ON POINT TO POINT REFLECTION OF HARMONIC FUNCTIONS ACROSS REAL-ANALYTIC HYPERSURFACES IN $\mathbb{R}^n$

By

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Abstract. Let Γ be a non-singular real-analytic hypersurface in some domain $U \subset \mathbb{R}^n$ and let $\mathrm{Har}_0(U,\Gamma)$ denote the linear space of harmonic functions in U that vanish on Γ. We seek a condition on $\mathbf{x}^0, \mathbf{x}^1 \in U \setminus \Gamma$ such that the reflection law

(RL) $\qquad u(\mathbf{x}^0) + Ku(\mathbf{x}^1) = 0, \qquad \forall u \in \mathrm{Har}_0(U,\Gamma)$

holds for some constant K. This is equivalent to the class $\mathrm{Har}_0(U,\Gamma)$ not separating the points $\mathbf{x}^0, \mathbf{x}^1$. We find that in odd-dimensional spaces (RL) *never* holds unless Γ is a sphere or a hyperplane, in which case there is a well known reflection generalizing the celebrated Schwarz reflection principle in two variables. In even-dimensional spaces the situation is different. We find a necessary and sufficient condition (denoted the SSR—strong Study reflection—condition), which we describe both analytically and geometrically, for (RL) to hold. This extends and complements previous work by e.g. P. R. Garabedian, H. Lewy, D. Khavinson and H. S. Shapiro.

0. Introduction

In this paper we study the following problem. Let Γ be a non-singular real-analytic hypersurface defined in some open set U in $\mathbb{R}^n$, i.e.

$$\Gamma = \{\mathbf{x} = (x_1, \dots, x_n) \in U : f(\mathbf{x}) = 0\}$$

where f is a real-analytic function in U with non-vanishing gradient on Γ. Consider the linear space $\mathrm{Har}_0(U,\Gamma)$ of harmonic functions in U that vanish on Γ. As is well known from results on elliptic regularity, all functions harmonic on "one side of Γ", i.e. in $U^+ = \{f(\mathbf{x}) > 0\}$, and vanishing on Γ extend into a fixed neighborhood $V^- \subset U^- = \{f(\mathbf{x}) < 0\}$ on "the other side of Γ" (also cf. [G], [L]). Now the question is whether there is any relationship between the values of such harmonic functions at a particular point in U^+ and the values at a certain other point in U^-. More precisely, given a point $\mathbf{x}^0 \in U^+$ does there exist another point $\mathbf{x}^1 \in U^-$ and a constant $K = K(\mathbf{x}^0, \Gamma)$ such that

$$u(\mathbf{x}^0) + Ku(\mathbf{x}^1) = 0 \tag{0.1}$$

for all $u \in \mathrm{Har}_0(U,\Gamma)$? In this paper we will make the neighborhood U small and investigate such a reflection law for points in a slightly smaller neighborhood $V \subset U$.

For $n = 2$ an affirmative answer is given by the celebrated Schwarz reflection principle, which actually claims much more (cf. e.g. [D], [Sh]).

Theorem 0.1 (Schwarz reflection principle) *Let $\Gamma \subset \mathbb{R}^2 \cong \mathbb{C}$ be a non-singular real-analytic curve in a neighborhood U of a point $z' = x' + iy'$. Then there exists a perhaps smaller neighborhood V of z' and an anti-conformal involution $R\colon V \mapsto V$ such that $R|_\Gamma = \mathrm{id}$ and*

$$u(z) + u(R(z)) = 0, \qquad z \in V \tag{0.2}$$

for all $u \in \mathrm{Har}_0(U, \Gamma)$.

Let us sketch here a proof (by no means the simplest!) blended from the arguments of Garabedian and Lewy ([G], [L]) that had grown out from an important idea of Study's [St] (see [D] and [Sh] for further discussion). This idea is the starting point of the present paper.

Sketch of proof We can write the equation of Γ in the form

$$\bar{z} = S(z),$$

where S is the so-called Schwarz function of Γ holomorphic in V, where $V \subset U$ is some neighborhood of z'. Shrinking V even further if necessary we can assume that S is also univalent in V. Imbed $\mathbb{R}^2$ into $\mathbb{C}^2 = \{(z, w)\colon z, w \in \mathbb{C}\}$ as the anti-holomorphic plane $\{w = \bar{z}\}$, so Γ is the intersection between the complex-analytic curve $\hat{\Gamma} = \{w = S(z)\}$ and $\mathbb{R}^2$. Fix a point $z^0 \in V \setminus \Gamma$ and let $u \in \mathrm{Har}_0(U, \Gamma)$. Then

$$u(z^0) = \frac{1}{2\pi} \int_{\gamma_\epsilon} \left(u(\zeta) \frac{\partial}{\partial n_\zeta} \log|z^0 - \zeta| - \frac{\partial u}{\partial n_\zeta} \log|z^0 - \zeta| \right) ds_\zeta, \tag{0.3}$$

where $\gamma_\epsilon = \{z \in \mathbb{C}\colon |z - z^0| = \epsilon\}$ is a sufficiently small circle centered at z^0, n_ζ is the outer normal and ds_ζ is the arclength. As is known, u extends as a holomorphic function of z and w into a certain fixed $\mathbb{C}^2$-neighborhood $\hat{U}$ of $\hat{\Gamma}$ which depends only on U and not on the function u. The form that we are integrating in (0.3) is a closed (multi-valued in $\mathbb{C}^2$!) 1-form that has a logarithmic singularity on the isotropic cone with vertex at the point z^0,

$$I_{z^0} = \{(z - z^0)(w - \bar{z}^0) = 0\}.$$

(I_{z^0} reduces to the two bicharacteristic complex lines $\{z = z^0\}$ and $\{w = \bar{z}^0\}$ passing through z^0.) Therefore we can move the contour γ_ϵ homotopically (not homologically due to the multi-valuedness) within $\hat{U} \setminus I_{z^0}$ without changing the value of the integral (0.3). Our first goal is to deform γ_ϵ to a contour on $\hat{\Gamma}$. For that purpose, note that γ_ϵ is homotopic in $\mathbb{C}^2 \setminus I_{z^0}$ to a contour γ_ϵ^1 that consists of two disjoint non-linked circles, each one surrounding the complex lines $\{z = z^0\}$ and $\{w = \bar{z}^0\}$ respectively, joined by a "handle". Indeed, $\mathbb{C}^2 \setminus I_{z^0}$ is homotopically equivalent to the three-sphere S^3 without two one-dimensional circles resulting from the intersection of I_{z^0} with S^3. Choosing a point on one of these circles and performing a stereographic projection with a pole at this point we obtain $\mathbb{R}^3$ with a line and a circle removed. The contour γ_ϵ becomes a loop tying them together. At this point it is easy to see that γ_ϵ is homotopic to the contour γ_ϵ^1 defined above. Having deformed γ_ϵ into γ_ϵ^1 we move γ_ϵ^1 continuously in $\mathbb{C}^2 \setminus I_{z^0}$ and lay it on $\hat{\Gamma}$. Thus, γ_ϵ in (0.3) can be replaced by a closed contour γ_ϵ^2 on $\hat{\Gamma}$ that consists of two small ϵ-circles σ_ϵ^1, σ_ϵ^2 surrounding the points $A = (S^{-1}(z^0), z^0)$ and $B = (z^0, S(z^0))$ and an arc σ_ϵ^3 joining σ_ϵ^1 and σ_ϵ^2 travelled twice. Since $u|_{\hat{\Gamma}} = 0$, the equation (0.3) can be written

$$u(z^0) = \frac{1}{4\pi i} \int_{\gamma_\epsilon^2} \log(z - z^0)(w - \bar{z}^0) \left(\frac{\partial u}{\partial z} dz - \frac{\partial u}{\partial w} dw \right). \tag{0.4}$$

Now observe that as $\epsilon \to 0$ the integrals in (0.4) over the circles σ_ϵ^1 and σ_ϵ^2 tend to zero, while that along σ_ϵ^3 tends to

$$u(z^0) = \frac{1}{2} \int_\gamma \left(\frac{\partial u}{\partial z} dz - \frac{\partial u}{\partial w} dw \right), \tag{0.5}$$

where $\gamma \subset \hat{\Gamma}$ is an arc joining the points A and B (from A to B). The equation (0.5) follows from (0.4) since the logarithm in (0.4) on "one side of σ_ϵ^3" differs from that on "the other side" by $2\pi i$. Finally applying the same argument to the point $z^1 = \overline{S(z^0)}$ we obtain

$$u(z^1) = \frac{1}{2} \int_{-\gamma} \left(\frac{\partial u}{\partial z} dz - \frac{\partial u}{\partial w} dw \right),$$

and the theorem follows with $R(z) = \overline{S(z)}$. □

Remark In the next section we make a similar deformation to the one of the contour γ_ϵ in an arbitrary number of dimensions. However, we do this in another way which yields more information about the intersection between $\hat{\Gamma}$ and the isotropic cone (in the above case, that intersection consists of only two points and, thus, is very simple).

Theorem 0.1 does not extend to higher dimensions unless Γ is a sphere or a hyperplane since any mapping preserving harmonic functions must be either conformal or anti-conformal [KS] and those mappings are extremely limited in $\mathbb{R}^n$, $n \geq 3$, in view of the Liouville theorem. (If Γ is a hyperplane, say $\{x_n = 0\}$, then

(0.2) holds with $R(\mathbf{x}) = (x_1, ..., x_{n-1}, -x_n)$, while for a sphere $\Gamma = \{|\mathbf{x}| = \rho\}$ the equation (0.1) holds with $\mathbf{x}^1 = R(\mathbf{x}^0) = \rho^2\mathbf{x}^0/|\mathbf{x}^0|^2$—the Kelvin transformation—and $K = |\mathbf{x}^0|^{2-n}$, cf. e.g. [Ke]). Recall (cf. the proof of Theorem 0.1 above; also, cf. [Stu] and [Sh]) that, for $n = 2$, the pairing between the points $\mathbf{x}^0 = z$ and $\mathbf{x}^1 = R(z)$ in Theorem 0.1 above is such that the isotropic cones (see below) emanating from the points $\mathbf{x}^0$ and $\mathbf{x}^1$ "meet" on the complexification $\hat{\Gamma}$ of Γ, i.e. on $\hat{\Gamma} = \{\mathbf{z} \in \mathbb{C}^n : f(\mathbf{z}) = 0\}$. It is therefore quite natural to conjecture that for (0.1) to hold in higher dimensions it is necessary that the isotropic cones

$$I_{\mathbf{x}^k} = \left\{\mathbf{z} \in \mathbb{C}^n : \sum_{j=1}^{n} (z_j - x_j^k)^2 = 0\right\},$$

for $k = 0, 1$, satisfy

$$I_{\mathbf{x}^0} \cap \hat{\Gamma} = I_{\mathbf{x}^1} \cap \hat{\Gamma} \tag{0.6}$$

(the Study relation, denoted SR in §3). In fact, if we enlarge the class of test functions $\mathrm{Har}_0(U, \Gamma)$ and allow polar singularities near Γ, the necessity of this condition is easy to prove [KS]. On the other hand, as it was shown in [KS], if for sufficiently many (e.g. a set of positive measure) points $\mathbf{x}^0$ near Γ we can find a Study related point $\mathbf{x}^1$ (i.e. such that (0.6) holds) then Γ must be a hyperplane or a sphere, provided that Γ is algebraic. Thus, we can expect the reflection law (0.1) to hold only for very special points $\mathbf{x}^0$, $\mathbf{x}^1$ near a generic analytic hypersurface Γ.

The reflection law (0.1) is equivalent to the separation of points question for the class $\mathrm{Har}_0(U, \Gamma)$. Indeed, we have the following simple proposition.

Proposition 0.1 *Let Γ, U, $\mathbf{x}^0$, $\mathbf{x}^1$ be as above. The following are equivalent:*
(a) *there is no constant K for which* (0.1) *holds for all $u \in \mathrm{Har}_0(U, \Gamma)$;*
(b) *there is a function $u \in \mathrm{Har}_0(U, \Gamma)$ such that*

$$\begin{cases} u(\mathbf{x}^0) &= 1, \\ u(\mathbf{x}^1) &= 0. \end{cases}$$

Proof The implication (b)⇒(a) is obvious. Assume that (a) holds. Then there are two functions $u_1, u_2 \in \mathrm{Har}_0(U, \Gamma)$ such that neither function vanishes at both points and such that

$$u_1(\mathbf{x}^0) + K_1 u_1(\mathbf{x}^1) = u_2(\mathbf{x}^0) + K_2 u_2(\mathbf{x}^1) = 0,$$

where $K_1 \neq K_2$. If one of them, say u_1, is zero at $\mathbf{x}^1$ then we are finished, because we can take $u = u_1/u_1(\mathbf{x}^0)$, so let us assume that neither function vanishes at $\mathbf{x}^1$. Then, as it is straightforward to verify, the function

$$u = \frac{1}{K_2 - K_1}\left(\frac{u_1}{u_1(\mathbf{x}^1)} - \frac{u_2}{u_2(\mathbf{x}^1)}\right)$$

satisfies

$$\begin{cases} u(\mathbf{x}^0) &= 1, \\ u(\mathbf{x}^1) &= 0. \end{cases}$$

This proves the implication (a)⇒(b). □

In [KS] it was shown that the functions in $\mathrm{Har}_0(U,\Gamma)$ separate points in $U \setminus \Gamma$ whenever Γ is a cylindrical, non-flat surface in $\mathbb{R}^3$. (Also, cf. the treatment of circular cylinders in [Sh].) In Section 2 we extend this result to all surfaces in $\mathbb{R}^3$ by showing that whenever (0.1) holds for a pair of points $\mathbf{x}^0$, $\mathbf{x}^1$ near Γ then Γ must be either a plane or a sphere (Theorem 2.1). In Section 5, Theorem 5.1, this is proven for arbitrary odd-dimensional spaces. Thus, in all odd-dimensional spaces the reflection (0.1) is strictly limited to the reflection in a plane and Kelvin transformation in a sphere for all pairs of points $\mathbf{x}^0$, $\mathbf{x}^1$. The situation for even-dimensional spaces is much more delicate and (not surprisingly) is reminiscent of the situation with the celebrated Huygens principle. In [Kh], (0.1) was shown to hold in $\mathbb{R}^4$ for axially symmetric surfaces and points $\mathbf{x}^0$, $\mathbf{x}^1$ on the axis of symmetry satisfying the Study relation (0.6). In view of this, it was suggested in [Kh] that (0.6) is, perhaps, a necessary and sufficient condition for (0.1) in even-dimensional spaces, as it is in $\mathbb{R}^2$ (cf. also the discussion in [KS]). However, it turns out that for all even $n \geq 4$ one needs a stronger condition (denoted the SSR—strong Study reflection—condition in this paper) which is necessary and sufficient for the reflection (0.1). Moreover, the SSR condition turns out to become more and more restrictive as the dimension n of the space increases.

Let us briefly describe the contents of this paper. In Section 1 we present the topological construction allowing us to represent the values of functions in $\mathrm{Har}_0(U,\Gamma)$ by integrals over certain cycles on $\hat{\Gamma}$ (the idea of doing this goes back to Garabedian [G]). In Section 2 we prove that no point to point reflection exists in $\mathbb{R}^3$ except across spheres and hyperplanes. In Section 3 we introduce the SSR condition in $\mathbb{R}^4$ and show that this condition is necessary and sufficient for the reflection principle to hold. In Section 4 we extend this to all even-dimensional spaces. Finally, in Section 5, using Hadamard's method of descent, we show that there is no reflection law in any odd-dimensional space. Thus, in a sense, we prove our major theorems twice. First for dimensions 3 and 4 and then for arbitrary odd- and even-dimensional spaces. We hope that this way of presenting the material will help to clarify the crucial points of the argument since the complexity and tediousness of the problem do increase with the dimensionality.

1. Some topology and notations

Let as above Γ be a nonsingular, real-analytic hypersurface through the origin in some neighborhood U of the origin in $\mathbb{R}^n$, $\Gamma = \{\mathbf{x} \in U : f(\mathbf{x}) = 0\}$. First observe that there is no loss of generality in assuming that the hyperplane $\{x_n = 0\}$ is tangent to Γ, because this paper deals with properties of harmonic functions in U and the class of harmonic functions is invariant under rotations. Consequently, in some smaller neighborhood, which we still denote by U, we may write

$$\Gamma = \{\mathbf{x} \in U : x_n = \phi(\mathbf{x}')\}$$

where $\mathbf{x}' = (x_1, ..., x_{n-1})$ and ϕ is some real-valued real-analytic function of $n-1$ variables with

$$\nabla\phi(\mathbf{0}) = \left(\frac{\partial\phi}{\partial x_1}, ..., \frac{\partial\phi}{\partial x_{n-1}}\right)(\mathbf{0}) = (0, ..., 0).$$

In the context of this paper there is no loss of generality in replacing U by a smaller neighborhood. In order to simplify the notation in certain formulas, we will use the notation

$$\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{j=1}^{k} a_j b_j,$$

where $\mathbf{a} = (a_1, ..., a_k)$ and $\mathbf{b} = (b_1, ..., b_k)$ are real or complex k dimensional vectors. The dimension k should be clear from the context. For real vectors, this expression coincides with the usual Euclidian scalar product and then we will use the notation $|\mathbf{a}|^2 = \langle \mathbf{a}, \mathbf{a} \rangle$ for the Euclidian norm. We will also use the convention that if $\mathbf{a} = (a_1, ..., a_k)$ is a given vector then $\mathbf{a}'$ denotes the vector $(a_1, ..., a_{k-1})$.

If $N(\mathbf{x})$ denotes the Newtonian potential of a point mass at the origin in $\mathbb{R}^n$, i.e.

$$N(\mathbf{x}) = \frac{c_n}{|\mathbf{x}|^{n-2}}$$

where c_n is some constant depending on the dimension n, then the value of any harmonic function u at a point $\mathbf{x}^0$ can be represented as

$$(1.1) \quad u(\mathbf{x}^0) = \int_{S^{n-1}(\mathbf{x}^0,\epsilon)} \sum_{j=1}^{n} (-1)^{j+1} \left(u(\mathbf{x}) \frac{\partial N}{\partial x_j}(\mathbf{x} - \mathbf{x}^0) - \frac{\partial u}{\partial x_j}(\mathbf{x}) N(\mathbf{x} - \mathbf{x}^0) \right) \omega_j,$$

where $S^{n-1}(\mathbf{x}^0, \epsilon)$ is the $n-1$ dimensional sphere of radius ϵ, for some sufficiently small $\epsilon > 0$, centered at $\mathbf{x}^0$ oriented by the outward normal and

$$\omega_j = dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n,$$

where $\widehat{dx_j}$ means omission of dx_j. Let us imbed $\mathbb{R}^n$ in $\mathbb{C}^n$ by using the coordinates

$$\mathbf{z} = (z_1, \ldots, z_n) = (x_1 + iy_1, \ldots, x_n + iy_n)$$

in $\mathbb{C}^n$ and letting $\mathbb{R}^n = \{\mathbf{z} \colon |\mathbf{y}| = 0\}$. Since ϕ is real-analytic near the origin in $\mathbb{R}^{n-1}$ it is an analytic function of the variables $\mathbf{z}'$ in some neighborhood of the origin in $\mathbb{C}^{n-1}$. We will use the notation $\hat{\nabla}$ for the complex gradient, i.e. the vector $(\partial/\partial z_1, \ldots, \partial/\partial z_k)$. The dimension k of this vector should be clear from the context, e.g. if we apply $\hat{\nabla}$ to ϕ then $k = n - 1$. We define $\hat{\Gamma}$, the complexification of Γ, as $\{\mathbf{z} \colon z_n = \phi(\mathbf{z}')\}$. This is an analytic hypersurface in some neighborhood $\hat{U}$ of the origin in $\mathbb{C}^n$ such that its restriction to $\mathbb{R}^n$ equals Γ. We choose $\hat{U}$ such that every harmonic function in $U = \hat{U} \cap \mathbb{R}^n$ extends as an analytic function into $\hat{U}$, i.e. such that $\hat{U}$ is the so-called hull of harmonicity of U.

Now, the Newtonian potential N extends as an analytic function, multi-valued if n is odd, into $\mathbb{C}^n \setminus I_{\mathbf{0}}$, where $I_{\mathbf{z}^0}$ denotes the isotropic cone with vertex at $\mathbf{z}^0$

$$I_{\mathbf{z}^0} = \{\mathbf{z} \colon \langle \mathbf{z} - \mathbf{z}^0, \mathbf{z} - \mathbf{z}^0 \rangle = 0\}$$

and, hence, the form we are integrating in (1.1) extends as a (multi-valued if n is odd) form into $\hat{U} \setminus I_{\mathbf{x}^0}$. Since $u(\mathbf{z})$ is harmonic in $\hat{U}$, i.e. satisfies

$$\Delta u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial z_j^2} = 0$$

and $N(\mathbf{z} - \mathbf{x}^0)$ is harmonic in $\mathbb{C}^n \setminus I_{\mathbf{x}^0}$ it follows that this form is closed. If n is even then the integral in (1.1) is independent of the cycle of integration as long as the cycle stays in the same homology class. If n is odd then the form we are integrating is single-valued in a neighborhood of $S^{n-1}(\mathbf{x}^0, \epsilon)$ and, therefore, the integral in (1.1) is independent of the cycle of integration as long as this cycle stays in the same homotopy class, in $\hat{U} \setminus I_{\mathbf{x}^0}$, as $S^{n-1}(\mathbf{x}^0, \epsilon)$. Note, however, that due to the multi-valuedness of the form the integral is *not* independent of the cycle if we only restrict it to the same *homology* class. Our main goal in this section is to prove that $S^{n-1}(\mathbf{x}^0, \epsilon)$ can be deformed continuously in $\hat{U} \setminus I_{\mathbf{x}^0}$ to a cycle surrounding (in a certain sense) part of the intersection $\hat{\Gamma} \cap I_{\mathbf{x}^0}$. This will enable us to replace the cycle of integration in (1.1) by one which is close to $\hat{\Gamma}$ and this, in turn, will enable us to "compare" different values $u(\mathbf{x}^0)$ and $u(\mathbf{x}^1)$. The precise statements of the topological results needed are presented in Lemmas 1.1, 1.2, and 1.3 below.

Let us write

$$\phi(\mathbf{z}') = \xi(\mathbf{z}') + i\eta(\mathbf{z}'),$$

where ξ and η are real-valued functions. Since $\hat{\nabla}\phi$ vanishes at the origin and the restriction of ϕ to $\mathbb{R}^{n-1}$ is real-valued, the coefficients in the Taylor expansion of ϕ at the origin are all real and the expansion starts with the quadratic terms. Consequently, we can write

$$\eta(\mathbf{x}' + i\mathbf{y}') = \langle \mathbf{y}', \mathbf{h}(\mathbf{x}' + i\mathbf{y}')\rangle = y_1 h_1(\mathbf{x}' + i\mathbf{y}') + \cdots + y_{n-1}h_{n-1}(\mathbf{x}' + i\mathbf{y}').$$

The functions $\mathbf{h} = (h_1, ..., h_{n-1})$ are not uniquely determined in general, but we can make them unique e.g. by demanding that h_j, for $j = 2, ..., n-1$, is independent of $y_1, ..., y_{j-1}$ (we let $y_1 h_1(\mathbf{x}' + i\mathbf{y}')$ consist of all terms in the Taylor expansion containing y_1, $y_2 h_2(\mathbf{x}' + i\mathbf{y}')$ consist of all remaining terms containing y_2, etc.).

Lemma 1.1 *There is a neighborhood V of the origin in $\mathbb{R}^n$, with $\overline{V} \subset \hat{U} \cap \mathbb{R}^n$, such that the following is true for all $\mathbf{x}^0 \in V \setminus \Gamma$:*

(a) *the intersection of $I_{\mathbf{x}^0}$ and the n dimensional manifold M defined by the equations*

$$\text{(1.2)} \qquad \begin{cases} \mathbf{x}' &= \mathbf{x}^{0\prime} + (x_n^0 - \xi)\mathbf{h} \\ x_n &= \xi \end{cases}$$

is a smooth $n-2$ dimensional manifold $\gamma_{\mathbf{x}^0}$, homeomorphic to the $n-2$ dimensional sphere, contained in $M \cap \hat{\Gamma}$, i.e. in the $n-1$ dimensional submanifold of M satisfying the equation

$$y_n = \eta;$$

(b) *the sphere $S^{n-1}(\mathbf{x}^0, \epsilon)$ is homotopic in $\hat{U} \backslash I_{\mathbf{x}^0}$ to the $n-1$ dimensional boundary $C_{\mathbf{x}^0}$ of a contractible neighborhood of $\gamma_{\mathbf{x}^0}$ in M.*

Remark Note that the first $n-1$ equations defining M are implicit and, hence, we have to make sure that we choose $\hat{U}$ and V so small, a priori, that M, defined in the lemma, is a manifold in $\hat{U}$.

Proof First, fix some $\mathbf{x}^0 \in \mathbb{R}^n \setminus \Gamma$, close to the origin. We will see how close, i.e. how small we have to make V, eventually. The idea of the proof is to deform $S^{n-1}(\mathbf{x}^0, \epsilon)$ along a family of n-planes starting with $\mathbb{R}^n$ and ending with one that approximates M, and then along a family of n-manifolds ending with M, all the while keeping control over the intersection between $I_{\mathbf{x}^0}$ and the n dimensional planes/manifolds.

As a first step, we consider the following continuous family of n dimensional planes Π_t, for $t \in [0, 1]$, in $\mathbb{C}^n$:

$$\begin{cases} (1-t)\mathbf{y}' &= t(\mathbf{x}' - \mathbf{x}^{0\prime}), \\ (1-t)y_n &= tx_n. \end{cases}$$

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$$\begin{cases} \mathbf{x}' = \mathbf{x}^{0\prime}, \\ x_n = 0, \end{cases}$$

which has the subspace $\{y_n = 0\}$ in common with $\{z_n = 0\}$, the tangent plane of $\hat{\Gamma}$. We rewrite the equation of $I_{\mathbf{x}^0}$ in terms of real variables:

$$(1.3) \qquad \begin{cases} |\mathbf{x} - \mathbf{x}^0|^2 = |\mathbf{y}|^2, \\ \langle \mathbf{x} - \mathbf{x}^0, \mathbf{y} \rangle = 0. \end{cases}$$

If we use the coordinates $\mathbf{x}$ on Π_t (this works well for $t \le 1/2$) then the intersection $\Pi_t \cap I_{\mathbf{x}^0}$ becomes

$$\begin{cases} |\mathbf{x} - \mathbf{x}^0|^2 = \dfrac{t^2}{(1-t)^2}(|\mathbf{x}' - \mathbf{x}^{0\prime}|^2 + x_n^2), \\ \dfrac{t^2}{(1-t)^2}(|\mathbf{x}' - \mathbf{x}^{0\prime}|^2 + (x_n - x_n^0)x_n) = 0. \end{cases}$$

Let us first assume that $t < 1/2$. A straightforward calculation shows that the latter can be written as

$$\begin{cases} |\mathbf{x}' - \mathbf{x}^{0\prime}|^2 + \left(x_n - \dfrac{(1-t)^2}{1-2t}x_n^0\right)^2 = \dfrac{t^2(1-t)^2}{(1-2t)^2}(x_n^0)^2, \\ |\mathbf{x}' - \mathbf{x}^{0\prime}|^2 + \left(x_n - \dfrac{1}{2}x_n^0\right)^2 = \dfrac{1}{4}(x_n^0)^2. \end{cases}$$

The second equation describes a fixed sphere of radius $x_n^0/2$ centered at $(\mathbf{x}^{0\prime}, x_n^0/2)$ in the moving plane Π_t. If we choose V so small that this sphere is contained in $\hat{U}$ for $t < 1/2$, then it is clear that we can move the sphere $S^{n-1}(\mathbf{x}^0, \epsilon)$ homotopically along with the planes Π_t such that this moving cycle—let us denote it by S_t—encloses the intersection $\Pi_t \cap I_{\mathbf{x}^0}$. For $t = 1/2$ the equation of the intersection can be written as

$$\begin{cases} x_n = \frac{1}{2}x_n^0, \\ |\mathbf{x}' - \mathbf{x}^{0\prime}|^2 + \left(x_n - \frac{1}{2}x_n^0\right)^2 = \frac{1}{4}(x_n^0)^2, \end{cases}$$

so we can move S_t all the way to $\Pi_{1/2}$. A similar argument for $t > 1/2$ (that we leave to the reader), using instead the coordinates $\mathbf{y}$ on Π_t, shows that we can move the cycle S_t all the way to a cycle S_1 in Π_1 that encloses the intersection $\Pi_1 \cap I_{\mathbf{x}^0}$. The latter is given by the equations

$$\begin{cases} |\mathbf{y}|^2 = (x_n^0)^2, \\ x_n^0 y_n = 0. \end{cases}$$

Next, we move the cycle with the n dimensional planes Π_{1+t}, for $t \in [0,1]$, defined by

$$\begin{cases} \mathbf{x}' = \mathbf{x}^{0\prime}, \\ x_n = t\xi(\mathbf{x}^{0\prime}). \end{cases}$$

The intersection $\Pi_{1+t} \cap I_{\mathbf{x}^0}$ is given by

$$\begin{cases} |\mathbf{y}|^2 = (t\xi(\mathbf{x}^{0\prime}) - x_n^0)^2, \\ (t\xi(\mathbf{x}^{0\prime}) - x_n^0)y_n = 0. \end{cases}$$

The same arguments as above show that we can move the cycle S_1 to a cycle S_2 in Π_2 surrounding the intersection

$$\begin{cases} |\mathbf{y}|^2 = (\xi(\mathbf{x}^{0\prime}) - x_n^0)^2, \\ (\xi(\mathbf{x}^{0\prime}) - x_n^0)y_n = 0, \end{cases}$$

which, since $\mathbf{x}^0 \notin \Gamma$ implies $\xi(\mathbf{x}^{0\prime}) \neq x_n^0$, can be written as

$$\begin{cases} |\mathbf{y}|^2 = (\xi(\mathbf{x}^{0\prime}) - x_n^0)^2, \\ y_n = 0, \end{cases}$$

i.e. a circle of radius $|\xi(\mathbf{x}^{0\prime}) - x_n^0|$ centered at the origin in the plane $y_n = 0$.

Now, let us consider the manifold M defined by (1.2). By the implicit function theorem, there is a neighborhood W of the origin in $\mathbb{R}^{n-1}$, a neighborhood V of the origin in $\mathbb{R}^n$, and functions $\mathbf{f} = (f_1, \ldots, f_n)$ independent of y_n such that M can be written as a graph in $\mathbb{C}^n \cong \mathbb{R}^{2n}$ over Π_2 as follows (note that according to our convention $\mathbf{f}'$ denotes the vector $(f_1, \ldots, f_{n-1})$ and not the derivative):

$$\begin{cases} \mathbf{x}' = \mathbf{x}^{0\prime} + \mathbf{f}'(\mathbf{y}'; \mathbf{x}^0), \\ x_n = \xi(\mathbf{x}^{0\prime}) + f_n(\mathbf{y}'; \mathbf{x}^0), \end{cases}$$

for $(\mathbf{y}', y_n) \in W \times \mathbb{R}$ and $\mathbf{x}^0 \in V$. Moreover, we have $\mathbf{f}(\mathbf{0};\mathbf{0}) = \mathbf{0}$. For the final step in the deformation of the cycle, we consider the continuous family of n dimensional manifolds M_{2+t}, for $t \in [0,1]$,

$$\begin{cases} \mathbf{x}' = \mathbf{x}^{0\prime} + t\mathbf{f}'(\mathbf{y}'; \mathbf{x}^0), \\ x_n = \xi(\mathbf{x}^{0\prime}) + tf_n(\mathbf{y}'; \mathbf{x}^0), \end{cases} \tag{1.4}$$

which moves from Π_2, at $t = 0$, to $M = M_3$, at $t = 1$. To control the intersection with $I_{\mathbf{x}^0}$ we need the following transversality results.

Assertion 1.1 *If B is a relatively compact domain in $\mathbb{C}^n \cong \mathbb{R}^{2n}$ then any manifold N in B that satisfies the following two conditions,* (i) *and* (ii), *meets $I_{\mathbf{x}_0}$ transversally.*

(i) $\mathbf{x}^0 \notin N \cap I_{\mathbf{x}^0}$;

(ii) *any unit conorm*
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To verify condi

(i) $\mathbf{x}^0 \notin N \cap I_{\mathbf{x}^0}$;

(ii) *any unit conormal (i.e. unit covector that annihilates the tangent space)* $\mathbf{r} \in (\mathbb{C}^n)^*$, $\mathbf{r} = \mathbf{u} + i\mathbf{v} = (u_1 + iv_1, \ldots, u_n + iv_n)$, *of N is such that* $|\mathbf{v}| < 1/\sqrt{2}$.

Proof of Assertion 1.1 That two manifolds N and N' in $\mathbb{R}^{2n}$ meet transversally means that the vector sum of the two tangent spaces at each point of intersection equals $\mathbb{R}^{2n}$. This is the same as saying that the intersection of the conormal spaces, at each point of intersection, contains only the zero covector which, in turn, is equivalent to the sets of unit conormals being disjoint. The hypersurface $I_{\mathbf{x}^0}$ has a singularity at $\mathbf{z} = \mathbf{x}^0$, but away from that point it is a manifold. Hypothesis (i) states that $\mathbf{x}^0$ is not in the intersection $N \cap I_{\mathbf{x}^0}$. The proof will be completed, in view of Hypothesis (ii), by showing that any unit conormal $\mathbf{r} = \mathbf{u} + i\mathbf{v}$ of $I_{\mathbf{x}^0} \setminus \{\mathbf{x}^0\}$ is such that $|\mathbf{v}| = 1/\sqrt{2}$. Since translations do not alter the set of unit conormals, we may assume that $\mathbf{x}^0 = 0$. Let us, just in this proof and for simplicity of notation, identify the cotangent space $(\mathbb{C}^n)^*$ with $\mathbb{R}^{2n}$ by saying that the point $\mathbf{r} = \mathbf{u} + i\mathbf{v} \in (\mathbb{C}^n)^*$ corresponds to $(u, v) \in \mathbb{R}^{2n}$ (this differs from the usual orientation by a factor $(-1)^n$). It follows immediately from the two real equations defining I_0 that any unit conormal in $\mathbb{R}^{2n}$ of I_0 at a point $\mathbf{z} = \mathbf{x} + i\mathbf{y}$ is of the form $(\mathbf{u}, \mathbf{v}) = a(\mathbf{x}, -\mathbf{y}) + b(\mathbf{y}, \mathbf{x})$, where $a, b \in \mathbb{R}$ are such that the vector has unit norm. Thus, $|\mathbf{v}|^2 = a^2|\mathbf{y}|^2 + b^2|\mathbf{x}^2|$ and since $\mathbf{z} \in I_0$ we get $|\mathbf{v}|^2 = (a^2 + b^2)|\mathbf{x}|^2$. The same holds for $|\mathbf{u}|^2$. Consequently, since $(\mathbf{u}, \mathbf{v})$ has unit norm, $|\mathbf{v}| = 1/\sqrt{2}$. □

Assertion 1.2 *If we choose $\hat{U}$ and V sufficiently small then the manifolds M_{2+t}, for every $t \in [0, 1]$, meet $I_{\mathbf{x}^0}$ transversally in $\hat{U}$ and the intersections $M_{2+t} \cap I_{\mathbf{x}^0}$ do not meet the boundary of $\hat{U}$, for every $\mathbf{x}^0 \in V \setminus \Gamma$.*

Remark Note that the family M_{2+t} also depends on $\mathbf{x}^0$.

Proof of Assertion 1.2 To prove the transversality it suffices to prove that M_{2+t} satisfies conditions (i) and (ii) of Assertion 1.1 in $\hat{U}$, if we choose $\hat{U}$ and V sufficiently small. Let us first verify that condition (i) holds. Assume the contrary, i.e. that $\mathbf{x}^0 \in M_{2+t}$ for some $t \in [0, 1]$. Then it follows from (1.4) that $\mathbf{f}'(\mathbf{0}; \mathbf{x}^0) = \mathbf{0}$. But, from (1.2), we have

$$f_n(\mathbf{0}; \mathbf{x}^0) = \xi(\mathbf{x}') - \xi(\mathbf{x}^{0'})$$

which equals zero, since $\mathbf{x}' = \mathbf{x}^{0'} + t\mathbf{f}'(\mathbf{0}; \mathbf{x}^0)$. Consequently, $x_n = \xi(\mathbf{x}^{0'})$ which does not equal x_n^0 since $\mathbf{x}^0 \notin \Gamma$. This is a contradiction and, thus, $\mathbf{x}^0 \notin M_{2+t}$ for all $t \in [0, 1]$.

To verify condition (ii) note that the conormal space of M_{2+t} at a point $\mathbf{z} \in M_{2+t}$

is spanned by the covectors

$$(u_1, v_1, \dots, u_n, v_n) = \begin{cases} \left(1, t\dfrac{\partial f_1}{\partial y_1}, 0, t\dfrac{\partial f_1}{\partial y_2}, \dots, 0, t\dfrac{\partial f_1}{\partial y_{n-1}}, 0, 0\right), \\ \vdots \\ \left(0, t\dfrac{\partial f_{n-1}}{\partial y_1}, \dots, 0, t\dfrac{\partial f_{n-1}}{\partial y_{n-2}}, -1, t\dfrac{\partial f_{n-1}}{\partial y_{n-1}}, 0, 0\right), \\ \left(0, t\dfrac{\partial f_n}{\partial y_1}, \dots, 0, t\dfrac{\partial f_n}{\partial y_{n-1}}, -1, 0\right). \end{cases}$$

Hence, the transversality follows e.g. if we can prove that there is a V and a W such that

$$\left|\frac{\partial f_i}{\partial y_j}(\mathbf{y}'; \mathbf{x}^0)\right| < \frac{1}{\sqrt{2n}}$$

for all $i = 1, \dots, n$, $j = 1, \dots, n-1$, $\mathbf{y}' \in W$ and $\mathbf{x}^0 \in V$. By continuity, this follows from the fact that

$$\frac{\partial f_i}{\partial y_j}(\mathbf{0}; \mathbf{0}) = 0.$$

The latter follows readily from (1.2) and the definition of M. The details are left to the reader.

We complete the proof by showing that the intersection $M_{2+t} \cap I_{\mathbf{x}^0}$ does not meet the boundary of $\hat{U}$ if we choose V sufficiently small. Clearly, we can make the intersections $M_{2+t} \cap \partial\hat{U}$ be contained in $\{\mathbf{z}: |\mathbf{x} - \mathbf{x}^0| < \epsilon|\mathbf{y}|\}$ for any $\epsilon > 0$ by making V small enough. Thus, $M_{2+t} \cap I_{\mathbf{x}^0} \cap \partial\hat{U}$ is empty in view of the first equation of (1.3).

□

Let us conclude the proof of Lemma 1.1. Assertion 1.2 implies that the intersections $M_{2+t} \cap I_{\mathbf{x}^0}$ are smooth and homeomorphic for any pair of $t, t' \in [0, 1]$. A straightforward calculation shows that the intersection $M \cap I_{\mathbf{x}^0}$ (recall that $M = M_3$) is given by the equations:

$$\begin{cases} |\mathbf{y}|^2 = (|\mathbf{h}|^2 + 1)(\xi - x_n^0)^2, \\ y_n = \eta. \end{cases}$$

This proves part (a) of the lemma. The same argument as before shows that we can move S_2 continuously along with M_{2+t} to a cycle on M which is homeomorphic to a $(n-1)$-sphere and which encloses the intersection $M \cap I_{\mathbf{x}^0}$. This finishes the proof of Lemma 1.1.

□

Remark At this point, the reader should keep the following picture in mind: $\gamma_{\mathbf{x}^0}$ is a topological $n-2$ sphere in the intersection $\hat{\Gamma} \cap I_{\mathbf{x}^0}$ "surrounded" by the topological $n-1$ sphere $C_{\mathbf{x}^0}$.

Clearly, the manifold $\gamma_{\mathbf{x}^0}$
and $u(\mathbf{x}^1)$, for which $I_{\mathbf{x}^0} \cap \hat{\Gamma}$
integration in each of the co
next lemma asserts that this

Lemma 1.2 *If $I_{\mathbf{x}^0} \cap \hat{\Gamma}$*
homotopic in $\hat{U} \setminus I_{\mathbf{x}^0}$ to the
neighborhood of $\gamma_{\mathbf{x}^1}$ in M_1,

Proof By Lemma 1.1, i
$C'_{\mathbf{x}^0}$. Let M_j be the manifold
the equations preceding (1.

$$\begin{cases} \mathbf{x}' \\ x_n \end{cases}$$

for $(\mathbf{y}', y_n) \in W \times \mathbb{R}$. Now,
$t \in [0, 1]$, and define it by t

$\mathbf{x}$

First, note that M_t satisfie
simply because both M_0 a
meet $\mathbf{x}^0$ for any $t \in [0, 1]$.

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$\gamma_{\mathbf{x}^1} = I_{\mathbf{x}^1} \cap$

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We conclude this sec
3 and Section 4.

Clearly, the manifold $\gamma_{\mathbf{x}^0}$ is not unique. When we compare two values $u(\mathbf{x}^0)$ and $u(\mathbf{x}^1)$, for which $I_{\mathbf{x}^0} \cap \hat{\Gamma} = I_{\mathbf{x}^1} \cap \hat{\Gamma}$, we want to be able to deform the cycle of integration in each of the corresponding integrals to cycles over the same set. Our next lemma asserts that this indeed is possible.

Lemma 1.2 *If $I_{\mathbf{x}^0} \cap \hat{\Gamma} = I_{\mathbf{x}^1} \cap \hat{\Gamma}$ as sets in $\hat{U}$ then the sphere $S^{n-1}(\mathbf{x}^0, \epsilon)$ is homotopic in $\hat{U} \setminus I_{\mathbf{x}^0}$ to the $n-1$ dimensional boundary $C'_{\mathbf{x}^0}$ of a contractible neighborhood of $\gamma_{\mathbf{x}^1}$ in M_1, the manifold M of Lemma 1.1 defined for the point $\mathbf{x}^1$.*

Proof By Lemma 1.1, it suffices to prove that $C_{\mathbf{x}^0}$ is homotopic in $\hat{U} \setminus I_{\mathbf{x}^0}$ to $C'_{\mathbf{x}^0}$. Let M_j be the manifold of Lemma 1.1 defined for the point $\mathbf{x}^j, j = 0, 1$. From the equations preceding (1.4) we see that M_j can be descibed by the equations

$$\begin{cases} \mathbf{x}' = \mathbf{x}^{j\prime} + \mathbf{f}'(\mathbf{y}'; \mathbf{x}^j) = \mathbf{g}'(\mathbf{y}'; \mathbf{x}^j), \\ x_n = \xi(\mathbf{x}^{j\prime}) + f_n(\mathbf{y}'; \mathbf{x}^j) = g_n(\mathbf{y}'; \mathbf{x}^j), \end{cases}$$

for $(\mathbf{y}', y_n) \in W \times \mathbb{R}$. Now, we deform M_0 to M_1. We call the deformation M_t, for $t \in [0, 1]$, and define it by the equations

$$\mathbf{x} = t\mathbf{g}(\mathbf{y}'; \mathbf{x}^1) + (1-t)\mathbf{g}(\mathbf{y}'; \mathbf{x}^0).$$

First, note that M_t satisfies the condition (ii) of Assertion 1.1 for every $t \in [0, 1]$ simply because both M_0 and M_1 do. Next, we want to make sure that M_t does not meet $\mathbf{x}^0$ for any $t \in [0, 1]$. If it does then we have that

$$\mathbf{x}^0 = t\mathbf{g}(\mathbf{0}; \mathbf{x}^1) + (1-t)\mathbf{g}(\mathbf{0}; \mathbf{x}^0)$$

for some $t \in [0, 1]$. In this case, we can modify the definition of M_t slightly, so as to avoid this point, by adding the term $t(1-t)\epsilon$ to the first equation of M_t. We choose ϵ so small that condition (ii) of Assertion 1.1 is still satisfied. Consequently, we deduce from Assertion 1.1 that the deformation M_t meets $I_{\mathbf{x}^0}$ transversally for each $t \in [0, 1]$. Also, we can make the intersection $M_t \cap I_{\mathbf{x}^0}$ avoid the boundary of $\hat{U}$ by the same argument as in the proof of Assertion 1.2. Hence, we can move $C_{\mathbf{x}^0}$ along with M_t to a cycle $C'_{\mathbf{x}^0}$ surrounding the intersection $M_1 \cap I_{\mathbf{x}^0}$ in M_1. This intersection must be $\gamma_{\mathbf{x}^1}$ since

$$\gamma_{\mathbf{x}^1} = I_{\mathbf{x}^1} \cap M_1 = I_{\mathbf{x}^1} \cap \hat{\Gamma} \cap M_1 = I_{\mathbf{x}^0} \cap \hat{\Gamma} \cap M_1 \subseteq I_{\mathbf{x}^0} \cap M_1$$

and since we know that both $M_1 \cap I_{\mathbf{x}^0}$ and $\gamma_{\mathbf{x}^1}$ are closed smooth manifolds of the same dimension. □

We conclude this section with the following lemma which will be used in Section 3 and Section 4.

Lemma 1.3 *If the neighborhoods $\hat{U}$ and V are chosen small enough then the intersection $I_{\mathbf{x}^0} \cap \hat{\Gamma}$ is a smooth connected manifold in $\hat{U}$, for $\mathbf{x}^0 \in V \setminus \Gamma$. In particular, it is irreducible.*

Proof First, note that the intersection between $I_{\mathbf{x}^0}$ and any complex hyperplane $\{z_n = a\}$, with $a \neq x_n^0$, is the smooth quadric given by the equation

$$\left\langle \mathbf{z}' - \mathbf{x}^{0'}, \mathbf{z}' - \mathbf{x}^{0'} \right\rangle + (a - x_n^0)^2 = 0.$$

It is easy to verify that the closure of the set of unit conormals of $I_{\mathbf{x}^0} \setminus \{\mathbf{x}^0\}$, in any compact set, does not meet the set $\{\mathbf{r} = (\mathbf{r}', r_n) : |\mathbf{r}'| = 0\}$. By continuity and the fact that the set of unit conormals is invariant under translations, it follows that there is a $\delta > 0$ such that any complex manifold, which does not meet the point $\mathbf{x}^0$ and for which the set of unit conormals is contained in the set $\{\mathbf{r} : |\mathbf{r}'| \leq \delta\}$, meets $I_{\mathbf{x}^0}$ transversally (cf. Assertion 1.1). Since $\hat{\Gamma} = \{z_n = \phi(\mathbf{z}')\}$, where ϕ has a vanishing gradient at the origin, it follows that, if we choose $\hat{U}$ sufficiently small, the intersection $\hat{\Gamma} \cap I_{\mathbf{x}^0}$ is a smooth manifold in $\hat{U}$. It could, a priori, be a disconnected manifold, though. To see that it is not we consider a deformation $N_t = \{z_n = (1-t)i\epsilon + t\phi(\mathbf{z}')\}$, for $t \in [0,1]$, from a plane $\{z_n = i\epsilon\}$ to $\hat{\Gamma}$. The number $\epsilon \geq 0$ is chosen small ($\epsilon = 0$ if possible) such that N_t does not meet $\mathbf{x}^0$ for any $t \in [0,1]$. The complex manifolds N_t all meet $I_{\mathbf{x}^0}$ transversally, but in order to make sure that no new components of the intersections $N_t \cap I_{\mathbf{x}^0}$ emerge in $\hat{U}$ or that the component we start with does not leave $\hat{U}$ as t runs from 0 to 1 we need to verify that the manifolds N_t meet the boundary component $\partial\hat{U} \cap I_{\mathbf{x}^0}$, for $\mathbf{x}^0$ in a sufficiently small domain V, transversally also. This would actually prove that the topological type of the intersections $N_t \cap I_{\mathbf{x}^0}$ does not change as t runs from 0 to 1. By continuity, it suffices to verify that the set of unit conormals of the boundary component $\partial\hat{U} \cap I_0$ at $\{z_n = 0\}$ does not meet the set $\{\mathbf{r} : |\mathbf{r}'| = 0\}$. This verification is straightforward and the details are omitted. □

2. The three-dimensional case

In this section we prove the following result on point to point reflection of harmonic functions vanishing on a hypersurface in $\mathbb{R}^3$. We keep the notation introduced in the previous section with $n = 3$.

Theorem 2.1 *Let Γ be a nonsingular, real-analytic hypersurface in some neighborhood U of the origin in $\mathbb{R}^3$. Suppose that Γ is neither part of a hyperplane nor a sphere. Then there is a neighborhood V, $\overline{V} \subset U$, such that for no pair of points $\mathbf{x}^0$, $\mathbf{x}^1$ in $V \setminus \Gamma$ is there a constant K satisfying*

$$u(\mathbf{x}^0) + Ku(\mathbf{x}^1) = 0, \qquad \forall u \in \mathrm{Har}_0(U, \Gamma),$$

where $\operatorname{Har}_0(U,\Gamma)$ *denotes the class of harmonic functions in* U *vanishing on* Γ.

Before we enter the proof of the theorem we need a lemma.

Lemma 2.1 *Let* $D_{\mathbf{x}^0}$ *be the relatively compact component of* $(\hat{\Gamma} \cap M) \setminus \gamma_{\mathbf{x}^0}$, *i.e. the domain in* $\hat{\Gamma} \cap M$ *bounded by the closed simple curve* $\gamma_{\mathbf{x}^0}$. *Then, for any* $u \in \operatorname{Har}_0(U,\Gamma)$ *and* $\mathbf{x}^0 \in V \setminus \Gamma$, *we have*

$$u(\mathbf{x}^0) = 2\int_{D_{\mathbf{x}^0}} \sum_{j=1}^{3} (-1)^j \frac{\partial u}{\partial z_j}(\mathbf{z}) N(\mathbf{z}-\mathbf{x}^0)\,\omega_j, \tag{2.1}$$

where N *is the Newtonian potential of a point mass at the origin as in* (1.1),

$$\omega_j = dz_1 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_3,$$

and the orientation of $D_{\mathbf{x}^0}$ *and the branch of the square root in* N *are chosen according to those in* (1.1).

Proof Let us fix $\mathbf{x}^0 \in V \setminus \Gamma$. It follows from Lemma 1.1 that

$$u(\mathbf{x}^0) = \int_{C_{\mathbf{x}^0}} \sum_{j=1}^{3} (-1)^{j+1} \left(u(\mathbf{z}) \frac{\partial N}{\partial z_j}(\mathbf{z}-\mathbf{x}^0) - \frac{\partial u}{\partial z_j}(\mathbf{z}) N(\mathbf{z}-\mathbf{x}^0) \right) \omega_j$$

holds for every $u \in \operatorname{Har}_0(U,\Gamma)$. In Lemma 1.1 we have chosen $\hat{U}$ so small that M is given as a graph over the real $\mathbf{y}$-plane. Consequently, we can make a change of coordinates such that M becomes a ball in $\mathbb{R}^3$ with coordinates $\mathbf{t}$. Also, the intersection $\hat{\Gamma} \cap M$ is given as the graph $y_3 = \eta$ in M. By changing coordinates in M (a ball in $\mathbb{R}^3$), we can assume that $\hat{\Gamma} \cap M$ is the plane $t_3 = 0$. Now, we can take $C = C_{\mathbf{x}^0}$ to be a sphere in M surrounding the closed simple curve $\gamma = \gamma_{\mathbf{x}^0}$. Let T_r be a tube of radius r in M around γ and let D_r be $D \setminus T_r$, where $D = D_{\mathbf{x}^0}$ is the domain in $\{t_3 = 0\}$ bounded by γ. Then the sphere C is homotopic to the cycle consisting of ∂T_r, a slightly deformed torus, and two copies of D_r the orientations of which differ by a sign (cf. [G]). Since $N(\cdot - \mathbf{x}^0)$ changes sign as we go around γ and the orientation of the two copies of D_r differ by a sign we get, for every sufficiently small $r > 0$,

$$u(\mathbf{x}^0) = \int_{\partial T_r} \sum_{j=1}^{3} (-1)^{j+1} \left(u \frac{\partial N}{\partial z_j}(\cdot - \mathbf{x}^0) - \frac{\partial u}{\partial z_j} N(\cdot - \mathbf{x}^0) \right) \omega_j$$
$$+ 2\int_{D_r} \sum_{j=1}^{3} (-1)^j \frac{\partial u}{\partial z_j} N(\cdot - \mathbf{x}^0)\,\omega_j.$$

We have used the fact that u vanishes on $\hat{\Gamma}$. Now, $D_r \to D$ in measure and the singularity of $N(\cdot - \mathbf{x}^0)$ at γ is integrable with respect to the two dimensional area measure on D, i.e. the integral in (2.1) converges absolutely. Consequently, the proof will be finished if we can show that the integral over ∂T_r tends to zero as r tends to zero. Since M and $I_{\mathbf{x}^0}$ meet transversally, the restriction of

$$g(\mathbf{z}) = \langle \mathbf{z} - \mathbf{x}^0, \mathbf{z} - \mathbf{x}^0 \rangle$$

to M, let us denote it by $\tilde{g}(\mathbf{t})$, has a non-vanishing gradient near γ. Moreover, this restriction vanishes only on γ. These two facts imply that we can write

$$\tilde{g}(\mathbf{t}) = \operatorname{dist}(\mathbf{t}, \gamma)\, k_1(\mathbf{t}),$$

where k_1 is a function satisfying $A < |k_1| < B$ for some positive constants A and B and $\operatorname{dist}(\mathbf{t}, \gamma)$ denotes the distance from $\mathbf{t}$ to γ. Also, since u vanishes on $\hat{\Gamma}$ we can write the restriction of u to M as $t_3 k_2(\mathbf{t})$ where k_2 is a bounded function. Finally, let us note that the restriction of ω_j to ∂T_r is a bounded function times the volume form on ∂T_r. If we put this together we obtain

$$\Big| \int_{\partial T_r} \sum_{j=1}^{3} (-1)^{j+1} \Big(u \frac{\partial N}{\partial z_j}(\cdot - \mathbf{x}^0) - \frac{\partial u}{\partial z_j} N(\cdot - \mathbf{x}^0) \Big) \omega_j \Big| \leq \int_{\partial T_r} \sum_{j=1}^{3} \Big(\frac{t_3 p_j(\mathbf{t})}{(\operatorname{dist}(\mathbf{t}, \gamma))^{3/2}} + \frac{q_j(\mathbf{t})}{(\operatorname{dist}(\mathbf{t}, \gamma))^{1/2}} \Big) dS,$$

where the p_j's and the q_j's are bounded functions. Now, the distance from a point on ∂T_r to γ is r, by definition of T_r, and since $t_3 \leq r$ on ∂T_r, we get

$$\left| \int_{\partial T_r} \sum_{j=1}^{3} (-1)^{j+1} \left(u \frac{\partial N}{\partial z_j}(\cdot - \mathbf{x}^0) - \frac{\partial u}{\partial z_j} N(\cdot - \mathbf{x}^0) \right) \omega_j \right| \leq \frac{K}{r^{1/2}} A(\partial T_r),$$

where $A(\partial T_r)$ denotes the area of ∂T_r and K is some constant independent of r. The area of ∂T_r is proportional to the radius r and, hence, the integral over ∂T_r tends to zero as $r^{1/2}$ when r tends to zero. □

We can now proceed with the proof of Theorem 2.1.

Proof of Theorem 2.1 During the course of proving Lemmas 1.1 and 1.2, we have chosen the neighborhoods U and V small. In the following, we may choose them even smaller. We pick two points $\mathbf{x}^0$ and $\mathbf{x}^1$ in $V \setminus \Gamma$. The proof splits into two cases:

(i) The sets $I_{\mathbf{x}^0} \cap \hat{\Gamma}$ and $I_{\mathbf{x}^1} \cap \hat{\Gamma}$ coincide in $\hat{U}$. By Lemma 1.2, we may choose the topological disks $D_{\mathbf{x}^0}$ and $D_{\mathbf{x}^1}$ to be the same. Let us denote this disk by D. We make the change of coordinates $w_3 = z_3 - \phi(\mathbf{z})$ and $\mathbf{w}' = \mathbf{z}'$ to make $\hat{\Gamma}$ the hyperplane $\{w_3 = 0\}$. As we have seen above, the disk D is the real-analytic image in $\{w_3 = 0\}$ under the mapping ($\mathbf{t}' = (t_1, t_2)$)

$$\mathbf{w}' = \mathbf{x}^{0\prime} + \mathbf{f}'(\mathbf{t}'; \mathbf{x}^0) + i\mathbf{t}' = \mathbf{g}'(\mathbf{t}'; \mathbf{x}^0) + i\mathbf{t}' \tag{2.2}$$

of a simply connected domain Ω with smooth boundary in the real $\mathbf{t}'$-plane. Denote by W a small neighborhood of $\mathbf{0}$ in the complex $\mathbf{t}'$-space (i.e. in $\mathbb{C}^2$) such that $\mathbf{g}'(\cdot; \mathbf{x}^0)$ extends holomorphically to W for each $\mathbf{x}^0$ in some neighborhood V of $\mathbf{0} \in \mathbb{R}^3$. Since $\hat{\nabla}_{\mathbf{t}'}\mathbf{g}'(\mathbf{0}; \mathbf{0}) = \mathbf{0}$, the equations (2.2) define, for each $\mathbf{x}^0 \in V$, a change of coordinates in $\tilde{U}$,

$$\tilde{U} = \hat{U} \cap \hat{\Gamma},$$

if we choose $\hat{U}$ and V small enough. Under this change of coordinates the disk D becomes the domain Ω in the real space $\mathbb{R}^2$ of $\hat{\Gamma} \cong \mathbb{C}^2$. By the Stone–Weierstrass theorem, we can approximate the continuous functions on Ω uniformly by analytic polynomials and, thus, we may approximate the continuous functions on D by functions analytic in $\tilde{U}$. Now, in order to get a contradiction, assume that there is a constant K such that

$$u(\mathbf{x}^0) + Ku(\mathbf{x}^1) = 0, \qquad \forall u \in \mathrm{Har}_0(U, \Gamma);$$

then it follows from (2.1) that

$$\int_D (N(\cdot - \mathbf{x}^0) \pm KN(\cdot - \mathbf{x}^1)) \sum_{j=1}^{3} (-1)^j \frac{\partial u}{\partial z_j} \omega_j = 0 \tag{2.3}$$

for every $u \in \mathrm{Har}_0(U, \Gamma)$. Which sign, $+$ or $-$, we should choose in (2.3) depends on which branch of the square root we choose in $N(\cdot - \mathbf{x}^j)$, $j = 0, 1$. A simple calculation shows that the restriction of the form

$$\sum_{j=1}^{3} (-1)^j \frac{\partial u}{\partial z_j} \omega_j$$

to $\hat{\Gamma} = \{w_3 = 0\}$ equals

$$\frac{\partial u}{\partial w_3} \left(1 + \left\langle \hat{\nabla}\phi, \hat{\nabla}\phi \right\rangle\right) dw_1 \wedge dw_2.$$

Note that the expression multiplying $\partial u/\partial w_3$ above vanishes at precisely the characteristic points (with respect to the Laplace operator) of $\hat{\Gamma}$ and, hence, not in a neighborhood of the origin (the characteristic points on a real-analytic hypersurface in $\mathbb{R}^n$ are precisely the singular ones). By the Cauchy–Kowalevskaya theorem (see [H1], ch. IX), there is, for every choice of analytic function ψ in $\tilde{U}$, a function u in $\mathrm{Har}_0(U,\Gamma)$, provided that we a priori choose U small enough, that solves the complex Cauchy problem

$$\begin{cases} \displaystyle\sum_{j=1}^{3} \frac{\partial^2 u}{\partial z_j^2} = 0, \\ \left.\begin{array}{l} u = 0 \\ \dfrac{\partial u}{\partial n} = \psi \end{array}\right\} \quad \text{on } \hat{\Gamma}, \end{cases}$$

where $\partial u/\partial n$ denotes the derivative of u in the conormal direction of $\hat{\Gamma}$, i.e. $\partial u/\partial w_3$ in the $\mathbf{w}$ coordinates. Since the analytic functions in $\tilde{U}$ approximate the continuous functions in D—note that we may subsequently have to shrink $\tilde{U}$ even more, but that this property holds for the functions analytic in the domain where $\mathbf{z} \to \mathbf{w}$ is a change of coordinates—it follows that (2.3) holds only if $N(\cdot - \mathbf{x}^0) \pm KN(\cdot - \mathbf{x}^1)$ vanishes in D. Since D is biholomorphic to Ω which has a non-empty interior in $\mathbb{R}^2$, it follows that $N(\cdot - \mathbf{x}^0) \pm KN(\cdot - \mathbf{x}^1)$ must vanish on all of $\hat{\Gamma}$. An explicit calculation with the Newtonian potentials shows that $\hat{\Gamma}$ must be contained in the algebraic hypersurface defined by the equation

$$\langle \mathbf{z} - \mathbf{x}^1, \mathbf{z} - \mathbf{x}^1 \rangle = K^2 \langle \mathbf{z} - \mathbf{x}^0, \mathbf{z} - \mathbf{x}^0 \rangle .$$

Consequently, Γ is a portion of either a hyperplane or a sphere. The proof of the first case is complete.

(ii) The sets $I_{\mathbf{x}^0} \cap \hat{\Gamma}$ and $I_{\mathbf{x}^1} \cap \hat{\Gamma}$ do not coincide. We can think of the point evaluations $u(\mathbf{x}^0)$ and $u(\mathbf{x}^1)$ as analytic functionals, T_0 and T_1, on the space of analytic functions in $\tilde{U}$, denoted by $\mathcal{O}(\tilde{U})$, realized as the corresponding integrals over $D_0 = D_{\mathbf{x}^0}$ and $D_1 = D_{\mathbf{x}^1}$. The fact, established above, that there is a biholomorphic change of variables in $\tilde{U}$ taking D_j to a simply connected domain Ω_j in $\mathbb{R}^2$ proves, by the Stone–Weierstrass theorem, that the continuous functions in D_j can be approximated uniformly by analytic functions in $\tilde{U}$ for $j = 0, 1$. Consequently, since the measure realizing T_j has support on all of D_j, no proper subset of D_j can carry T_j, i.e. there is no measure supported on a proper subset of D_j that can represent T_j. One says that $K_j = \overline{D_j}$ is a support for T_j. Now assume, in order to get a contradiction, that there is a constant such that $T_0 + KT_1 = 0$. This implies that both K_0 and K_1 are carriers of T_0, because $T_0 = -KT_1$. By a theorem of C. O.

Kiselman (see corollary 2.6 in [Ki]), the analytic functional T_0 must be carried by the set

$$K_0 \cap (\overline{L \setminus K_0} \cup K_1),$$

where L is the $\mathcal{O}(\tilde{U})$- hull of the union $K_0 \cup K_1$. If we can prove that there is a point in K_0 and a neighborhood U', in $\hat{\Gamma} \cong \mathbb{C}^2$, of that point such that $L \cap U' = K_0 \cap U'$ then $\overline{L \setminus K_0}$ does not meet $K_0 \cap U''$ for some smaller neighborhood $U'' \subset U'$ and, thus, $K_0 \cap (\overline{L \setminus K_0} \cup K_1)$ is a proper subset of K_0. This contradicts the fact that no proper subset of K_0 carries T_0 and the proof would be finished. The way to prove that $L \cap U' = K_0 \cap U'$ is to use the Oka–Stolzenberg theorem. This theorem asserts (see e.g. [Sto]) that $L \cap U' = K_0 \cap U'$ if we are able to pass an analytic hypersurface through each point of $U' \setminus K_0$ and move it off to infinity without intersecting the set $K_0 \cup K_1$.

For the remainder of this proof, the arguments will essentially take place in two dimensions. Therefore, to help the reader better visualize the geometric picture, we will avoid the vector notation used above. We consider the domains Ω_0 and Ω_1 in the real (t_1, t_2)-plane, as in case (i) above, such that D_0 and D_1 are given as the images of these domains under the mappings (2.2). There are two possibilities:

(ii.a) $\Omega_0 = \Omega_1$. We make the change of coordinates taking D_0 to $\Omega = \Omega_0 = \Omega_1$ in the real (t_1, t_2)-plane. We denote the new variables by $\zeta_j = t_j + is_j$. A simple calculation shows that the disk D_1 is given by the mapping

$$\begin{cases} \zeta_1 = t_1 + i(g_1^1 - g_1^0), \\ \zeta_2 = t_2 + i(g_2^1 - g_2^0), \end{cases} \tag{2.4}$$

where we, for brevity, use the notation $g_j^k(t_1, t_2) = g_j(t_1, t_2; \mathbf{x}^k)$ for $k = 0, 1$ and $j = 1, 2$. Also, let $h(t_1, t_2)$ denote the function defining the domain Ω, i.e. the real part of $g(w_1, w_2; \mathbf{x}^0) = (w_1 - x_1^0)^2 + (w_2 - x_2^0)^2 + (\phi - x_3^0)^2$, evaluated at $(w_1, w_2) = (t_1 + ig_1^0, t_2 + ig_2^0)$, or equivalently $g(t_1, t_2; \mathbf{x}^1)$ evaluated at $(w_1, w_2) = (t_1 + ig_1^1, t_2 + ig_2^1)$. First, we claim that we can assume that

$$\frac{\partial h}{\partial t_1}(g_1^1 - g_1^0) + \frac{\partial h}{\partial t_2}(g_2^1 - g_2^0) \tag{2.5}$$

restricted to $\partial\Omega$ is not identically zero. To see this, note, as we noted in Lemma 1.2, that the disk D_1 and the curve $\gamma_1 = \gamma_{\mathbf{x}^1}$ are not canonically defined. In the proof of Lemma 1.1 we could have continued to deform the surface M to the surface M_ϵ defined by the equations

$$\begin{cases} x_1 = g_1^1(y_1, y_2) + \epsilon y_2, \\ x_2 = g_2^1(y_1, y_2) - \epsilon y_1, \\ x_3 = \xi(x_1 + iy_1, x_2 + iy_2). \end{cases} \tag{2.6}$$

Here ϵ is a small number or a small function with small gradient. By choosing $\hat{U}$ and V sufficiently small we can get a uniform, for $\mathbf{x} \in V$, allowed size on ϵ and its gradient. In this notation, which differs slightly from the one used in Lemma 1.1, M is M_0. The form of the perturbation is chosen such that the intersection $I_{\mathbf{x}^1} \cap M_\epsilon$ is contained in $\hat{\Gamma}$. Note that the perturbation is in the tangent space of a circle, centered at the origin, through the point (y_1, y_2). By the same argument as in Lemma 1.1, the intersection $I_{\mathbf{x}^1} \cap M_\epsilon$ is a smooth curve ${}^\epsilon\gamma_1$ contained in $\hat{\Gamma}$. Under this perturbation either ${}^\epsilon\Omega_1$, the projection of ${}^\epsilon D_1$ (defined in the obvious way) as above, changes to become different from Ω or it does not. If it does we proceed as in case (b) below. If it does not then we claim that the restriction to $\partial\Omega$ of

$$\frac{\partial h}{\partial t_1}(g_1^1 - g_1^0 + \epsilon t_2) + \frac{\partial h}{\partial t_2}(g_2^1 - g_2^0 - \epsilon t_1)$$

changes. Assume, in order to get a contradiction, that it does not change; then it would follow from (2.5) that

$$\frac{\partial h}{\partial t_1} t_2 - \frac{\partial h}{\partial t_2} t_1 = 0$$

on $\partial\Omega$. It is easy to see that this implies that Ω is a ball. Hence, using equation (2.5) again we deduce that

$$\begin{cases} g_1^1(t_1,t_2) = g_1^0(t_1,t_2) + \lambda(t_1,t_2)t_2, \\ g_2^1(t_1,t_2) = g_2^0(t_1,t_2) - \lambda(t_1,t_2)t_1, \end{cases}$$

for some function λ which is small with small gradient, on the circle $\partial\Omega$. In this case, we can modify (g_1^1, g_2^1) in the same manner as above. We set

$$\begin{cases} {}^\tau g_1^1(t_1,t_2) = g_1^1(t_1,t_2) - \tau\lambda(t_1,t_2)t_2, \\ {}^\tau g_2^1(t_1,t_2) = g_2^1(t_1,t_2) + \tau\lambda(t_1,t_2)t_1. \end{cases}$$

Since we have assumed that ${}^\tau\Omega_1$ does not change under such a perturbation, this deforms the disk D_1 to the disk D_0 as τ runs from 0 to 1. This contradicts the fact that $I_{\mathbf{x}^0} \cap \hat{\Gamma}$ does not equal $I_{\mathbf{x}^1} \cap \hat{\Gamma}$, because the common boundary of the disks D_0 and the deformed D_1 would be contained in the intersection $I_{\mathbf{x}^0} \cap I_{\mathbf{x}^1} \cap \hat{\Gamma}$. Consequently, we may assume that (2.5) is not identically zero on $\partial\Omega$. By real-analyticity, it is then zero only at a finite number of points on $\partial\Omega$. It follows that there is a point $\mathbf{t}^0 \in \partial\Omega$ such that the real line tangent to $\partial\Omega$ at $\mathbf{t}^0$,

$$at_1 + bt_2 = c,$$

does not meet Ω (let us assu
that $a(g_1^1 - g_1^0) + b(g_2^1 - g_2^0)$
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The proof of

does not meet Ω (let us assume that all points of Ω satisfy $at_1 + bt_2 < c$) and such that $a(g_1^1 - g_1^0) + b(g_2^1 - g_2^0) \neq 0$ at $\mathbf{t}^0$. By moving this tangent line slightly into Ω we cut off an open subset of Ω, namely

$$\omega = \Omega \cap \{at_1 + bt_2 > c'\} \tag{2.7}$$

for some $c' < c$, in which $|a(g_1^1 - g_1^0) + b(g_2^1 - g_2^0)| > \delta$ for some $\delta > 0$. Let U' denote the domain

$$\{(\zeta_1, \zeta_2) = (t_1 + is_1, t_2 + is_2) \colon (t_1, t_2) \in \omega \text{ and } |as_1 + bs_2| < \delta\}$$

in $\mathbb{C}^2$. It is easy to see that through each point (ζ_1^0, ζ_2^0) in $U' \setminus \Omega$ we can pass the following curve of complex lines,

$$L_r = \{a\zeta_1 + b\zeta_2 = a\zeta_1^0 + b\zeta_2^0 + r\}, \tag{2.8}$$

and that this curve of complex lines moves off to infinity, as $r \to \infty$, without intersecting neither $\overline{\Omega} = K_0$ nor K_1. By the Oka–Stolzenberg theorem, we get that $U' \setminus \Omega$ does not meet L, the $\mathcal{O}(\bar{U})$-hull of $K_0 \cup K_1$ defined above. This finishes the proof of possibility (a), by the theorem of Kiselman mentioned above.

(ii.b) $\Omega_0 \neq \Omega_1$. By real-analyticity, the boundaries $\partial\Omega_0$ and $\partial\Omega_1$ intersect only at a finite number of points. It follows that there is a point $\mathbf{t}^0$ on one of the boundaries, say on $\partial\Omega_0$, which is not on the other boundary $\partial\Omega_1$ and such that the real tangent line

$$at_1 + bt_2 = c$$

of $\partial\Omega_0$ at $\mathbf{t}^0$ meets neither Ω_0 nor Ω_1; let us assume that $\Omega_0 \cup \Omega_1$ is contained in $\{at_1 + bt_2 < c\}$. As above, we make the change of coordinates taking D_0 to Ω_0 and denote the new variables by $\zeta_j = t_j + is_j$. Move the tangent line at $\mathbf{t}^0$ slightly into Ω_0 and let ω be the open subset of Ω_0 defined as in (2.7). We choose $c - c'$ so small that Ω_1 is contained in $\{at_1 + bt_2 < c'\}$. We can take U' to be the cylinder

$$\{(\zeta_1, \zeta_2) \colon (t_1, t_2) \in \omega\}.$$

Again, it is easy to see that through each point (ζ_1^0, ζ_2^0) in $U' \setminus \Omega$ we can pass the curve of complex lines defined by (2.8) and that this curve moves off to infinity, as $r \to \infty$, without intersecting neither $\overline{\Omega_0} = K_0$ nor K_1. This finishes the proof of possibility (b) by combining the Oka–Stolzenberg theorem and the Kiselman theorem exactly as in the proof of possibility (a).

The proof of case (ii) and, hence, of the theorem is complete. □

Remark As we already noted in the introduction, the argument in case (ii) of the proof above, i.e. the case when $I_{\mathbf{x}^0} \cap \hat{\Gamma}$ and $I_{\mathbf{x}^1} \cap \hat{\Gamma}$ do not coincide (the points are not SR points with respect to Γ according to Definition 3.1 below), could be shortened considerably if we were to consider a stronger point to point reflection law than the one considered in Theorem 2.1. Namely, if we were to demand that "reflection of singularities" should hold as well, i.e. that any harmonic function vanishing on Γ and with a pole at $\mathbf{x}^0$, e.g. a Green's function on one side of Γ with pole at $\mathbf{x}^0$, should have a pole also at $\mathbf{x}^1$ then it is clear that $I_{\mathbf{x}^0} \cap \hat{\Gamma}$ and $I_{\mathbf{x}^1} \cap \hat{\Gamma}$ must coincide. To see this, note that the singularity of such a harmonic function propagates along the isotropic cone emanating from the pole so in order for the function to vanish on Γ the singularities of the function must cancel on $\hat{\Gamma}$, i.e. the isotropic cones $I_{\mathbf{x}^0} \cap \hat{\Gamma}$ and $I_{\mathbf{x}^1} \cap \hat{\Gamma}$ coincide. This was noted in [KS].

3. The four-dimensional case

Before we state the theorem on point to point reflection of harmonic functions in $\mathbb{R}^4$ we need some definitions.

Definition 3.1 Two points $\mathbf{x}^0$ and $\mathbf{x}^1$ in $V \setminus \Gamma$ are said to be SR points, Study reflection points, with respect to Γ if the intersection $I_{\mathbf{x}^0} \cap \hat{\Gamma}$ and $I_{\mathbf{x}^1} \cap \hat{\Gamma}$ are equal as sets in $\hat{U}$ (cf. [KS]).

It was shown in [KS] that, in $\mathbb{R}^n$ with $n > 2$, the set of points for which there is a Study reflected point is an algebraic set of codimension at least 1, unless Γ is a sphere or a hyperplane in which case there is always a reflected point (the ordinary "mirror" reflection in a plane and the Kelvin transformation for a sphere). In $\mathbb{R}^2$, the Study reflection coincides with the well known Schwarz reflection.

In [Kh], it is shown that there is a point to point reflection of functions in $\mathrm{Har}_0(U,\Gamma)$, where Γ is an axially symmetric surface in $\mathbb{R}^4$, between SR points on the axis of symmetry, i.e. for each pair of SR points $\mathbf{x}^0$ and $\mathbf{x}^1$ on the axis of symmetry there is a constant K such that

$$u(\mathbf{x}^0) + Ku(\mathbf{x}^1) = 0, \qquad \forall u \in \mathrm{Har}_0(U,\Gamma). \tag{3.1}$$

It was shown in the previous section that nothing like this is true in $\mathbb{R}^3$, but one might hope that it is true in general in $\mathbb{R}^4$. However, this turns out to be false. We have to introduce a stronger reflection relation, namely

Definition 3.2 Two points $\mathbf{x}^0$ and $\mathbf{x}^1$ in $V \setminus \Gamma$ are said to be SSR points, strong Study reflection points, with respect to Γ and with constant λ if they are SR points and if

$$dg(\cdot - \mathbf{x}^0)|_{\hat{\Gamma}} = \lambda dg(\cdot - \mathbf{x}^1)|_{\hat{\Gamma}},$$

where g is the defining function f

and λ is a complex constant, on t

Of course, the notion of SSR g as well as the "mirror" reflectior spheres. Moreover, the reader c each pair of SR points that, in satisfy the SSR condition (cf. [Kh] mentioned above are sim the main result in this section.

Theorem 3.1 *Let Γ be c neighborhood U of the origin with $\overline{V} \subset U$, such that, giver equivalent:*

(a) *there is a constant K su*

(b) *the points $\mathbf{x}^0$ and $\mathbf{x}^1$ ar λ equal to $1/K$.*

Before we prove this, we c of SSR. To provide an intui condition as well as to mot heuristically in the perhaps wave equation.

Suppose that our surface $\tilde{\Gamma}$ in the real subspace

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Let $\tilde{I}_0 = I_{\mathbf{x}^0} \cap W$, $\tilde{I}_1 = I_{\mathbf{x}^1}$ $\mathbf{x}^1$ that meet on $\tilde{\Gamma}$ along where g is the defining function for I_0, i.e.

$$g(\mathbf{z}) = \langle \mathbf{z}, \mathbf{z} \rangle$$

and λ is a complex constant, on the common intersection $I_{\mathbf{x}^0} \cap \hat{\Gamma} = I_{\mathbf{x}^1} \cap \hat{\Gamma}$.

Of course, the notion of SSR generalizes the Schwarz reflection in two variables as well as the "mirror" reflection in hyperplanes and the Kelvin transformation for spheres. Moreover, the reader can readily verify that when Γ is axially symmetric each pair of SR points that, in addition, lie on the axis of symmetry does indeed satisfy the SSR condition (cf. also Proposition 3.1 below). Thus, the results of [Kh] mentioned above are simple corollaries of the following theorem, which is the main result in this section.

Theorem 3.1 *Let Γ be a nonsingular, real-analytic hypersurface in some neighborhood U of the origin in $\mathbb{R}^4$. Then there is a neighborhood V of the origin, with $\overline{V} \subset U$, such that, given two points $\mathbf{x}^0$ and $\mathbf{x}^1$ in $V \setminus \Gamma$, the following are equivalent:*

(a) *there is a constant K such that* (3.1) *holds*;

(b) *the points $\mathbf{x}^0$ and $\mathbf{x}^1$ are SSR points with respect to Γ and with the constant λ equal to* $1/K$.

Before we prove this, we digress momentarily to discuss and motivate the notion of SSR. To provide an intuitive insight for the reader into the nature of the SSR condition as well as to motivate its definition, let us briefly discuss the problem heuristically in the perhaps more familiar context of the Huygens principle for the wave equation.

Suppose that our surface $\hat{\Gamma} = \{\mathbf{z} \in \mathbb{C}^4 : f(\mathbf{z}) = 0\}$ appears as a real hypersurface $\tilde{\Gamma}$ in the real subspace

$$W = i\mathbb{R}^3 \times \mathbb{R} = \{(i\mathbf{x}', x_4) : x_j \in \mathbb{R}\}$$

(in general, this intersection has codimension 2). Let $\mathbf{x}^0 = (0,0,0,t)$ and $\mathbf{x}^1 = (0,0,0,-t)$ be SR points. Obviously, we have $I_{\mathbf{x}^0} \cap I_{\mathbf{x}^1} \subset \{z_4 = 0\}$. Any $u \in \mathrm{Har}_0(U,\Gamma)$, extended as a holomorphic function into $\mathbb{C}^4$, satisfies in W the wave equation

$$\sum_{j=1}^{3} \frac{\partial^2 u}{\partial x_j^2} - \frac{\partial^2 u}{\partial x_4^2} = 0.$$

Let $\tilde{I}_0 = I_{\mathbf{x}^0} \cap W$, $\tilde{I}_1 = I_{\mathbf{x}^1} \cap W$ be the ordinary "light cones" emanating from $\mathbf{x}^0$ and $\mathbf{x}^1$ that meet on $\tilde{\Gamma}$ along the 2-dimensional sphere $S_t = \{(i\mathbf{x}', 0) : |\mathbf{x}'| = t\}$. Now, by

Kirchoff's formula (see e.g. [J], chapter 5), the value of u at $\mathbf{x}^0$ can be calculated in terms of the Cauchy data on the plane $\{x_4 = 0\}$ along the sphere S_t only:

$$u(\mathbf{x}^0) = \frac{C}{t^2} \int_{S_t} \left(t \frac{\partial u}{\partial x_4} + \sum_{j=1}^{3} \frac{\partial u}{\partial x_j} x_j \right) dS,$$

where C is some constant and dS denotes the Lebesgue measure on S_t taken with positive (with respect to the outward normal) orientation. Note that we have used the fact that u vanishes on S_t. Since $-\nabla g(\cdot - \mathbf{x}^0) = 2(\mathbf{x}', t)$ on the plane $\{x_4 = 0\}$ we can rewrite the above equation in the form

$$\begin{aligned} u(\mathbf{x}^0) &= -\frac{C}{2t^2} \int_{S_t} \langle \nabla u, \nabla g(\cdot - \mathbf{x}^0) \rangle \, dS \\ &= -\frac{C}{2t^2} \int_{S_t} \frac{\partial u}{\partial n} \langle \mathbf{n}, \nabla g(\cdot - \mathbf{x}^0) \rangle \, dS, \end{aligned}$$

where $\mathbf{n}$ denotes the unit normal to $\tilde{\Gamma}$. Similarly, we have

$$u(\mathbf{x}^1) = -\frac{C}{2t^2} \int_{S_t} \frac{\partial u}{\partial n} \langle \mathbf{n}, \nabla g(\cdot - \mathbf{x}^1) \rangle \, dS.$$

Now, if (3.1) holds then, in view of equations above and since $\partial u / \partial n$ runs over a dense set of the continuous functions on S_t as u runs over $\mathrm{Har}_0(U, \Gamma)$, it follows that

$$\langle \nabla g(\cdot - \mathbf{x}^0) + K \nabla g(\cdot - \mathbf{x}^1), \mathbf{n} \rangle = 0$$

on S_t. To fix the ideas, assume that $\tilde{\Gamma}$ can be written as $\{x_4 = \phi(\mathbf{x}')\}$. Then, since the intersection of $\tilde{\Gamma}$ with the plane $\{x_4 = 0\}$ contains the sphere S_t it follows that $\nabla \phi = \psi(\mathbf{x}')\mathbf{x}'$ on S_t. Combining this with the equation above and the definition of S_t, we find that ψ is constant on S_t,

$$\psi(\mathbf{x}') = \frac{1}{t} \frac{1-K}{1+K}.$$

Noting that

$$\begin{cases} -dg(\cdot - \mathbf{x}^0)|_{\tilde{\Gamma}} = 2 \langle \mathbf{x}' + t\nabla\phi, d\mathbf{x}' \rangle \\ -dg(\cdot - \mathbf{x}^1)|_{\tilde{\Gamma}} = 2 \langle \mathbf{x}' - t\nabla\phi, d\mathbf{x}' \rangle \end{cases}$$

we obtain that

$$dg(\cdot - \mathbf{x}^0)|_{\tilde{\Gamma}} = \lambda dg(\cdot - \mathbf{x}^1)|_{\tilde{\Gamma}},$$

where $\lambda = 1/K$, on S_t and, therefore, everywhere on $I_{\mathbf{x}^0} \cap \hat{\Gamma} = I_{\mathbf{x}^1} \cap \hat{\Gamma}$. This is precisely the SSR condition in Definition 3.2.

Before we proceed with the proo
geometric description of SSR points
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translation and rotation if necessa
$\mathbf{x}^1 = (0,0,0,-a)$; let us denote the
them from the fixed coordinates $\mathbf{z}$.

Proposition 3.1 *The two poir*
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$$\Sigma = \left\{ \langle \mathbf{s}', \mathbf{s}' \rangle \right.$$

where

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$$\Lambda = \{s_4 =$$

If $\lambda = 1$ then $c = 0$ and the set Σ

Remark The set Λ is easily
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Proof First assume that $\mathbf{x}^0$
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denote the function $g(\cdot - \mathbf{x}^0)$ a

$$\tfrac{1}{2} dg$$

Taking the restriction to $\hat{\Gamma}$ and

$$\tfrac{1}{2} dg_{\pm}$$

From this equation and Defin

on Λ, where

Before we proceed with the proof of the main result let us give an equivalent geometric description of SSR points. Fix two points $\mathbf{x}^0$ and $\mathbf{x}^1$ in $V \setminus \Gamma$ and define a to be half the distance between the points. Without loss of generality, performing translation and rotation if necessary, we can assume that $\mathbf{x}^0 = (0,0,0,a)$ and $\mathbf{x}^1 = (0,0,0,-a)$; let us denote the coordinates by $\mathbf{s} = (s_1, s_2, s_3, s_4)$ to distinguish them from the fixed coordinates $\mathbf{z}$.

Proposition 3.1 *The two points $\mathbf{x}^0$ and $\mathbf{x}^1$ are SSR points with respect to Γ and with constant λ if and only if $\hat{\Gamma}$ is tangent to the complex sphere*

$$\Sigma = \left\{ \langle \mathbf{s}', \mathbf{s}' \rangle + \left(s_4 + \frac{1}{c} \right)^2 + a^2 = \frac{1}{c^2} \right\},$$

where

$$c = \frac{1-\lambda}{a + a\lambda},$$

along the set

$$\Lambda = \{s_4 = 0\} \cap \{\langle \mathbf{s}', \mathbf{s}' \rangle + a^2 = 0\}.$$

If $\lambda = 1$ then $c = 0$ and the set Σ becomes the hyperplane $\{s_4 = 0\}$.

Remark The set Λ is easily seen to be the intersection between the isotropic cones $I_{\mathbf{x}^0}$ and $I_{\mathbf{x}^1}$.

Proof First assume that $\mathbf{x}^0 = (0,0,0,a)$ and $\mathbf{x}^1 = (0,0,0,-a)$ are SSR points with respect to Γ and with constant λ. It is easy to check that the intersection $I_0 \cap I_1$, where $I_j = I_{\mathbf{x}^j}$, is the irreducible set Λ. Since the points, in particular, are SR points it is clear that $\hat{\Gamma} \cap I_j$, for $j = 0, 1$, must be contained in Λ and since this set is irreducible we have equality. To fix the ideas, assume that $\hat{\Gamma}$ can be written $s_4 = \psi(\mathbf{s}')$ near each point on Λ (the general case being similar). Also, let g_- denote the function $g(\cdot - \mathbf{x}^0)$ and g_+ the function $g(\cdot - \mathbf{x}^1)$. It follows that

$$\tfrac{1}{2} dg_\pm = \langle \mathbf{s}', d\mathbf{s}' \rangle + (s_4 \mp a) ds_4.$$

Taking the restriction to $\hat{\Gamma}$ and using the coordinates $\mathbf{s}'$ there we obtain

$$\tfrac{1}{2} dg_\pm|_{\hat{\Gamma}} = \left\langle \mathbf{s}' + (s_4 \mp a)\hat{\nabla}\psi, d\mathbf{s}' \right\rangle.$$

From this equation and Definition 3.2 we see that if the points are SSR points then

$$d\psi = -c \langle \mathbf{s}', d\mathbf{s}' \rangle$$

on Λ, where

$$c = \frac{1-\lambda}{a + a\lambda}.$$

This is equivalent to $\hat{\Gamma}$ being tangent to Σ along Λ. This proves the "only if" part of the proposition. The other part follows from observing that all the steps above are reversible and using the fact, established in Lemma 1.3, that $\hat{\Gamma} \cap I_j$ is irreducible in $\hat{U}$ to deduce that $\hat{\Gamma} \cap I_j$ must equal Λ if Σ is tangent to $\hat{\Gamma}$ along Λ. □

Loosely speaking, Proposition 3.1 states that the condition for two points to be SSR points with respect to Γ means that $\hat{\Gamma}$ has an infinitesimal axial symmetry near the set Λ about the line connecting the two points.

Let us now return to Theorem 3.1. Before we actually enter the proof of it, we prove the following lemma. We state and prove it in the general even dimensional space, because we will need this result in Section 4. Thus, the setting for Lemma 3.1 below is $\mathbb{R}^n$ with n even. Recall from Section 1 that $\hat{\Gamma} = \{z_n = \phi(\mathbf{z}')\}$ and that the coordinates $\mathbf{w}$ are defined by $\mathbf{w}' = \mathbf{z}'$ and $w_n = z_n - \phi(\mathbf{z}')$.

Lemma 3.1 *For any $u \in \mathrm{Har}_0(U, \Gamma)$ and $\mathbf{x}^0 \in V \setminus \Gamma$, we have*

$$u(\mathbf{x}^0) = \int_{\gamma_{\mathbf{x}^0}} \omega_{\mathbf{x}^0}, \tag{3.2}$$

where $\omega_{\mathbf{x}^0}$ is a $n-2$ form on $I_{\mathbf{x}^0} \cap \hat{\Gamma}$ and the orientation of $\gamma_{\mathbf{x}^0}$ is chosen accordingly to the one in (1.1). *Near each point on $I_{\mathbf{x}^0} \cap \hat{\Gamma}$ at which*

$$\frac{1}{2}\frac{\partial g}{\partial w_k}(\cdot - \mathbf{x}^0)\Big|_{\hat{\Gamma}} = w_k - x_k^0 + \frac{\partial \phi}{\partial w_k}(\phi - x_n^0) \neq 0,$$

we have

$$\omega_{\mathbf{x}^0} = \frac{2\pi i c_n}{(p-1)!}\left[\left(\frac{\partial/\partial w_k}{\partial g/\partial w_k}\right)^{p-1}\left(\frac{\partial u/\partial w_n\left(1+\left\langle\hat{\nabla}\phi, \hat{\nabla}\phi\right\rangle\right)}{\partial g/\partial w_k}\right)\right]\omega_k, \tag{3.3}$$

where $p = (n-2)/2$, $\omega_k = (-1)^{k+1} dw_1 \wedge \cdots \wedge \widehat{dw_k} \wedge \cdots \wedge dw_{n-1}$ and where we, by a slight abuse of notation, write g instead of $g(\cdot - \mathbf{x}^0)$.

Proof We know from Lemma 1.1 that $u(\mathbf{x}^0)$ can be obtained as an integral over $C = C_{\mathbf{x}^0}$ of the form α in (1.1). Since the number of dimensions is even, this form is single-valued and, hence, we may replace C by any cycle homologous to C in $M \setminus I$, where $I = I_{\mathbf{x}^0}$. As in the proof of Lemma 2.1, we let T be a tube in M around $\gamma = \gamma_{\mathbf{x}^0}$. Then we may integrate over ∂T instead of over C. Also, $\partial T = \delta\gamma$, where δ denotes the Leray coboundary operator (see e.g. [AY], Chapter III.16). Consequently, we get

$$u(\mathbf{x}^0) = 2\pi i \int_\gamma \operatorname*{Res}_I \alpha, \tag{3.4}$$

where Res means the Leray res

then the proof is finished once

$\alpha = c_n \sum_{j=1}^{n} (-1)^j \Big(u \frac{2p(}{}$

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Proof of Theorem
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where Res means the Leray residue class (see [AY], Chapter III.16). If we set

$$\omega_{\mathbf{x}^0} = 2\pi i \operatorname*{Res}_{I} \alpha$$

then the proof is finished once we show that $\omega_{\mathbf{x}^0}$ has the form (3.3). Note that α is

$$\alpha = c_n \sum_{j=1}^{n} (-1)^j \left(u \frac{2p(z_j - x_j^0)}{g^{p+1}} + \frac{1}{g^p} \frac{\partial u}{\partial z_j} \right) dz_1 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_n,$$

where we, as above, write g instead of $g(\cdot - \mathbf{x}^0)$. This should cause no confusion. We can write this as

$$\alpha = \frac{u\beta}{g^{p+1}} + \frac{\beta'}{g^p} \tag{3.5}$$

for some regular forms β and β'. Since γ is contained in $I \cap \hat{\Gamma}$ we may calculate ω by first taking the restriction (pullback) on $\hat{\Gamma}$ and then taking the residue at $I \cap \hat{\Gamma}$. It is an easy exercise to show that taking the residue commutes with taking restrictions. Since u vanishes on $\hat{\Gamma}$, the first term in (3.5) vanishes when we take the restriction to $\hat{\Gamma}$. The second term becomes, in the coordinates $\mathbf{w}$,

$$c_n \frac{\partial u}{\partial w_n} \frac{1 + \left\langle \hat{\nabla}\phi, \hat{\nabla}\phi \right\rangle}{g^p} \, dw_1 \wedge \cdots \wedge dw_{n-1}$$

on $\hat{\Gamma}$. Now, we can reduce the order of the pole on $I \cap \hat{\Gamma}$ as described in [AY], p. 114, by noting that near points on $I \cap \hat{\Gamma}$ at which $\partial g/\partial w_k \neq 0$ we have

$$dw_1 \wedge \cdots \wedge dw_{n-1} = (-1)^{k+1} dg|_{\hat{\Gamma}} \wedge \frac{dw_1 \wedge \cdots \wedge \widehat{dw_k} \wedge \cdots \wedge dw_{n-1}}{\partial g/\partial w_k}.$$

In the end, we have a form with a simple pole on $I \cap \hat{\Gamma}$ and, since this form has maximal degree on $\hat{\Gamma}$, the residue can be calculated using the Poincaré formula. The result is (3.3). □

Note that when $n = 4$, the differential operator in (3.3) is just the constant 1. We are now ready to prove the theorem.

Proof of Theorem 3.1 Fix two points $\mathbf{x}^0$ and $\mathbf{x}^1$ in $V \setminus \Gamma$. Let us first note that there can be no constant K such that (3.1) holds if $\mathbf{x}^0$ and $\mathbf{x}^1$ are not SR points. The proof of this is practically the same as case (ii) in the proof of Theorem 2.1. The objects corresponding to the disks D_0 and D_1 are the domains in M_0 and M_1, the manifolds denoted by M in Lemma 1.1 defined for the points $\mathbf{x}^0$ and $\mathbf{x}^1$

respectively, bounded by $\gamma_{\mathbf{x}^0}$ and $\gamma_{\mathbf{x}^1}$. Let us denote these domains by B_0 and B_1. Just note that in this situation the analytic functionals T_0 and T_1 are realized by measures supported on $\gamma_{\mathbf{x}^0}$ and $\gamma_{\mathbf{x}^1}$ instead of by measures supported on B_0 and B_1. However, the argument of case (ii) carries through with this modification also. We leave the details to the reader.

We are left with the case when the points $\mathbf{x}^0$ and $\mathbf{x}^1$ are SR points. By Lemma 1.2, we may assume that $\gamma_{\mathbf{x}^0} = \gamma_{\mathbf{x}^1} = \gamma$. Assume that there is a constant K such that (3.1) holds. The continuous functions on γ can be uniformly approximated, on γ, by analytic functions in $\hat{U}$ (by the same argument as in the proof of Theorem 2.1). By the Cauchy–Kowalevskaya theorem, exactly as in the proof of Theorem 2.1, it follows from (3.2) and (3.3), since $n = 4$ and hence $p = 1$, that

$$\omega'_{\mathbf{x}^0} - K\omega'_{\mathbf{x}^1} = 0 \tag{3.6}$$

on γ, where

$$\omega'_{\mathbf{x}^j} = \frac{\omega_{\mathbf{x}^j}}{\partial u/\partial w_4}.$$

The minus sign in the equation (3.6) is due to the fact that the cycles $\gamma_{\mathbf{x}^0}$ and $\gamma_{\mathbf{x}^1}$ have opposite orientation. To see this, note that the intersection between the isotropic cones $I_{\mathbf{x}^0}$ and $I_{\mathbf{x}^1}$ is contained in the (complexified) hyperplane Π perpendicular to and bisecting the line connecting the two points. For simplicity, let us just consider the case when Γ is the hyperplane $\{x_4 = 0\}$ (the general case is similar). In this case, the deformation of the spheres $S^3(\mathbf{x}^0, \epsilon)$ and $S^3(\mathbf{x}^1, \epsilon)$ carried out in Section 1 takes place in a 5 (real) dimensional subspace W of $\mathbb{C}^4$. The intersection $W \cap \Pi$ (in this case, when Γ is a hyperplane, $\Pi = \hat{\Gamma}$ of course) is a hyperplane in W of which the two points $\mathbf{x}^0$ and $\mathbf{x}^1$ are on opposite sides. Since the two spheres $S^3(\mathbf{x}^0, \epsilon)$ and $S^3(\mathbf{x}^1, \epsilon)$ have positive orientation with respect to the cones $I_{\mathbf{x}^0}$ and $I_{\mathbf{x}^1}$, respectively, it is clear that the deformed spheres surrounding the intersection $W \cap I_{\mathbf{x}^0} \cap I_{\mathbf{x}^1}$ have opposite orientation with respect to that intersection. Hence, the cycles $\gamma_{\mathbf{x}^0}$ and $\gamma_{\mathbf{x}^1}$ have opposite orientation.

It follows from (3.3) that

$$dg(\cdot - \mathbf{x}^0)|_{\hat{\Gamma}} - \frac{1}{K} dg(\cdot - \mathbf{x}^1)|_{\hat{\Gamma}} = 0 \tag{3.7}$$

on γ. Since there is a biholomorphic change of coordinates in $\hat{\Gamma} = \{w_4 = 0\}$ taking γ to a set of real codimension one in the real subspace of $\{w_4 = 0\}$ and since the intersection $I_{\mathbf{x}^0} \cap \hat{\Gamma}$ is irreducible, it follows that (3.7) holds on all of $I_{\mathbf{x}^0} \cap \hat{\Gamma}$. This proves the implication (a) $\Rightarrow$ (b). The opposite implication is obvious. □

4. The general even-dimensional case

Let us see what we have to do to generalize Theorem 3.1 to higher even dimensions. Let us first look at Lemma 3.1 in the case where the number of dimensions is even but greater than 4. We notice a difference from the four dimensional case, namely that $\partial u/\partial w_n$ does not appear as a factor in front of the form we are integrating. This causes some problems, because we want to vary the function $\partial u/\partial w_n$ and have the corresponding form $\omega_{\mathbf{x}^0}$ vary over a sufficiently large subspace of the forms with merely continuous coefficients in order to deduce that the analytic functionals T_0 and T_1, used in the proofs of Theorems 2.1 and 3.1, are not supported by any proper subset of $\gamma_{\mathbf{x}^0}$; in this case the functionals are represented by distributions on $\gamma_{\mathbf{x}^0}$ instead of by measures as in the four dimensional case, but this makes no difference. Because of the rather complicated dependence on $\partial u/\partial w_n$ we cannot use the Stone–Weierstrass theorem directly as in the proof of Theorem 3.1. However, once we show that we can make the forms $\omega_{\mathbf{x}^j}$, for $j = 0, 1$, range over a sufficiently large subspace of the forms with continuous coefficients as we let $\partial u/\partial w_n$ range over the analytic functions in $\tilde{U}$ (recall $\tilde{U} = \hat{U} \cap \hat{\Gamma}$), the proof that the existence of a constant K such that (3.1) holds implies that $\mathbf{x}^0$ and $\mathbf{x}^1$ are SR points (the Definition 3.1 immediately generalizes to higher dimensions) can be carried out exactly as in the proof of Theorem 3.1.

The next question is what the generalization of SSR points to higher dimensions should be. It is clear from (3.3) that the same definition as in the four dimensional case is not sufficient. If we expand (3.3) using the Leibnitz formula, it is straightforward to check that the following definition of SSR points is sufficient for the implication (b)⇒(a) in Theorem 3.1 to hold in the higher dimensional case.

Definition 4.1 Two points $\mathbf{x}^0$ and $\mathbf{x}^1$ in $V \setminus \Gamma$ are said to be SSR points, strong Study reflection points, with respect to Γ in $\mathbb{R}^n$ if they are SR points (the Definition 3.1 immediately extends to arbitrary dimensions) and if there is a complex constant λ such that

$$g(\cdot - \mathbf{x}^0)|_{\hat{\Gamma}} \doteq \left((\lambda)^{1/p} + h'\left(g(\cdot - \mathbf{x}^1)|_{\hat{\Gamma}}\right)^p\right) g(\cdot - \mathbf{x}^1)|_{\hat{\Gamma}},$$

where p is as in Lemma 3.1 and h' is some analytic function in $\tilde{U}$.

It is clear that this definition coincides with Definition 3.2 in the case $n = 4$. Similarly to the situation in $\mathbb{R}^4$, *Definition* 4.1 *is equivalent to* $\hat{\Gamma}$ *meeting a certain complex sphere (or hyperplane, cf. Proposition* 3.1*) with center on the line through the points* $\mathbf{x}^0$ *and* $\mathbf{x}^1$ *with p-th order tangency along the intersection between the isotropic cones* $I_{\mathbf{x}^0}$ *and* $I_{\mathbf{x}^1}$. At the end of this section, we illustrate the SSR condition by giving a couple of examples.

Now, if we would like to prove that this definition also gives the full opposite implication in Theorem 3.1, i.e. (a)⇒(b), then we have to know (similar to the situation described above) that as we let $\partial u/\partial w_n$ range over the analytic functions in $\tilde{U}$ the form $\omega_{\mathbf{x}^0} - K\omega_{\mathbf{x}^1}$ ranges over such a large subspace of the forms with continuous coefficients that (a) implies that the differential operator in the expression for $\omega_{\mathbf{x}^0} - K\omega_{\mathbf{x}^1}$ must be identically zero. Given this, the proof of the implication (a)⇒(b) in Theorem 3.1 for general even dimensions would consist of verifying that the differential operator vanishes identically only if $\mathbf{x}^0$ and $\mathbf{x}^1$ are SSR points. This is easy to see if we write $\omega_{\mathbf{x}^0} - K\omega_{\mathbf{x}^1}$ in the form (4.1) below.

The desired properties of $\omega_{\mathbf{x}^0}$ and $\omega_{\mathbf{x}^0} - K\omega_{\mathbf{x}^1}$ mentioned above follow from Assertion 4.1 below and the fact that the analytic functions in $\tilde{U}$, and hence the analytic functions in $\tilde{U} \cap I_{\mathbf{x}^j}$ for $j = 0, 1$, approximate the continuous functions on $\gamma_{\mathbf{x}^j}$. Hence, Assertion 4.1 in combination with the proof of Theorem 3.1 proves Theorem 3.1 in the general even dimensional case with the Definition 4.1 of SSR points.

In order to state and prove Assertion 4.1 we need to make some simplifications and introduce some more notation. As we have noted before, the function

$$1 + \left\langle \hat{\nabla}\phi, \hat{\nabla}\phi \right\rangle$$

does not vanish in $\tilde{U}$. Consequently, there is a one-to-one correspondence between choices of $\partial u/\partial w_n$ and $\partial u/\partial w_n(1+ < \hat{\nabla}\phi, \hat{\nabla}\phi >)$. Let us denote the choice of $\partial u/\partial w_n(1+ < \hat{\nabla}\phi, \hat{\nabla}\phi >)$ by ψ. Let φ_j be the mapping taking an analytic function ψ in $\tilde{U}$ to the analytic $n-2$ form $\omega_{\mathbf{x}^j}$ on $\tilde{U} \cap I_{\mathbf{x}^j}$ for $j = 0, 1$. We write $\mathcal{O}(\tilde{U})$ for the space of analytic functions in $\tilde{U}$ and $\Omega_j^{n-2}(I_{\mathbf{x}^j} \cap \tilde{U})$ for the space of analytic $n-2$ forms on $I_{\mathbf{x}^j} \cap \tilde{U}$. The mapping φ_j can be described as follows:

$$\varphi_j(\psi) = \operatorname*{Res}_{I_{\mathbf{x}^j} \cap \hat{\Gamma}} \frac{\psi}{(g(\cdot - \mathbf{x}^j))^p} dw_1 \wedge \cdots \wedge dw_{n-1}.$$

Assume that $\mathbf{x}^0$ and $\mathbf{x}^1$ are SR points, that $\gamma_{\mathbf{x}^0} = \gamma_{\mathbf{x}^1} = \gamma$ and let Y denote the common intersection $I_{\mathbf{x}^0} \cap \hat{\Gamma} = I_{\mathbf{x}^1} \cap \hat{\Gamma}$. The fact that $\mathbf{x}^0$ and $\mathbf{x}^1$ are SR points implies that the spaces $\Omega_j^{n-2}(\tilde{U} \cap I_{\mathbf{x}^j})$ coincide for $j = 0, 1$. We denote this space by $\Omega_Y^{n-2}(\tilde{U} \cap Y)$. We denote the space of analytic functions in $\tilde{U} \cap Y$ by $\mathcal{O}_Y(\tilde{U} \cap Y)$.

Assertion 4.1 *For every constant K, the image of the mapping $\varphi = \varphi_0 - K\varphi_1$ is an $\mathcal{O}_Y(\tilde{U} \cap Y)$-submodule of $\Omega_Y^{n-2}(\tilde{U} \cap Y)$; in particular, if $\eta \in \operatorname{Im}\varphi$ then $t\eta \in \operatorname{Im}\varphi$ for every $t \in \mathcal{O}_Y(\tilde{U} \cap Y)$. The image is not trivial, i.e. not equal to $\{0\}$, unless the differential operator in the local expression for φ (see (4.1) below) is identically zero.*

Proof Let us write the formula (3.3) in a more canonical form. At each point A on $\tilde{U} \cap Y$, we make a change of coordinates $\mathbf{s} = \mathbf{s}(\mathbf{w}')$ on $\hat{\Gamma}$ such that

$$g(\mathbf{w}'(\mathbf{s}) - \mathbf{x}^1) = s_{n-1}.$$

Since

$g(\cdot$

for some non-vanishing analytic

φ at A:

(4.1)

$\varphi(\psi) = \frac{ds_1 \wedge \cdots \wedge ds_{n-2}}{(p-1)!} \times$

$\left[\left(\frac{\partial/\partial s_{n-1}}{h + s_{n-1}\partial h/\partial s_{n-}}\right.\right.$

where D is the determinant of
that any function $t \in \mathcal{O}_Y(\tilde{U} \cap$
only on $(s_1, ..., s_{n-2})$. Hence, s
derivatives with respect to the
any $t \in \mathcal{O}_Y(\tilde{U} \cap Y)$ we have

(4.2)

near A, where $t\psi$ means, by a
the submanifold Y as a funct
conclude that $t\varphi(\psi)$ belongs
(4.2), as defined above, is o
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where $\tilde{\Omega}_Y^{n-2}$ denotes the tr
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145). Then Im$\tilde{\varphi}$ is a subs
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of $\tilde{\Omega}_Y^{n-2}$, where $\tilde{\mathcal{O}}_Y$ is th
$\Omega_Y^{n-2}(\tilde{U} \cap Y)$, and Im$\varphi$ is
trivial, unless the differe
the differential operator
is identically zero). Thi

Since

$$g(\cdot - \mathbf{x}^0) = hg(\cdot - \mathbf{x}^1)$$

for some non-vanishing analytic function h in $\tilde{U}$, we get the following formula for φ at A:

(4.1)

$$\varphi(\psi) = \frac{ds_1 \wedge \cdots \wedge ds_{n-2}}{(p-1)!} \times \left[\left(\frac{\partial / \partial s_{n-1}}{h + s_{n-1} \partial h / \partial s_{n-1}} \right)^{p-1} \left(\frac{\psi D}{h + s_{n-1} \partial h / \partial s_{n-1}} \right) - K \frac{\partial^{p-1}(\psi D)}{\partial s_{n-1}^{p-1}} \right] \Bigg|_Y ,$$

where D is the determinant of the Jacobian of the coordinate change. Now, note that any function $t \in \mathcal{O}_Y(\tilde{U} \cap Y)$ expressed near A in the coordinates $\mathbf{s}$ depends only on $(s_1, ..., s_{n-2})$. Hence, since the differential operator in (4.1) only involves derivatives with respect to the s_{n-1} variable, it is clear that for any $\psi \in \mathcal{O}(\tilde{U})$ and any $t \in \mathcal{O}_Y(\tilde{U} \cap Y)$ we have

$$\varphi(t\psi) = t\varphi(\psi) \tag{4.2}$$

near A, where $t\psi$ means, by a slight abuse of notation, the function t extended from the submanifold Y as a function independent of s_{n-1} multiplied by ψ. In order to conclude that $t\varphi(\psi)$ belongs to Imφ in $\Omega_Y^{n-2}(\tilde{U} \cap Y)$ (note that the function $t\psi$ in (4.2), as defined above, is only defined locally, although it is possible to extend t globally from Y such that (4.2) holds; this, however, requires some calculations and we leave this alternative conclusion of the proof to the reader), we consider the different spaces introduced above as sheaves. We refer the reader e.g. to [GR], ch. IV, or [B], Appendix A:II, for the basics of sheaf theory. We let $\tilde{\varphi}$ denote the linear sheaf map induced by φ, mapping

$$\tilde{\varphi}\colon \mathcal{O} \longmapsto \tilde{\Omega}^{n-2},$$

where $\tilde{\Omega}_Y^{n-2}$ denotes the trivial extension of Ω_Y^{n-2}, i.e. the stalk of $\tilde{\Omega}_Y^{n-2}$ at a point $A \in \tilde{U}$ equals the stalk of Ω_Y^{n-2} at A if $A \in \tilde{U} \cap Y$ and $\{0\}$ if $A \notin \tilde{U} \cap Y$ (see [GR], p. 145). Then Im$\tilde{\varphi}$ is a subsheaf of $\tilde{\Omega}_Y^{n-2}$. By (4.2), the germ of $t\tilde{\varphi}(\psi)$ at each point A of $\tilde{U} \cap Y$ belongs to Im$\tilde{\varphi}(A)$ (the stalk of Im$\tilde{\varphi}$ at A). For A not on $\tilde{U} \cap Y$, the germ is zero and, hence, also in Im$\tilde{\varphi}(A)$. This means, since Im$\tilde{\varphi}$ is a subsheaf of $\tilde{\Omega}_Y^{n-2}$, that $t\tilde{\varphi}(\psi)$ belongs to Im$\tilde{\varphi}(\tilde{U})$. It also follows that Im$\tilde{\varphi}$ is an $\tilde{\mathcal{O}}_Y$-submodule of $\tilde{\Omega}_Y^{n-2}$, where $\tilde{\mathcal{O}}_Y$ is the trivial extension of $\mathcal{O}_Y$. Thus, $t\varphi(\psi)$ belongs to Imφ in $\Omega_Y^{n-2}(\tilde{U} \cap Y)$, and Im$\varphi$ is an $\mathcal{O}_Y(\tilde{U} \cap Y)$-submodule of $\tilde{\Omega}_Y^{n-2}$. Obviously, Imφ is not trivial, unless the differential operator in (4.1) is identically zero (if it is trivial then the differential operator must, in particular, annihilate all polynomials and, thus, it is identically zero). This completes the proof. □

Let us summarize the results of Section 3 and Section 4 by restating Theorem 3.1 in the general case:

Theorem 4.1 *Let Γ be a a nonsingular, real-analytic hypersurface in some neighborhood U of the origin in $\mathbb{R}^n$, with n even. Then there is a neighborhood V of the origin, with $\overline{V} \subset U$, such that, given two points $\mathbf{x}^0$ and $\mathbf{x}^1$ in $V \setminus \Gamma$, the following are equivalent:*

(a) *there is a constant K such that* (3.1) *holds;*

(b) *the points $\mathbf{x}^0$ and $\mathbf{x}^1$ are SSR points with respect to Γ with the constant λ in the Definition* 4.1 *equal to $1/K$; the number p is as in Lemma* 4.1.

In view of Proposition 0.1, we have the following:

Corollary 4.1 *Assume that Γ is neither part of a hyperplane nor a sphere. Then the functions from* $\mathrm{Har}_0(U,\Gamma)$ *separate the points $\mathbf{x}^0, \mathbf{x}^1 \in V \setminus \Gamma$ if and only if they are not SSR points with respect to Γ.*

We conclude this section with the following examples.

Example 4.1 Let us consider axially symmetric hypersurfaces in $\mathbb{R}^n$, with $n \geq 4$ even. By choosing coordinates appropriately, such a surface Γ can be expressed by an equation of the form $f(x_1, \rho) = 0$, where f is a real analytic function of two variables with nonvanishing gradient along its zero locus and where $\rho^2 = x_2^2 + \cdots + x_n^2$. Geometrically, we can think of Γ as being obtained by revolving a curve

$$\gamma = \{(x_1, \rho) \in \mathbb{R}^2 : f(x_1, \rho) = 0\},$$

symmetric with respect to the x_1-axis, around the x_1-axis in $\mathbb{R}^n$. (We imbed $\mathbb{R}^2$ as a two-plane in $\mathbb{R}^n$ such that the x_1-axis in $\mathbb{R}^2$ coincides with the x_1-axis in $\mathbb{R}^n$.) We call γ the meridian curve and $\mathbb{R}^2$ with coordinates (x_1, ρ) the meridian plane. As mentioned earlier, this situation with $n = 4$ was studied in [Kh]. The result in that paper is that for each point $\mathbf{x}^0$ on the x_1-axis there is a point $\mathbf{x}^1$, also on the x_1-axis, such that (3.1) holds. If we identify $\mathbf{x}^0$ with its corresponding point $(t, 0) \in \mathbb{R}^2$ (where $t = x_1^0$) then the point $\mathbf{x}^1$ corresponds to $(S(t), 0) \in \mathbb{R}^2$, where $S(x_1 + i\rho)$ denotes the Schwarz function of the meredian curve γ, and the constant K in (3.1) equals $-S'(t)$. Let us verify that $\mathbf{x}^0$ and $\mathbf{x}^1$ so defined satisfy the SSR condition with respect to Γ in $\mathbb{R}^4$. Introducing coordinates $\zeta = z_1 + i\rho$, $\zeta^* = z_1 - i\rho$ in the complexified meridian plane (z_1, ρ) (we allow ρ to be complex here), we can write the equations of the isotropic cones $I_{\mathbf{x}^0}, I_{\mathbf{x}^1}$ $(t \in \mathbb{R}!)$ as follows:

$$(4.3) \qquad \begin{cases} g(\cdot - \mathbf{x}^0) = (z_1 - t)^2 + \rho^2 = (\zeta - t)(\zeta^* - t) = 0, \\ g(\cdot - \mathbf{x}^1) = (z_1 - S(t))^2 + \rho^2 = (\zeta - S(t))(\zeta^* - S(t)) = 0. \end{cases}$$

$I_{\mathbf{x}^0}$ and $I_{\mathbf{x}^1}$ meet on the plane $z_1 = (t + S(t))/2$, and the intersections of $I_{\mathbf{x}^j}$ with $\hat{\Gamma}$ meet the complexified meridian plane (ζ, ζ^*) at the two points $(S^{-1}(t), t)$ and $(t, S(t))$; note that $\hat{\gamma} \subset \mathbb{C}^2 := \{(\zeta, \zeta^*)\}$ is given by the equation $\{\zeta^* = S(\zeta)\}$. Hence, in view of (4.3) we have on $I_{\mathbf{x}^0} \cap \hat{\Gamma} = I_{\mathbf{x}^1} \cap \hat{\Gamma}$

$$\begin{cases} dg(\cdot - \mathbf{x}^0)|_{\hat{\Gamma}} = ((S(\zeta) - t) + (\zeta - t)S'(\zeta))d\zeta, \\ dg(\cdot - \mathbf{x}^1)|_{\hat{\Gamma}} = ((S(\zeta) - S(t)) + (\zeta - S(t))S'(\zeta))d\zeta. \end{cases}$$

Thus, the SSR condition in $\mathbb{R}^4$ (in its equivalent formulation Definition 3.2) for $\mathbf{x}^0$ and $\mathbf{x}^1$ at $(t, S(t))$ and $(S^{-1}(t), t)$ is fulfilled with

$$\lambda = -\frac{1}{S'(t)}$$

(for the second point we must keep in mind that $S = S^{-1}$, which follows from the fact that γ is symmetric with respect to the x_1-axis), as claimed above, and the result in [Kh] is seen to agree with Theorems 3.1 and 4.1 of the present paper.

Recall that, geometrically, the SSR condition in $\mathbb{R}^4$ means that $\hat{\Gamma}$ is tangent, at the appropriate points, to the sphere given in Proposition 3.1. In higher even dimensions, $n \geq 6$, the SSR condition is more restrictive (it calls for higher order of contact between $\hat{\Gamma}$ and the sphere), and the corresponding statement for axially symmetric hypersurfaces is not true in general as the following examples illustrate.

(i) Let $\gamma \subset \mathbb{R}^2$ be the ellipse

$$x_1^2 + \frac{\rho_1^2}{a^2} = 1,$$

where $a^2 \neq 1$, and let Γ be the corresponding axially symmetric ellipsoidal surface in $\mathbb{R}^n$, $n \geq 4$ even. Since $\hat{\Gamma}$ is a quadric, it cannot have an order of contact greater than one with another quadric unless the two quadrics are the same. Hence, since $a^2 \neq 1$, no two points are SSR with respect to Γ in $\mathbb{R}^n$, with $n \geq 6$ even.

(ii) Let $\gamma \subset \mathbb{R}^2$ be the curve

$$x_1 = c(\rho^2 + 1)^k,$$

where $k \geq 2$ is an integer and $c > 0$ is small, and let Γ be the corresponding surface of revolution in $\mathbb{R}^n$, with $n \geq 4$ even. We claim that the points $\mathbf{x}^0 = (1, 0, ..., 0)$ and $\mathbf{x}^1 = (-1, 0, ..., 0)$ are SSR points with respect to Γ and with constant $\lambda = 1$ if and only if $n \leq 2k$. To prove this, we shall verify that the complex hyperplane $\{z_1 = 0\}$ in $\mathbb{C}^n$ meets $\hat{\Gamma}$ with order of contact $k - 1$ along the intersection with the isotropic cones $I_{\mathbf{x}^0}$ and $I_{\mathbf{x}^1}$, in accordance with the geometric interpretation of the SSR condition. By axial symmetry, it suffices to check that $\hat{\gamma}$ in the complex meridian plane $\mathbb{C}^2$ meets the two points $(0, i)$ and $(0, -i)$ (the points in $\mathbb{C}^2$ that generate the intersection $I_{\mathbf{x}^0} \cap I_{\mathbf{x}^1}$ in $\mathbb{C}^n$ by revolution) and, moreover, that $\hat{\gamma}$ meets the hyperplane $\{z_1 = 0\}$ with order of contact $k-1$ at $(0, i)$ and $(0, -i)$. If we denote the variables in $\mathbb{C}^2$ by (z_1, ρ) (as above, we allow ρ to be complex here) then $\hat{\gamma}$ is given by

$$z_1 = c(\rho^2 + 1)^k.$$

Obviously, the two points $(0, i)$ and $(0, -i)$ are on $\hat{\gamma}$ and the order of contact between $\hat{\gamma}$ and $\{z_1 = 0\}$ at each of the two points is, as claimed, $k-1$. This proves the claim above, since $\mathbf{x}^0$ and $\mathbf{x}^1$ are SSR with respect to Γ and with constant $\lambda = 1$ if and only if $(n-2)/2 \le k-1$.

5. The general odd-dimensional case

Let us finally consider the general odd dimensional case. First we need a representation lemma similar to Lemmas 2.1 and 3.1. Recall, as in the previous sections, that $\hat{\Gamma} = \{z_n = \phi(\mathbf{z}')\}$ and that the coordinates $\mathbf{w}$ are defined by $w_j = z_j$ for $j = 1, \ldots, n-1$ and $w_n = z_n - \phi(\mathbf{z}')$.

Lemma 5.1 *Let $D_{\mathbf{x}^0}$ be the relatively compact component of $(\hat{\Gamma} \cap M) \setminus \gamma_{\mathbf{x}^0}$, i.e. the domain in $\hat{\Gamma} \cap M$ bounded by the deformed sphere $\gamma_{\mathbf{x}^0}$. Then, for any $u \in \mathrm{Har}_0(U, \Gamma)$ and $\mathbf{x}^0 \in V \setminus \Gamma$, we have*

$$u(\mathbf{x}^0) = 2 \int_{D_{\mathbf{x}^0}} \omega_{\mathbf{x}^0},$$

where $\omega_{\mathbf{x}^0}$ is a $n-1$ form in $D_{\mathbf{x}^0}$ and the orientation of $D_{\mathbf{x}^0}$ is chosen according to the one in (1.1). The form $\omega_{\mathbf{x}^0}$ can be written as

$$\omega_{\mathbf{x}^0} = c_n \frac{\partial u}{\partial w_n} \frac{1 + \left\langle \hat{\nabla}\phi, \hat{\nabla}\phi \right\rangle}{g(\cdot - \mathbf{x}^0)^{(n-2)/2}} \, dw_1 \wedge \cdots \wedge dw_{n-1}, \tag{5.1}$$

where $g(\cdot - \mathbf{x}^0)$ is the defining function for $I_{\mathbf{x}^0}$ as before.

Remark Note that the form $\omega_{\mathbf{x}^0}$ has a singularity on the boundary $\gamma_{\mathbf{x}^0}$. However, this singularity is integrable.

Proof If we would try to prove this directly using a shrinking tube as in the proof of Lemma 2.1 we would run into difficulties. It is not even clear from the rough estimates used in that proof that the two integrals appearing there converge, for $n \ge 5$, as the radius of the tube tends to zero; much less that the integral over the torus tends to zero. We use another approach. Since we have calculated the representation for the general even dimensional case we can use Hadamard's

method of descent (cf. e.g. [J], (
denote the variables in $\mathbb{R}^{n+1}$ by $'\mathbf{x}$
with $s = s' + is''$. The function u (
that does not depend on the variat
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point $(0, \mathbf{x}^0)$ in $\mathbb{R}^{n+1}$. By Lemma

Since $\hat{\Gamma}$ is a cylinder in $\mathbb{C}^{n+1}$ i
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(5.2) $h_\pm = \pm i\sqrt{(}$

($\mathbf{y}$ is the imaginary part of $\mathbf{z}$).

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coordinates. Recall that the
$1, \ldots, n-1$ and $w_n = z_n - \phi($
equation (3.1) using the fact

which is never zero on $\gamma_{'\mathbf{x}}$
$D_{\mathbf{x}^0}$, and the facts that u ar
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$\omega_{\mathbf{x}^0} = h_+^* \omega_{'\mathbf{x}^0} = \frac{2\pi i c_n}{(p -}$

$\times \Big[$

method of descent (cf. e.g. [J], chapter 5). We consider the space $\mathbb{R}^{n+1}$. We denote the variables in $\mathbb{R}^{n+1}$ by $'\mathbf{x} = (s', \mathbf{x})$ and the variables in $\mathbb{C}^{n+1}$ by $'\mathbf{z} = (s, \mathbf{z})$ with $s = s' + is''$. The function u can be extended as a harmonic function in $\mathbb{R}^{n+1}$ that does not depend on the variable s'. We consider also the hypersurface Γ as a cylinder in $\mathbb{R}^{n+1}$. The fact that we use the same notation for the objects considered in the higher dimensional space should cause no confusion. Now, the situation is exactly the one in the previous section because $n+1$ is even. Let $'\mathbf{x}^0$ denote the point $(0, \mathbf{x}^0)$ in $\mathbb{R}^{n+1}$. By Lemma 3.1, we have

$$u(\mathbf{x}^0) = \int_{\gamma_{'\mathbf{x}^0}} \omega_{'\mathbf{x}^0}.$$

Since $\hat{\Gamma}$ is a cylinder in $\mathbb{C}^{n+1}$ it is defined by the equation $\{z_n = \phi(\mathbf{z}')\}$, where ϕ does not depend on s. It follows, from the calculations in Section 1, that $\gamma_{'\mathbf{x}^0}$ can be written as the union $\gamma_+ \cup \gamma_-$ where $\gamma_\pm$ can be written as a graph over $D_{\mathbf{x}^0}$, considered as lying in the subspace $\{s = 0\}$, given by the equation $s = h_\pm(\mathbf{z}')$ and

$$h_\pm = \pm i\sqrt{(|\mathbf{h}|^2+1)(\xi - x_n^0) - (|\mathbf{y}'|^2 + \eta^2)} \tag{5.2}$$

($\mathbf{y}$ is the imaginary part of $\mathbf{z}$). It follows that

$$u(\mathbf{x}^0) = 2\int_{D_{\mathbf{x}^0}} h_+^* \omega_{'\mathbf{x}^0},$$

where h_+^* denotes the pullback by the mapping induced by h_+ and the orientation of $D_{\mathbf{x}^0}$ is chosen in accordance with (1.1). Let us change to another set of coordinates which we have used previously, the coordinates we have denoted by $\mathbf{w}$. In this case though, it is convenient to use the notation $(t, \mathbf{w}) = (t, w_1, ..., w_n)$ for these coordinates. Recall that these coordinates are given by $t = s$, $w_j = z_j$ for $j = 1, ..., n-1$ and $w_n = z_n - \phi(\mathbf{z}')$. The form $h_+^* \omega_{'\mathbf{x}^0}$ is now readily calculated from equation (3.1) using the fact, notation as in Lemma 3.1, that

$$\frac{\partial g(\cdot - '\mathbf{x}^0)}{\partial t} = 2t$$

which is never zero on $\gamma_{'\mathbf{x}^0}$ except on the intersection with $\gamma_{\mathbf{x}^0}$, the boundary of $D_{\mathbf{x}^0}$, and the facts that u and ϕ are independent of the variable t. Also, note that the function under the square root sign in (5.2) equals the restriction of $g(\cdot - \mathbf{x}^0)$ to $D_{\mathbf{x}^0}$. This calculation shows that

$$\begin{aligned}\omega_{\mathbf{x}^0} = h_+^*\omega_{'\mathbf{x}^0} = & \frac{2\pi i c_{n+1}}{(p-1)!}\frac{\partial u}{\partial w_n}\left(1 + \left\langle \hat{\nabla}\phi, \hat{\nabla}\phi \right\rangle\right) \\ & \times \left[\left(\frac{\partial/\partial t}{2t}\right)^{p-1}\left(\frac{1}{2t}\right)\right]\Bigg|_{t = i\sqrt{g(\mathbf{w}' - \mathbf{x}^0)}} dw_1 \wedge \cdots \wedge dw_{n-1},\end{aligned}$$

where $p = [(n+1)-2]/2 = (n-1)/2$ is the integer appearing in Lemma 3.1. It is straightforward to verify that

$$\left(\frac{\partial/\partial t}{2t}\right)^{p-1}\left(\frac{1}{2t}\right) = (-1)^{p+2}\frac{(2p-3)!!}{2^p t^{2p-1}}.$$

If we put this into the expression for ω_{x^0} above, use the definition of the integer p and the fact (cf. [J], page 97) that

$$c_{n+1} = \frac{2^{(n-1)/2}((n-3)/2)!}{2\pi(n-4)!!}c_n$$

we obtain equation (5.1). □

Now, we formulate and prove the last theorem in this paper. This theorem, together with Theorem 4.1, completes the investigation of reflection properties of harmonic functions in $\mathbb{R}^n$.

Theorem 5.1 *Let Γ be a nonsingular, real-analytic hypersurface in some neighborhood U of the origin in $\mathbb{R}^n$, with n odd. Suppose that Γ is neither part of a hyperplane nor a sphere. Then there is a neighborhood V, $\overline{V} \subset U$, such that for no pair of points $\mathbf{x}^0$, $\mathbf{x}^1$ in $V \setminus \Gamma$ is there a constant K satisfying*

$$u(\mathbf{x}^0) + Ku(\mathbf{x}^1) = 0, \qquad \forall u \in \mathrm{Har}_0(U,\Gamma), \tag{5.3}$$

where $\mathrm{Har}_0(U,\Gamma)$ denotes the class of harmonic functions in U vanishing on Γ.

Proof Note that the representation of a harmonic function in $\mathbb{R}^n$, with n odd, given by Lemma 5.1 is of the same form as in $\mathbb{R}^3$, i.e. the normal derivative of the function at $\hat{\Gamma}$ appears as a factor in the form that we are integrating. Hence, the proof of Theorem 2.1 immediately carries over to this case to prove that if there is a constant such that (5.3) holds then we must have

$$\frac{1}{g(\cdot - \mathbf{x}^0)^{(n-2)/2}} - K\frac{1}{g(\cdot - \mathbf{x}^1)^{(n-2)/2}} = 0 \tag{5.4}$$

on Γ. Consequently, Γ is contained in the real-algebraic hypersurface

$$\langle \mathbf{z} - \mathbf{x}^0, \mathbf{z} - \mathbf{x}^0 \rangle = K^{2/(n-2)} \langle \mathbf{z} - \mathbf{x}^0, \mathbf{z} - \mathbf{x}^0 \rangle,$$

i.e. Γ is contained in either a hyperplane or a sphere. □

As before, combining Theorem 5.1 with Proposition 0.1 we obtain:

Corollary 5.1 *If, for $\mathbf{x}^0$, $\mathbf{x}^1 \in V \setminus \Gamma$, the functions from $\mathrm{Har}_0(U,\Gamma)$ do not separate those points then Γ is either part of a sphere or a hyperplane.*

6. A final remark

Thus, Theorems 4.1 and 5.1
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6. A final remark

Thus, Theorems 4.1 and 5.1 completely solve the problem of point to point reflection of harmonic functions in higher dimensions. As in the situation with the Huygens principle (the Huygens principle in the strong form does not hold for general hyperbolic equations), point to point reflection for solutions of more general elliptic equations almost always fails. For example if, for a pair of points near a real-analytic curve $\Gamma \subset \mathbb{R}^2$, the equation (0.1) holds for all solutions of the Helmholtz equation vanishing on Γ then Γ must be a straight line (see [KS] for the proof of this). On the other hand, Garabedian suggested a weaker reflection principle in [G]. Namely, point to compact set reflection, which may hold for solutions of rather general partial differential equations $P(x,D)u = 0$, i.e. that for each point $\mathbf{x}^0$ there is a compact set $K = K(\mathbf{x}^0, \Gamma, P(x,D))$ "on the other side" of the hypersurface Γ and a measure (or a distribution) $\mu = \mu(\mathbf{x}^0, \Gamma, P(x,D))$ supported on K such that

$$u(\mathbf{x}^0) = \int_K u(\mathbf{x})\,d\mu$$

for all solutions of $P(x,D)u = 0$ with $u|_\Gamma = 0$. An explicit calculation confirming this conjecture for the Helmholtz operator in two dimensions has been done in [SSS].

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