)}{2\pi}$ *if, and only if,* Ω *is a disk.*

Proof. By Theorem 2,

$$\|[T_z^*, T_z]\|_{A^2(\Omega)\to A^2(\Omega)} = \|[T_\varphi^*, T_\varphi]\|_{A^2(\mathbb{D})\to A^2(\mathbb{D})}$$

where φ is the conformal map from $\mathbb{D}$ onto the simply connected domain Ω. The corollary now immediately follows from Theorem 6. $\square$

Remark. Just as the results in [10] and [3] can be combined to give a new proof of the St. Venant inequality, corollary 8 gives another proof that equality in the St. Venant inequality characterizes disks.

5. Concluding Remarks

It must be noted that Olsen-Reguera proof only applies to Toeplitz operators whose symbol is in the Dirichlet space, and the upper bound is in terms of $\|\varphi'\|_{A^2}$ rather than the area of $\varphi(\mathbb{D})$. Example 4 however leads us to believe that the upper bound in terms of the area $\varphi(\mathbb{D})$ without multiplicity, i.e. the spectrum of T_φ is, perhaps, the correct one. If so, the result should be extendable to all Toeplitz operators with bounded analytic symbol. One of the consequences of the Olsen-Reguera result is a unique sharp upper bound on all extremal problems of the type

$$\sup_{u\in(A^2(\mathbb{D}))^\perp,\|u\|\leq 1} \left|\int_{\mathbb{D}} \bar{\varphi} u dA\right|$$

since by a standard duality argument, we have that

$$\inf_{f\in A^2(\mathbb{D})} \|\bar{\varphi}-f\|_{L^2(\mathbb{D})} = \sup\{\left|\int_{\mathbb{D}} \bar{\varphi} u dA\right| :\ u\in(A^2(\mathbb{D}))^\perp,\ \|u\|\leq 1\}$$

(cf. [7, §4]). We believe examining extremal problems of this type could very well lead to a proof of the Olsen-Reguera result which doesn't depend on power series calculations. It would be interesting to be able to determine the sharp "isoperimetric" bounds for these types of extremal problems, similar to Putnam's inequality and [9]. It would also be worthwhile to investigate finitely connected domains. While the result of Olsen-Reguera is certainly applicable, it would be nice to establish the deficiency based on the number of boundary components, as in [9]. Finally, it would be interesting to see which, if any, other so-called "isoperimetric sandwiches" could be expressed in terms of $[T^*, T]$ acting on other function spaces, e.g. the Dirichlet space.

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