ON POINT TO POINT REFLECTION OF HARMONIC FUNCTIONS ACROSS REAL-ANALYTIC HYPERSURFACES IN $\mathbb{R}^n$

By

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Abstract. Let $\Gamma$ be a non-singular real-analytic hypersurface in some domain $U \subset \mathbb{R}^n$ and let $\text{Har}_0(U, \Gamma)$ denote the linear space of harmonic functions in $U$ that vanish on $\Gamma$. We seek a condition on $x^0, x^1 \in U \setminus \Gamma$ such that the reflection law

$$u(x^0) + Ku(x^1) = 0, \quad \forall u \in \text{Har}_0(U, \Gamma)$$

holds for some constant $K$. This is equivalent to the class $\text{Har}_0(U, \Gamma)$ not separating the points $x^0, x^1$. We find that in odd-dimensional spaces (RL) never holds unless $\Gamma$ is a sphere or a hyperplane, in which case there is a well-known reflection generalizing the celebrated Schwarz reflection principle in two variables. In even-dimensional spaces the situation is different. We find a necessary and sufficient condition (denoted the SSR—strong Study reflection—condition), which we describe both analytically and geometrically, for (RL) to hold. This extends and complements previous work by e.g. P. R. Garabedian, H. Lewy, D. Khavinson and H. S. Shapiro.

0. Introduction

In this paper we study the following problem. Let $\Gamma$ be a non-singular real-analytic hypersurface defined in some open set $U$ in $\mathbb{R}^n$, i.e.

$$\Gamma = \{ x = (x_1, ..., x_n) \in U : f(x) = 0 \}$$

where $f$ is a real-analytic function in $U$ with non-vanishing gradient on $\Gamma$. Consider the linear space $\text{Har}_0(U, \Gamma)$ of harmonic functions in $U$ that vanish on $\Gamma$. As is well known from results on elliptic regularity, all functions harmonic on “one side of $\Gamma$”, i.e. in $U^+ = \{ f(x) > 0 \}$, and vanishing on $\Gamma$ extend into a fixed neighborhood $V^- \subset U^- = \{ f(x) < 0 \}$ on “the other side of $\Gamma$” (also cf. [G], [L]). Now the question is whether there is any relationship between the values of such harmonic functions at a particular point in $U^+$ and the values at a certain other point in $U^-$. More precisely, given a point $x^0 \in U^+$ does there exist another point $x^1 \in U^-$ and a constant $K = K(x^0, \Gamma)$ such that

$$u(x^0) + Ku(x^1) = 0 \quad (0.1)$$

for all $u \in \text{Har}_0(U, \Gamma)$? In this paper we will make the neighborhood $U$ small and investigate such a reflection law for points in a slightly smaller neighborhood $V \subset U$.

For \( n = 2 \) an affirmative answer is given by the celebrated Schwarz reflection principle, which actually claims much more (cf. e.g. [D], [Sh]).

**Theorem 0.1 (Schwarz reflection principle)** Let \( \Gamma \subset \mathbb{R}^2 \cong \mathbb{C} \) be a non-singular real-analytic curve in a neighborhood \( U \) of a point \( z' = x' + iy' \). Then there exists a perhaps smaller neighborhood \( V \) of \( z' \) and an anti-conformal involution \( R: V \mapsto V \) such that \( R|_V = \text{id} \) and

\[
(0.2) \quad u(z) + u(R(z)) = 0, \quad z \in V
\]

for all \( u \in \text{Har}_0(U, \Gamma) \).

Let us sketch here a proof (by no means the simplest!) blended from the arguments of Garabedian and Lewy ([G], [L]) that had grown out from an important idea of Study's [St] (see [D] and [Sh] for further discussion). This idea is the starting point of the present paper.

**Sketch of proof** We can write the equation of \( \Gamma \) in the form

\[
\bar{z} = S(z),
\]

where \( S \) is the so-called Schwarz function of \( \Gamma \) holomorphic in \( V \), where \( V \subset U \) is some neighborhood of \( z' \). Shrinking \( V \) even further if necessary we can assume that \( S \) is also univalent in \( V \). Imbed \( \mathbb{R}^2 \) into \( \mathbb{C}^2 = \{ (z, w); z, w \in \mathbb{C} \} \) as the anti-holomorphic plane \( \{ w = \bar{z} \} \), so \( \Gamma \) is the intersection between the complex-analytic curve \( \hat{\Gamma} = \{ w = S(z) \} \) and \( \mathbb{R}^2 \). Fix a point \( z^0 \in V \setminus \Gamma \) and let \( u \in \text{Har}_0(U, \Gamma) \). Then

\[
(0.3) \quad u(z^0) = \frac{1}{2\pi} \int_{\gamma_\epsilon} \left( u(\zeta) \frac{\partial}{\partial n_\zeta} \log |z^0 - \zeta| - \frac{\partial u}{\partial n_\zeta} \log |z^0 - \zeta| \right) ds_\zeta,
\]

where \( \gamma_\epsilon = \{ z \in \mathbb{C} : |z - z^0| = \epsilon \} \) is a sufficiently small circle centered at \( z^0 \), \( n_\zeta \) is the outer normal and \( ds_\zeta \) is the arclength. As is known, \( u \) extends as a holomorphic function of \( z \) and \( w \) into a certain fixed \( C^2 \)-neighborhood \( \hat{U} \) of \( \hat{\Gamma} \) which depends only on \( U \) and not on the function \( u \). The form that we are integrating in (0.3) is a closed (multi-valued in \( C^2 \)) 1-form that has a logarithmic singularity on the isotropic cone with vertex at the point \( z^0 \),

\[
\Lambda^0 = \{ (z - z^0)(w - z^0) = 0 \}.
\]

(\( \Lambda^0 \) reduces to the two bicharacteristic complex lines \( \{ z = z^0 \} \) and \( \{ w = z^0 \} \) passing through \( z^0 \).) Therefore we can move the contour \( \gamma_\epsilon \) homotopically (not homologically due to the multi-valuedness) within \( \hat{U} \setminus \Lambda^0 \) without changing the value of the integral (0.3). Our first goal is to deform \( \gamma_\epsilon \) to a contour on \( \hat{\Gamma} \). For that purpose, note that \( \gamma \) has two disjoint non-linked \( \gamma^1 \) and \( \gamma^2 \) respectively equivalent to the three-sides from the intersection of performing a stereographic a line and a circle remove. At this point it is easy to having deformed \( \gamma_\epsilon \) into \( \gamma^1 \) and \( \gamma^2 \). Thus, \( \gamma_\epsilon \) in (0.3) can be small \( \epsilon \)-circles \( \sigma^1_\epsilon \), \( \sigma^2_\epsilon \) su and an arc \( \sigma^3_\epsilon \) joining \( \sigma^1_\epsilon \) \( \sigma^2_\epsilon \) can be written

\[
(0.4) \quad u(z^0) = \frac{1}{4\pi},
\]

Now observe that as \( \epsilon \to 0 \), while that along \( \sigma^3_\epsilon \)

\[
(0.5) \quad u(z^0) = \frac{1}{4\pi},
\]

where \( \gamma \subset \hat{\Gamma} \) is an arc jo follows from (0.4) since that on “the other side” \( z^1 = S(z^0) \) we obtain

and the theorem follow.

**Remark** In the ne the contour \( \gamma_\epsilon \) in an at another way which yield the isotopic cone (in ti and, thus, is very simpl.

Theorem 0.1 does r a hyperplane since any conformal or anti-conf n \( \geq 3 \), in view of the L
that purpose, note that \( \gamma_\varepsilon \) is homotopic in \( \mathbb{C}^2 \backslash I_c \) to a contour \( \gamma^1_c \) that consists of two disjoint non-linked circles, each one surrounding the complex lines \( \{ z = z^0 \} \) and \( \{ w = w^0 \} \) respectively, joined by a “handle”.\n
Indeed, \( \mathbb{C}^2 \backslash I_c \) is homotopically equivalent to the three-sphere \( S^3 \) without two one-dimensional circles resulting from the intersection of \( I_c \) with \( S^3 \). Choosing a point on one of these circles and performing a stereographic projection with a pole at this point we obtain \( \mathbb{R}^3 \) with a line and a circle removed. The contour \( \gamma_\varepsilon \) becomes a loop tying them together.\n
At this point it is easy to see that \( \gamma_\varepsilon \) is homotopic to the contour \( \gamma^1_c \) defined above.\n
Having deformed \( \gamma_\varepsilon \) into \( \gamma^1_c \) we move \( \gamma^1_c \) continuously in \( \mathbb{C}^2 \backslash I_c \) and lay it on \( \hat{\Gamma} \).\n
Thus, \( \gamma_\varepsilon \) in (0.3) can be replaced by a closed contour \( \gamma^2_\varepsilon \) on \( \hat{\Gamma} \) that consists of two small \( \varepsilon \)-circles \( \sigma^1_\varepsilon \), \( \sigma^2_\varepsilon \) surrounding the points \( A = (S^{-1}(z^0), z^0) \) and \( B = (z^0, S(z^0)) \) and an arc \( \sigma^3_\varepsilon \) joining \( \sigma^1_\varepsilon \) and \( \sigma^2_\varepsilon \) travelled twice. Since \( u|_B = 0 \), the equation (0.3) can be written\n
\[
(0.4) \quad u(z^0) = \frac{1}{4\pi i} \int_{\sigma^2_\varepsilon} \log(z - z^0)(w - z^0) \left( \frac{\partial u}{\partial z} \, dz - \frac{\partial u}{\partial w} \, dw \right).
\]

Now observe that as \( \varepsilon \to 0 \) the integrals in (0.4) over the circles \( \sigma^1_\varepsilon \) and \( \sigma^2_\varepsilon \) tend to zero, while that along \( \sigma^3_\varepsilon \) tends to\n
\[
(0.5) \quad u(z^0) = \frac{1}{2} \int_\gamma \left( \frac{\partial u}{\partial z} \, dz - \frac{\partial u}{\partial w} \, dw \right),
\]

where \( \gamma \subset \hat{\Gamma} \) is an arc joining the points \( A \) and \( B \) (from \( A \) to \( B \)). The equation (0.5) follows from (0.4) since the logarithm in (0.4) on “one side of \( \sigma^3_\varepsilon \)” differs from that on “the other side” by \( 2\pi i \). Finally applying the same argument to the point \( z^1 = S(z^0) \) we obtain\n
\[
(0.6) \quad u(z^1) = \frac{1}{2} \int_{-\gamma} \left( \frac{\partial u}{\partial z} \, dz - \frac{\partial u}{\partial w} \, dw \right),
\]

and the theorem follows with \( R(z) = \overline{S(z)} \). \( \square \)

**Remark** In the next section we make a similar deformation to the one of the contour \( \gamma_\varepsilon \) in an arbitrary number of dimensions. However, we do this in another way which yields more information about the intersection between \( \hat{\Gamma} \) and the isotropic cone (in the above case, that intersection consists of only two points and, thus, is very simple).

Theorem 0.1 does not extend to higher dimensions unless \( \Gamma \) is a sphere or a hyperplane since any mapping preserving harmonic functions must be either conformal or anti-conformal (KS) and those mappings are extremely limited in \( \mathbb{R}^n \), \( n \geq 3 \), in view of the Liouville theorem. (If \( \Gamma \) is a hyperplane, say \( \{ x_n = 0 \} \), then
(0.2) holds with \( R(x) = (x_1, \ldots, x_{n-1}, -x_n) \), while for a sphere \( \Gamma = \{ |x| = \rho \} \) the equation (0.1) holds with \( x^1 = R(x^0) = \rho^2 x^0/|x^0|^2 \)—the Kelvin transformation—and \( K = |x^0|^{2-n} \), cf. e.g. [Ke]). Recall (cf. the proof of Theorem 0.1 above; also, cf. [Stu] and [Sh]) that, for \( n = 2 \), the pairing between the points \( x^0 = z \) and \( x^1 = R(z) \) in Theorem 0.1 above is such that the isotropic cones (see below) emanating from the points \( x^0 \) and \( x^1 \) “meet” on the complexification \( \hat{\Gamma} \) of \( \Gamma \), i.e. on \( \hat{\Gamma} = \{ z \in \mathbb{C}^n : f(z) = 0 \} \). It is therefore quite natural to conjecture that for (0.1) to hold in higher dimensions it is necessary that the isotropic cones

\[
I_{x^k} = \{ z \in \mathbb{C}^n : \sum_{j=1}^n (z_j - x_j^k)^2 = 0 \},
\]

for \( k = 0, 1 \), satisfy

\[
(0.6) \quad I_{x^0} \cap \hat{\Gamma} = I_{x^1} \cap \hat{\Gamma}
\]

(the Study relation, denoted SR in §3). In fact, if we enlarge the class of test functions \( \text{Har}_0(U, \Gamma) \) and allow polar singularities near \( \Gamma \), the necessity of this condition is easy to prove [KS]. On the other hand, as it was shown in [KS], if for sufficiently many (e.g. a set of positive measure) points \( x^0 \) near \( \Gamma \) we can find a Study related point \( x^1 \) (i.e. such that (0.6) holds) then \( \Gamma \) must be a hyperplane or a sphere, provided that \( \Gamma \) is algebraic. Thus, we can expect the reflection law (0.1) to hold only for very special points \( x^0, x^1 \) near a generic analytic hypersurface \( \Gamma \).

The reflection law (0.1) is equivalent to the separation of points question for the class \( \text{Har}_0(U, \Gamma) \). Indeed, we have the following simple proposition.

**Proposition 0.1** Let \( \Gamma, U, x^0, x^1 \) be as above. The following are equivalent:

(a) there is no constant \( K \) for which (0.1) holds for all \( u \in \text{Har}_0(U, \Gamma) \);

(b) there is a function \( u \in \text{Har}_0(U, \Gamma) \) such that

\[
\begin{aligned}
& u(x^0) = 1, \\
& u(x^1) = 0.
\end{aligned}
\]

**Proof** The implication (b) \( \Rightarrow \) (a) is obvious. Assume that (a) holds. Then there are two functions \( u_1, u_2 \in \text{Har}_0(U, \Gamma) \) such that neither function vanishes at both points and such that

\[
u_1(x^0) + K_1 u_1(x^1) = u_2(x^0) + K_2 u_2(x^1) = 0,
\]

where \( K_1 \neq K_2 \). If one takes \( u = u_1/u_2 \), then, as it is straightforward to check, it satisfies

\[
I_{x^0} \cap \Gamma = I_{x^1} \cap \Gamma.
\]

This proves the implication (b) \( \Rightarrow \) (a).

In [KS] it was shown that whenever \( \Gamma \) is a cylinder of circular cylinders in \( \mathbb{R}^4 \) by showing that \( w \) is independent of \( z \), \( \Gamma \) must be either a plumb or a plane. This is proven for arbitrary \( \Gamma \) in the case where \( \text{Har}_0(U, \Gamma) \) consists of reflection (transformation in a suitable dimensional space) is a spherical reflection in a hyperplane parallel to \( \Gamma \) satisfying the Study relation (0.6). It is, perhaps, a new phenomenon that for all even \( n \geq 1 \), the Study reflection—coisometric (0.1) is Morita—almost as the dimension of \( \mathbb{R}^4 \) increases.

Let us briefly discuss the topological consequences: the harmonic analysis of \( \text{Har}_0(U, \Gamma) \) by integrals of several hyperplane functions in \( \mathbb{R}^3 \) except across the critical condition in \( \mathbb{R}^4 \) and reflection principle in \( \mathbb{R}^3 \). The same holds for even-dimensional spaces. Finally, in \( \mathbb{R}^3 \) there is no reflection principle, and in \( \mathbb{R}^4 \) there is a major theorem in even-dimensional spaces.
where \( K_1 \neq K_2 \). If one of them, say \( u_1 \), is zero at \( x^1 \) then we are finished, because we can take \( u = u_1 / u_1(x^0) \), so let us assume that neither function vanishes at \( x^1 \). Then, as it is straightforward to verify, the function

\[
    u = \frac{1}{K_2 - K_1} \left( \frac{u_1}{u_1(x^1)} - \frac{u_2}{u_2(x^1)} \right)
\]

satisfies

\[
    \begin{cases}
        u(x^0) = 1, \\
        u(x^1) = 0.
    \end{cases}
\]

This proves the implication (a) \( \Rightarrow \) (b). \( \square \)

In [KS] it was shown that the functions in \( \operatorname{Har}_0(U, \Gamma) \) separate points in \( U \setminus \Gamma \) whenever \( \Gamma \) is a cylindrical, non-flat surface in \( \mathbb{R}^3 \). (Also, cf. the treatment of circular cylinders in [Shh].) In Section 2 we extend this result to all surfaces in \( \mathbb{R}^3 \) by showing that whenever (0.1) holds for a pair of points \( x^0, x^1 \) near \( \Gamma \) then \( \Gamma \) must be either a plane or a sphere (Theorem 2.1). In Section 5, Theorem 5.1, this is proven for arbitrary odd-dimensional spaces. Thus, in all odd-dimensional spaces the reflection (0.1) is strictly limited to the reflection in a plane and Kelvin transformation in a sphere for all pairs of points \( x^0, x^1 \). The situation for even-dimensional spaces is much more delicate and (not surprisingly) is reminiscent of the situation with the celebrated Huygens principle. In [Kh], (0.1) was shown to hold in \( \mathbb{R}^4 \) for axially symmetric surfaces and points \( x^0, x^1 \) on the axis of symmetry satisfying the Study relation (0.6). In view of this, it was suggested in [Kh] that (0.6) is, perhaps, a necessary and sufficient condition for (0.1) in even-dimensional spaces, as it is in \( \mathbb{R}^2 \) (cf. also the discussion in [KS]). However, it turns out that for all even \( n \geq 4 \) one needs a stronger condition (denoted the SSR—strong Study reflection—condition in this paper) which is necessary and sufficient for the reflection (0.1). Moreover, the SSR condition turns out to become more and more restrictive as the dimension \( n \) of the space increases.

Let us briefly describe the contents of this paper. In Section 1 we present the topological construction allowing us to represent the values of functions in \( \operatorname{Ham}(U, \Gamma) \) by integrals over certain cycles on \( \hat{\Gamma} \) (the idea of doing this goes back to Garabedian [G]). In Section 2 we prove that no point to point reflection exists in \( \mathbb{R}^3 \) except across spheres and hyperplanes. In Section 3 we introduce the SSR condition in \( \mathbb{R}^4 \) and show that this condition is necessary and sufficient for the reflection principle to hold. In Section 4 we extend this to all even-dimensional spaces. Finally, in Section 5, using Hadamard’s method of descent, we show that there is no reflection law in any odd-dimensional space. Thus, in a sense, we prove our major theorems twice. First for dimensions 3 and 4 and then for arbitrary odd- and even-dimensional spaces. We hope that this way of presenting the material
will help to clarify the crucial points of the argument since the complexity and
tediousness of the problem do increase with the dimensionality.

1. Some topology and notations

Let as above $\Gamma$ be a nonsingular, real-analytic hypersurface through the origin
in some neighborhood $U$ of the origin in $\mathbb{R}^n$, $\Gamma = \{ x \in U : f(x) = 0 \}$. First observe
that there is no loss of generality in assuming that the hyperplane $\{ x_n = 0 \}$ is
tangent to $\Gamma$, because this paper deals with properties of harmonic functions in $U$
and the class of harmonic functions is invariant under rotations. Consequently, in
some smaller neighborhood, which we still denote by $U$, we may write

$$\Gamma = \{ x \in U : x_n = \phi(x') \}$$

where $x' = (x_1, ..., x_{n-1})$ and $\phi$ is some real-valued real-analytic function of $n - 1$
variables with

$$\nabla \phi(0) = \left( \frac{\partial \phi}{\partial x_1}, ..., \frac{\partial \phi}{\partial x_{n-1}} \right)(0) = (0, ..., 0).$$

In the context of this paper there is no loss of generality in replacing $U$ by a smaller
neighborhood. In order to simplify the notation in certain formulas, we will use the
notation

$$\textbf{0} = \sum_{j=1}^{k} a_j b_j,$$

where $\textbf{a} = (a_1, ..., a_k)$ and $\textbf{b} = (b_1, ..., b_k)$ are real or complex $k$
dimensional vectors. The dimension $k$ should be clear from the context. For real vectors, this
expression coincides with the usual Euclidean scalar product and then we will use
the notation $|\textbf{a}|^2 = \langle \textbf{a}, \textbf{a} \rangle$ for the Euclidean norm. We will also use the convention
that if $\textbf{a} = (a_1, ..., a_k)$ is a given vector then $\textbf{a}'$ denotes the vector $(a_1, ..., a_{k-1})$.

If $N(x)$ denotes the Newtonian potential of a point mass at the origin in $\mathbb{R}^n$, i.e.,

$$N(x) = \frac{c_n}{|x|^{n-2}}$$

where $c_n$ is some constant depending on the dimension $n$, then the value of any
harmonic function $u$ at a point $x^0$ can be represented as

$$(1.1) \quad u(x^0) = \int_{S^{n-1}(x^0)} \sum_{j=1}^{n} (-1)^{j+1} \left( u(x) \frac{\partial N}{\partial x_j}(x - x^0) - \frac{\partial u}{\partial x_j}(x) N(x - x^0) \right) \omega_j,$$

where $S^{n-1}(x^0, \epsilon)$ is the $n - 1$ dimensional sphere of radius $\epsilon$, for some sufficiently
small $\epsilon > 0$, centered at $x^0$ oriented by the outward normal and

$$\omega_j = dx_1 \wedge ... \wedge \widehat{dx_j} \wedge ... \wedge dx_n,$$

where $\widehat{dx_j}$ means omission of

$$z = (z' , \epsilon) \in \mathbb{C}^n$$

in $\mathbb{C}^n$ and letting $\mathbb{R}^n = \{ z' : |y| < \epsilon \}$
it is an analytic function of
in $\mathbb{C}^{n-1}$. We will use the
$(\theta/\partial x_1, ..., \theta/\partial x_n)$. The dim
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$\{ z : z_n = \phi(z') \}$. This is an
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Let us write

$$(\xi, \eta) \in \mathbb{R}^n$$

where $\xi$ and $\eta$ are rea
restriction of $\phi$ to $\mathbb{R}^n$
where \( \Delta \gamma \) means omission of \( d\gamma \). Let us imbed \( \mathbb{R}^n \) in \( \mathbb{C}^n \) by using the coordinates
\[
z = (z_1, \ldots, z_n) = (x_1 + iy_1, \ldots, x_n + iy_n)
\]
in \( \mathbb{C}^n \) and letting \( \mathbb{R}^n = \{ z : |y| = 0 \} \). Since \( \phi \) is real-analytic near the origin in \( \mathbb{R}^{n-1} \) it is an analytic function of the variables \( z' \) in some neighborhood of the origin in \( \mathbb{C}^{n-1} \). We will use the notation \( \nabla \) for the complex gradient, i.e. the vector \( (\partial/\partial z_1, \ldots, \partial/\partial z_n) \). The dimension \( k \) of this vector should be clear from the context, e.g. if we apply \( \nabla \) to \( \phi \) then \( k = n - 1 \). We define \( \hat{\Gamma} \), the complexification of \( \Gamma \), as \( \{ z : z_n = \phi(z') \} \). This is an analytic hypersurface in some neighborhood \( \hat{U} \) of the origin in \( \mathbb{C}^n \) such that its restriction to \( \mathbb{R}^n \) equals \( \Gamma \). We choose \( \hat{U} \) such that every harmonic function in \( U = \hat{U} \cap \mathbb{R}^n \) extends as an analytic function into \( \hat{U} \), i.e. such that \( \hat{U} \) is the so-called hull of harmonicity of \( U \).

Now, the Newtonian potential \( N \) extends as an analytic function, multi-valued if \( n \) is odd, into \( \mathbb{C}^n \setminus I_0 \), where \( I_0 \) denotes the isotropic cone with vertex at \( z^0 \)
\[
I_0 = \{ z : \langle z - z^0, z - z^0 \rangle = 0 \}
\]
and, hence, the form we are integrating in (1.1) extends as a (multi-valued if \( n \) is odd) form into \( \hat{U} \setminus I_0 \). Since \( u(z) \) is harmonic in \( \hat{U} \), i.e. satisfies
\[
\Delta u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial z_j^2} = 0
\]
and \( N(z - x^0) \) is harmonic in \( \mathbb{C}^n \setminus I_0 \) it follows that this form is closed. If \( n \) is even then the integral in (1.1) is independent of the cycle of integration as long as the cycle stays in the same homology class. If \( n \) is odd then the form we are integrating is single-valued in a neighborhood of \( S^{n-1}(x^0, \epsilon) \) and, therefore, the integral in (1.1) is independent of the cycle of integration as long as this cycle stays in the same homotopy class, in \( \hat{U} \setminus I_0 \), as \( S^{n-1}(x^0, \epsilon) \). Note, however, that due to the multi-valuedness of the form the integral is not independent of the cycle if we only restrict it to the same homology class. Our main goal in this section is to prove that \( S^{n-1}(x^0, \epsilon) \) can be deformed continuously in \( \hat{U} \setminus I_0 \) to a cycle surrounding (in a certain sense) part of the intersection \( \hat{\Gamma} \cap I_0 \). This will enable us to replace the cycle of integration in (1.1) by one which is close to \( \hat{\Gamma} \) and this, in turn, will enable us to "compare" different values \( u(x^0) \) and \( u(x^1) \). The precise statements of the topological results needed are presented in Lemmas 1.1, 1.2, and 1.3 below.

Let us write
\[
\phi(z') = \xi(z') + i\eta(z'),
\]
where \( \xi \) and \( \eta \) are real-valued functions. Since \( \nabla \phi \) vanishes at the origin and the restriction of \( \phi \) to \( \mathbb{R}^{n-1} \) is real-valued, the coefficients in the Taylor expansion
of \( \phi \) at the origin are all real and the expansion starts with the quadratic terms. Consequently, we can write

\[
\eta(x' + iy') = (y', h(x' + iy')) = y_1 h_1(x' + iy') + \cdots + y_{n-1} h_{n-1}(x' + iy').
\]

The functions \( h = (h_1, \ldots, h_{n-1}) \) are not uniquely determined in general, but we can make them unique e.g. by demanding that \( h_j \), for \( j = 2, \ldots, n - 1 \), is independent of \( y_1, \ldots, y_{j-1} \) (we let \( y_1 h_1(x' + iy') \) consist of all terms in the Taylor expansion containing \( y_1 \), \( y_2 h_2(x' + iy') \) consist of all remaining terms containing \( y_2 \), etc.).

**Lemma 1.1** There is a neighborhood \( V \) of the origin in \( \mathbb{R}^n \), with \( V \subset \hat{U} \cap \mathbb{R}^n \), such that the following is true for all \( x^0 \in V \setminus \Gamma \):

(a) the intersection of \( I_{x^0} \) and the \( n \) dimensional manifold \( M \) defined by the equations

\[
\begin{align*}
(x' &= x^0' + (x_n^0 - \xi)h \\
x_n &= \xi
\end{align*}
\]

is a smooth \( n - 2 \) dimensional manifold \( \gamma_{x^0} \), homeomorphic to the \( n - 2 \) dimensional sphere, contained in \( M \cap \Gamma \), i.e. in the \( n - 1 \) dimensional submanifold of \( M \) satisfying the equation

\[
y_n = y_i;
\]

(b) the sphere \( S^{n-1}(x^0, \varepsilon) \) is homotopic in \( \hat{U} \setminus I_{x^0} \) to the \( n - 1 \) dimensional boundary \( C_{x^0} \) of a contractible neighborhood of \( \gamma_{x^0} \) in \( M \).

**Remark** Note that the first \( n - 1 \) equations defining \( M \) are implicit and, hence, we have to make sure that we choose \( \hat{U} \) and \( V \) so small, a priori, that \( M \), defined in the lemma, is a manifold in \( \hat{U} \).

**Proof** First, fix some \( x^0 \in \mathbb{R}^n \setminus \Gamma \), close to the origin. We will see how close, i.e. how small we have to make \( V \), eventually. The idea of the proof is to deform \( S^{n-1}(x^0, \varepsilon) \) along a family of \( n \)-planes starting with \( \mathbb{R}^n \) and ending with one that approximates \( M \), and then along a family of \( n \)-manifolds ending with \( M \), all the while keeping control over the intersection between \( I_{x^0} \) and the \( n \) dimensional planes/manifolds.

As a first step, we consider the following continuous family of \( n \) dimensional planes \( \Pi_t \), for \( t \in [0, 1] \), in \( \mathbb{C}^n \):

\[
\begin{align*}
(1 - t)y' &= t(x' - x^0') \\
(1 - t)y_n &= tx_n.
\end{align*}
\]

Note that \( \Pi_0 = \mathbb{R}^n \) and \( \Pi_1 \), which has the subspace \( \hat{I} \). We rewrite the equation

\[
(1.3)
\]

If we use the coordinates \( \Pi_t \cap I_{x^0} \) becomes

\[
\begin{align*}
|x' - : \\
|x' - |
\end{align*}
\]

Let us first assume that can be written as

\[
\begin{align*}
\left| x' - : \right| \\
| x' - |
\end{align*}
\]

The second equation, in the moving plane \( I \) for \( t < 1/2 \), then it is along with the plane encloses the intersection between \( I_{x^0} \) and the \( n \) dimensional planes/manifolds.

so we can move \( S(\cdot) \), leave to the reader), the cycle \( S_1 \) all the

The latter is given by
Note that \( \Pi_0 = \mathbb{R}^n \) and \( \Pi_1 \) is the plane
\[
\begin{cases}
  x' = x'^0, \\
  x_n = 0,
\end{cases}
\]
which has the subspace \( \{y_n = 0\} \) in common with \( \{z_n = 0\} \), the tangent plane of \( \hat{\Gamma} \). We rewrite the equation of \( I_x^\circ \) in terms of real variables:
\[
\begin{cases}
  |x - x^0|^2 = |y|^2, \\
  \langle x - x^0, y \rangle = 0.
\end{cases}
\tag{1.3}
\]
If we use the coordinates \( x \) on \( \Pi_t \) (this works well for \( t \leq 1/2 \)) then the intersection \( \Pi_t \cap I_x^\circ \) becomes
\[
\begin{cases}
  |x - x^0|^2 = \frac{t^2}{(1-t)^2}(|x' - x'^0|^2 + x_n^2), \\
  \frac{t^2}{(1-t)^2}(|x' - x'^0|^2 + (x_n - x_n^0)x_n) = 0.
\end{cases}
\]
Let us first assume that \( t < 1/2 \). A straightforward calculation shows that the latter can be written as
\[
\begin{cases}
  |x' - x'^0|^2 + \left(x_n - \frac{(1-t)^2}{1-2t}x_n^0\right)^2 = \frac{t^2(1-t)^2}{(1-2t)^2}(x_n^0)^2, \\
  |x' - x'^0|^2 + \left(x_n - \frac{1}{2}x_n^0\right)^2 = \frac{1}{4}(x_n^0)^2.
\end{cases}
\]
The second equation describes a fixed sphere of radius \( x_n^0/2 \) centered at \( (x_n^0, x_n^0/2) \) in the moving plane \( \Pi_t \). If we choose \( V \) so small that this sphere is contained in \( \hat{U} \) for \( t < 1/2 \), then it is clear that we can move the sphere \( S^{n-1}(x^0, \epsilon) \) homotopically along with the planes \( \Pi_t \) such that this moving cycle—let us denote it by \( S_t \)—encloses the intersection \( \Pi_t \cap I_x^\circ \). For \( t = 1/2 \) the equation of the intersection can be written as
\[
\begin{cases}
  x_n = \frac{1}{2}x_n^0, \\
  |x' - x'^0|^2 + (x_n - \frac{1}{2}x_n^0)^2 = \frac{1}{4}(x_n^0)^2,
\end{cases}
\]
so we can move \( S_t \) all the way to \( \Pi_{1/2} \). A similar argument for \( t > 1/2 \) (that we leave to the reader), using instead the coordinates \( y \) on \( \Pi_t \), shows that we can move the cycle \( S_t \) all the way to a cycle \( S_1 \) in \( \Pi_1 \) that encloses the intersection \( \Pi_1 \cap I_x^\circ \). The latter is given by the equations
\[
\begin{cases}
  |y|^2 = (x_n^0)^2, \\
  x_n^0y_n = 0.
\end{cases}
\]
Next, we move the cycle with the \( n \) dimensional planes \( \Pi_{1+t} \), for \( t \in [0, 1] \), defined by

\[
\begin{align*}
x' &= x^0', \\
x_n &= t\xi(x^0').
\end{align*}
\]

The intersection \( \Pi_{1+t} \cap I_k \) is given by

\[
\begin{align*}
|y|^2 &= (t\xi(x^0') - x_n^0)^2, \\
(\xi(x^0') - x_n^0)y_n &= 0.
\end{align*}
\]

The same arguments as above show that we can move the cycle \( S_1 \) to a cycle \( S_2 \) in \( \Pi_2 \) surrounding the intersection

\[
\begin{align*}
|y|^2 &= (\xi(x^0') - x_n^0)^2, \\
(\xi(x^0') - x_n^0)y_n &= 0,
\end{align*}
\]

which, since \( x^0 \notin \Gamma \) implies \( \xi(x^0') \neq x_n^0 \), can be written as

\[
\begin{align*}
|y|^2 &= (\xi(x^0') - x_n^0)^2, \\
y_n &= 0,
\end{align*}
\]

i.e. a circle of radius \( |\xi(x^0') - x_n^0| \) centered at the origin in the plane \( y_n = 0 \).

Now, let us consider the manifold \( M \) defined by (1.2). By the implicit function theorem, there is a neighborhood \( W \) of the origin in \( \mathbb{R}^n \), a neighborhood \( V \) of the origin in \( \mathbb{R}^n \), and functions \( f = (f_1, \ldots, f_n) \) independent of \( y_n \) such that \( M \) can be written as a graph in \( \mathbb{C}^n \cong \mathbb{R}^{2n} \) over \( \Pi_2 \) as follows (note that according to our convention \( \xi' \) denotes the vector \( (f_1, \ldots, f_{n-1}) \) and not the derivative):

\[
\begin{align*}
x' &= x^0' + f'(y', x^0), \\
x_n &= \xi(x^0') + f_n(y', x^0),
\end{align*}
\]

for \((y', y_n) \in W \times \mathbb{R} \) and \( x^0 \in V \). Moreover, we have \( f(0, 0) = 0 \). For the final step in the deformation of the cycle, we consider the continuous family of \( n \) dimensional manifolds \( M_{2+t} \), for \( t \in [0, 1] \),

\[
\begin{align*}
x' &= x^0' + tf'(y', x^0), \\
x_n &= \xi(x^0') + tf_n(y', x^0),
\end{align*}
\]

which moves from \( \Pi_2 \), at \( t = 0 \), to \( M = M_3 \), at \( t = 1 \). To control the intersection with \( I_k \), we need the following transversality results.

**Assertion 1.1** If \( B \) is a relatively compact domain in \( \mathbb{C}^n \cong \mathbb{R}^{2n} \) then any manifold \( N \) in \( B \) that satisfies the following two conditions, \( (i) \) and \( (ii) \), meets \( I_{k_0} \) transversally.

(i) \( x^0 \notin N \cap I_{k_0} \);

(ii) any unit conormal \( r \in (\mathbb{C}^n)^* \), \( r = u \)

**Proof of Assertion** means that the vector equals \( \mathbb{R}^{2n} \). This is the at each point of intersection equivalent to the sets a singularity at \( z = \).

(ii) states that \( x^0 \) is not in view of Hypothesis \( I_{k_0} \setminus \{x^0\} \) is such that unit conormals, we simplify the notation for the point \( r = u + iv \); orientation by a fact defining \( I_0 \) that any cross \( (u, v) = a(x, y) + i \).

Thus, \( |v|^2 = a^2 |y|^2 + \) holds for \( |u|^2 \).

**Assertion 1.2** if for every \( t \in [0, 1] \), \( t \) not meet the boundary.

**Remark** Note \( t \)

**Proof of Asser**

\( M_{2+t} \) satisfies cond sufficiently small, i.e. that \( x^0 \in M_{2+t} \). But, from (1.2), we

which equals zero does not equal \( x_n^0 \) \( t \in [0, 1] \).

To verify cond
(i) $x^0 \notin N \cap I_{x^0}$;

(ii) any unit conormal (i.e., unit covector that annihilates the tangent space) $r \in (C^n)^*, r = u + iv = (u_1 + iv_1, \ldots, u_n + iv_n)$, of $N$ is such that $|v| < 1/\sqrt{2}$.

**Proof of Assertion 1.1** That two manifolds $N$ and $N'$ in $\mathbb{R}^{2n}$ meet transversally means that the vector sum of the two tangent spaces at each point of intersection equals $\mathbb{R}^{2n}$. This is the same as saying that the intersection of the conormal spaces, at each point of intersection, contains only the zero covector which, in turn, is equivalent to the sets of unit conormals being disjoint. The hypersurface $I_{x^0}$ has a singularity at $z = x^0$, but away from that point it is a manifold. Hypothesis (i) states that $x^0$ is not in the intersection $N \cap I_{x^0}$. The proof will be completed, in view of Hypothesis (ii), by showing that any unit conormal $r = u + iv$ of $I_{x^0} \setminus \{x^0\}$ is such that $|v| = 1/\sqrt{2}$. Since translations do not alter the set of unit conormals, we may assume that $x^0 = 0$. Let us, just in this proof and for simplicity of notation, identify the cotangent space $(C^n)^*$ with $\mathbb{R}^{2n}$ by saying that the point $r = u + iv \in (C^n)^*$ corresponds to $(u, v) \in \mathbb{R}^{2n}$ (this differs from the usual orientation by a factor $(-1)^n$). It follows immediately from the two real equations defining $I_0$ that any unit conormal in $\mathbb{R}^{2n}$ of $I_0$ at a point $z = x + iy$ is of the form $(u, v) = a(x, -y) + b(y, x)$, where $a, b \in \mathbb{R}$ are such that the vector has unit norm. Thus, $|v|^2 = a^2|y|^2 + b^2|x|^2$ and since $z \in I_0$ we get $|v|^2 = (a^2 + b^2)|x|^2$. The same holds for $|u|^2$. Consequently, since $(u, v)$ has unit norm, $|v| = 1/\sqrt{2}$. □

**Assertion 1.2** If we choose $\bar{U}$ and $V$ sufficiently small then the manifolds $M_{2, +t}$, for every $t \in [0, 1]$, meet $I_{x^0}$ transversally in $\bar{U}$ and the intersections $M_{2, +t} \cap I_{x^0}$ do not meet the boundary of $\bar{U}$, for every $x^0 \in V \setminus \Gamma$.

**Remark** Note that the family $M_{2, +t}$ also depends on $x^0$.

**Proof of Assertion 1.2** To prove the transversality it suffices to prove that $M_{2, +t}$ satisfies conditions (i) and (ii) of Assertion 1.1 in $\bar{U}$, if we choose $\bar{U}$ and $V$ sufficiently small. Let us first verify that condition (i) holds. Assume the contrary, i.e., that $x^0 \in M_{2, +t}$ for some $t \in [0, 1]$. Then it follows from (1.4) that $f'(0; x^0) = 0$. But, from (1.2), we have

$$f_n(0; x^0) = \xi(x') - \xi(x'^0)$$

which equals zero, since $x' = x'^0 + tf'(0; x^0)$. Consequently, $x_n = \xi(x'^0)$ which does not equal $x^0_n$ since $x^0 \notin \Gamma$. This is a contradiction and, thus, $x^0 \notin M_{2, +t}$ for all $t \in [0, 1]$.

To verify condition (ii) note that the conormal space of $M_{2, +t}$, at a point $z \in M_{2, +t}$
is spanned by the covectors

\[
(u_1, v_1, \ldots, u_n, v_n) = \begin{cases}
(1, i \frac{\partial f_1}{\partial y_1}, 0, i \frac{\partial f_1}{\partial y_2}, \ldots, 0, i \frac{\partial f_1}{\partial y_{n-1}}, 0, 0), \\
(0, i \frac{\partial f_{n-1}}{\partial y_1}, \ldots, 0, i \frac{\partial f_{n-1}}{\partial y_{n-2}}, -1, i \frac{\partial f_{n-1}}{\partial y_{n-1}}, 0, 0), \\
(0, i \frac{\partial f_n}{\partial y_1}, \ldots, 0, i \frac{\partial f_n}{\partial y_{n-1}}, -1, 0).
\end{cases}
\]

Hence, the transversality follows e.g. if we can prove that there is a \( V \) and a \( W \) such that

\[
\left| \frac{\partial f_i}{\partial y_j}(y'; x^0) \right| < \frac{1}{\sqrt{2n}}
\]

for all \( i = 1, \ldots, n, j = 1, \ldots, n - 1, y' \in W \) and \( x^0 \in V \). By continuity, this follows from the fact that

\[
\frac{\partial f_i}{\partial y_j}(0, 0) = 0.
\]

The latter follows readily from (1.2) and the definition of \( M \). The details are left to the reader.

We complete the proof by showing that the intersection \( M_{2+t} \cap I_{x^0} \) does not meet the boundary of \( \bar{U} \) if we choose \( V \) sufficiently small. Clearly, we can make the intersections \( M_{2+t} \cap \partial \bar{U} \) be contained in \( \{ z : |z - x^0| < \varepsilon |y| \} \) for any \( \varepsilon > 0 \) by making \( V \) small enough. Thus, \( M_{2+t} \cap I_{x^0} \cap \partial \bar{U} \) is empty in view of the first equation of (1.3).

Let us conclude the proof of Lemma 1.1. Assertion 1.2 implies that the intersections \( M_{2+t} \cap I_{x^0} \) are smooth and homeomorphic for any pair of \( t, t' \in [0, 1] \). A straightforward calculation shows that the intersection \( M \cap I_{x^0} \) (recall that \( M = M_3 \)) is given by the equations:

\[
\begin{cases}
|y|^2 = (|t|^2 + 1)(\xi - x^0)^2, \\
y_n = \eta.
\end{cases}
\]

This proves part (a) of the lemma. The same argument as before shows that we can move \( S_2 \) continuously along with \( M_{2+t} \) to a cycle on \( M \) which is homeomorphic to a \((n - 1)\)-sphere and which encloses the intersection \( M \cap I_{x^0} \). This finishes the proof of Lemma 1.1.

**Remark** At this point, the reader should keep the following picture in mind: \( \gamma_{x^0} \) is a topological \( n - 2 \) sphere in the intersection \( \bar{\Gamma} \cap I_{x^0} \) "surrounded" by the topological \( n - 1 \) sphere \( C_{x^0} \).

Clearly, the manifold \( \gamma_{x^0} \) and \( \mu(x^1) \), for which \( I_{x^0} \cap \bar{\Gamma} \) integration in each of the cone next lemma asserts that this:

**Lemma 1.2** If \( I_{x^0} \cap \bar{\Gamma} \) homotopic in \( \bar{U} \setminus I_{x^0} \) to the neighborhood of \( \gamma_{x^1} \) in \( M_1 \),

**Proof** By Lemma 1.1, \( i_{C_{x^0}} \). Let \( M_T \) be the manifold the equations preceding (1.3)

\[
\begin{cases}
x' \\
x
\end{cases}
\]

for \( (y', y_n) \in W \times \mathbb{R} \). Now, \( t \in [0, 1] \), and define it by \( t \)

First, note that \( M_t \) satisfies simply because both \( M_0 \) a meet \( x^0 \) for any \( t \in [0, 1] \).

for some \( t \in [0, 1] \). In this avoid this point, by adding \( \varepsilon \) so small that condition deduce from Assertion 1

\[
\gamma_{x^1} = I_{x^1} \cap \\
\text{and since we know that same dimension.}
\]

We conclude this sec 3 and Section 4.
Clearly, the manifold $\gamma_{\varphi}$ is not unique. When we compare two values $u(x^0)$ and $u(x^1)$, for which $I_{x^0} \cap \hat{\Gamma} = I_{x^1} \cap \hat{\Gamma}$, we want to be able to deform the cycle of integration in each of the corresponding integrals to cycles over the same set. Our next lemma asserts that this indeed is possible.

**Lemma 1.2** If $I_{x^0} \cap \hat{\Gamma} = I_{x^1} \cap \hat{\Gamma}$ as sets in $\hat{U}$ then the sphere $S^{n-1}(x^0, \epsilon)$ is homotopic in $\hat{U} \setminus I_{x^0}$ to the $n-1$ dimensional boundary $C'_{x^0}$ of a contractible neighborhood of $\gamma_{x^1}$ in $M_1$, the manifold M of Lemma 1.1 defined for the point $x^1$.

**Proof** By Lemma 1.1, it suffices to prove that $C_{x^0}$ is homotopic in $\hat{U} \setminus I_{x^0}$ to $C'_{x^0}$. Let $M_j$ be the manifold of Lemma 1.1 defined for the point $x^j$, $j = 0, 1$. From the equations preceding (1.4) we see that $M_j$ can be described by the equations

$$
\begin{cases}
x' = x^j + t'(y^j, x^j) = g'(y^j, x^j), \\
x = t(x^j) + t_n(y^j, x^j) = g_n(y^j, x^j),
\end{cases}
$$

for $(y^j, x^j) \in W \times \mathbb{R}$. Now, we deform $M_0$ to $M_1$. We call the deformation $M_t$, for $t \in [0, 1]$, and define it by the equations

$$
x = rg(y^j, x^j) + (1 - t)g(y^j, x^0).
$$

First, note that $M_t$ satisfies the condition (ii) of Assertion 1.1 for every $t \in [0, 1]$ simply because both $M_0$ and $M_1$ do. Next, we want to make sure that $M_t$ does not meet $x^0$ for any $t \in [0, 1]$. If it does then we have that

$$
x^0 = rg(0, x^j) + (1 - t)g(0, x^0)
$$

for some $t \in [0, 1]$. In this case, we can modify the definition of $M_t$ slightly, so as to avoid this point, by adding the term $t(1 - t)\epsilon$ to the first equation of $M_t$. We choose $\epsilon$ so small that condition (ii) of Assertion 1.1 is still satisfied. Consequently, we deduce from Assertion 1.1 that the deformation $M_t$ meets $I_{x^0}$ transversally for each $t \in [0, 1]$. Also, we can make the intersection $M_t \cap I_{x^0}$ avoid the boundary of $\hat{U}$ by the same argument as in the proof of Assertion 1.2. Hence, we can move $C_{x^0}$ along with $M_t$ to a cycle $C'_{x^0}$ surrounding the intersection $M_1 \cap I_{x^0}$ in $M_1$. This intersection must be $\gamma_{x^1}$ since

$$
\gamma_{x^1} = I_{x^1} \cap M_1 = I_{x^1} \cap \hat{\Gamma} \cap M_1 = I_{x^0} \cap \hat{\Gamma} \cap M_1 \subseteq I_{x^0} \cap M_1
$$

and since we know that both $M_1 \cap I_{x^0}$ and $\gamma_{x^1}$ are closed smooth manifolds of the same dimension. □

We conclude this section with the following lemma which will be used in Section 3 and Section 4.
Lemma 1.3 If the neighborhoods \( \hat{U} \) and \( V \) are chosen small enough then the intersection \( I_{\phi} \cap \hat{\Gamma} \) is a smooth connected manifold in \( \hat{U} \), for \( x^0 \in V \setminus \Gamma \). In particular, it is irreducible.

Proof First, note that the intersection between \( I_{\phi} \) and any complex hyperplane \( \{z_n = a\} \), with \( a \neq x_n^0 \), is the smooth quadric given by the equation

\[
\left( z' - x_n^0, z' - x_n^0 \right) + (a - x_n^0)^2 = 0.
\]

It is easy to verify that the closure of the set of unit conormals of \( I_{\phi} \setminus \{x^0\} \), in any compact set, does not meet the set \( \{r = (r', r_n): |r'| = 0\} \). By continuity and the fact that the set of unit conormals is invariant under translations, it follows that there is a \( \delta > 0 \) such that any complex manifold, which does not meet the point \( x^0 \) and for which the set of unit conormals is contained in the set \( \{r: |r'| \leq \delta\} \), meets \( I_{\phi} \) transversally (cf. Assertion 1.1). Since \( \hat{\Gamma} = \{z_n = \phi(z')\} \), where \( \phi \) has a vanishing gradient at the origin, it follows that, if we choose \( \hat{U} \) sufficiently small, the intersection \( \hat{\Gamma} \cap I_{\phi} \) is a smooth manifold in \( \hat{U} \). It could, a priori, be a disconnected manifold, though. To see that it is not we consider a deformation \( N_t = \{z_n = (1 - t)\bar{z} + t\phi(z')\} \), for \( t \in [0, 1] \), from a plane \( \{z_n = \bar{z}\} \) to \( \hat{\Gamma} \). The number \( \epsilon \geq 0 \) is chosen small (\( \epsilon = 0 \) if possible) such that \( N_t \) does not meet \( x^0 \) for any \( t \in [0, 1] \). The complex manifolds \( N_t \) all meet \( I_{\phi} \) transversally, but in order to make sure that no new components of the intersections \( N_t \cap I_{\phi} \) emerge in \( \hat{U} \) or that the component we start with does not leave \( \hat{U} \) as \( t \) runs from 0 to 1 we need to verify that the manifolds \( N_t \) meet the boundary component \( \partial \hat{U} \cap I_{\phi} \), for \( x^0 \) in a sufficiently small domain \( V \), transversally also. This would actually prove that the topological type of the intersections \( N_t \cap I_{\phi} \) does not change as \( t \) runs from 0 to 1. By continuity, it suffices to verify that the set of unit conormals of the boundary component \( \partial \hat{U} \cap I_{\phi} \) at \( \{z_n = 0\} \) does not meet the set \( \{r: |r'| = 0\} \). This verification is straightforward and the details are omitted. \( \square \)

2. The three-dimensional case

In this section we prove the following result on point to point reflection of harmonic functions vanishing on a hypersurface in \( \mathbb{R}^3 \). We keep the notation introduced in the previous section with \( n = 3 \).

Theorem 2.1 Let \( \Gamma \) be a nonsingular, real-analytic hypersurface in some neighborhood \( U \) of the origin in \( \mathbb{R}^3 \). Suppose that \( \Gamma \) is neither part of a hyperplane nor a sphere. Then there is a neighborhood \( V, \ V \subset U \), such that for no pair of points \( x^0, x^1 \) in \( V \setminus \Gamma \) is there a constant \( K \) satisfying

\[
u(x^0) + Ku(x^1) = 0, \quad \forall u \in \text{Har}_0(U, \Gamma),
\]

where \( \text{Har}_0(U, \Gamma) \) denotes the clc.

Before we enter the proof of this theorem we state:

Lemma 2.1 Let \( D_{\phi} \) be the i.e. the domain in \( \hat{\Gamma} \cap M \) boundary \( u \in \text{Har}_0(U, \Gamma) \) and \( x^0 \in V \setminus \Gamma \),

\[
u(x^0) = 2 \int_I u(x^0) = 2 \int_I \int_{C_{\phi}} \sum_{j=1}^3 (-1)^j
\]

where \( N \) is the Newtonian potential

\[
u_j = \omega_j
\]

and the orientation of \( D_{\phi} \) is according to those in (1.1).

Proof Let us fix \( x^0 \in V \setminus \Gamma \)

\[
u(x^0) = \int_{C_{\phi}} \sum_{j=1}^3 (-1)^j
\]

holds for every \( u \in \text{Har}_0(U, \Gamma) \) given as a graph over the of coordinates such that \( M \) intersection \( \hat{\Gamma} \cap M \) is given : \( M \) (a ball in \( \mathbb{R}^3 \)), we can assume \( C = C_{\phi} \) to be a sphere in \( \mathbb{R}^3 \) be a tube of radius \( r \) in \( M \) domain in \( \{x_3 = 0\} \) boundary consisting of \( \partial Tr \), a slightly of which differ by a sign \( (-) \) and the orientation of \( \Gamma \) sufficiently small \( r > 0 \),

\[
u(x^0) = \int_{\partial Tr} \sum_{j} (-1)^j + 2 \]

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\[ u(x) = \int_{\gamma} -\sum_{j=1}^{3} (-1)^{j+1} \left( \frac{\partial N}{\partial y_j} (x - x_0) - \frac{\partial y_j}{\partial x} N(x - x_0) \right) dy_j. \]

\[ u(x_0) = \int_{\gamma} -\sum_{j=1}^{3} (-1)^{j+1} \frac{\partial N}{\partial y_j} (x - x_0) dy_j, \]

where \( \gamma \) is the boundary of \( D \) and the branch of the square root in \( N \) are chosen according to those in (1.1).

Proof. Let us fix \( x_0 \in \gamma \). It follows from Lemma 1 that

\[ u(x_0) = 2 \int_{\gamma} \sum_{j=1}^{3} (-1)^{j+1} \frac{\partial N}{\partial y_j} (y - x_0) dy_j = 0, \]

where \( N \) is the Newtonian potential of a point mass at the origin as in (1.1).

Before we enter the proof of the theorem we need a lemma.

**Lemma 2.1.** Let \( D \) be the relatively compact component of \( \mathbb{R}^3 \setminus M \), i.e., the domain in \( \mathbb{R}^3 \) bounded by the closed simple curve \( \gamma \). Then, for any \( u \in H^1(\mathbb{R}^3) \) and \( \phi \in \mathcal{V}(\gamma) \), we have

\[ u(x_0) = \int_{\gamma} -\sum_{j=1}^{3} (-1)^{j+1} \frac{\partial N}{\partial y_j} (x - x_0) dy_j. \]
We have used the fact that $u$ vanishes on $\hat{\Gamma}$. Now, $D_r \to D$ in measure and the singularity of $N(\cdot - x^0)$ at $\gamma$ is integrable with respect to the two dimensional area measure on $D$, i.e. the integral in (2.1) converges absolutely. Consequently, the proof will be finished if we can show that the integral over $\partial T_r$ tends to zero as $r$ tends to zero. Since $M$ and $I_{x^0}$ meet transversally, the restriction of \( g(z) = (z - x^0, z - x^0) \) to $M$, let us denote it by $\bar{g}(t)$, has a non-vanishing gradient near $\gamma$. Moreover, this restriction vanishes only on $\gamma$. These two facts imply that we can write

\[
\bar{g}(t) = \text{dist}(t, \gamma) k_1(t),
\]

where $k_1$ is a function satisfying $A < |k_1| < B$ for some positive constants $A$ and $B$ and $\text{dist}(t, \gamma)$ denotes the distance from $t$ to $\gamma$. Also, since $u$ vanishes on $\hat{\Gamma}$ we can write the restriction of $u$ to $M$ as $t_3k_2(t)$ where $k_2$ is a bounded function. Finally, let us note that the restriction of $\omega_j$ to $\partial T_r$ is a bounded function times the volume form on $\partial T_r$. If we put this together we obtain

\[
\left| \int_{\partial T_r} \sum_{j=1}^{3} (-1)^{j+1} \left( u \frac{\partial N(\cdot - x^0)}{\partial x_j} - \frac{\partial u}{\partial x_j} N(\cdot - x^0) \right) \omega_j \right| 
\leq \int_{\partial T_r} \sum_{j=1}^{3} \left( \frac{t_3p_j(t)}{(\text{dist}(t, \gamma))^3/2} + \frac{q_j(t)}{(\text{dist}(t, \gamma))^{1/2}} \right) dS,
\]

where the $p_j$'s and the $q_j$'s are bounded functions. Now, the distance from a point on $\partial T_r$ to $\gamma$ is $r$, by definition of $T_r$, and since $t_3 \leq r$ on $\partial T_r$, we get

\[
\left| \int_{\partial T_r} \sum_{j=1}^{3} (-1)^{j+1} \left( u \frac{\partial N(\cdot - x^0)}{\partial x_j} - \frac{\partial u}{\partial x_j} N(\cdot - x^0) \right) \omega_j \right| \leq \frac{K}{r^{1/2}} A(\partial T_r),
\]

where $A(\partial T_r)$ denotes the area of $\partial T_r$ and $K$ is some constant independent of $r$. The area of $\partial T_r$ is proportional to the radius $r$ and, hence, the integral over $\partial T_r$ tends to zero as $r^{1/2}$ when $r$ tends to zero.

We can now proceed with the proof of Theorem 2.1.

**Proof of Theorem 2.1** During the course of proving Lemmas 1.1 and 1.2, we have chosen the neighborhoods $U$ and $V$ small. In the following, we may choose them even smaller. We pick two points $x^0$ and $x^1$ in $V \setminus \Gamma$. The proof splits into two cases:

(i) The sets $I_{x^0} \cap \hat{\Gamma}$ and $I_{x^1} \cap \hat{\Gamma}$ are the topological disks $D_{x^0}$ and $D_{x^1}$, respectively. We make the change of coordinate $w'$ to $w^0$ of a simply connected domain $W$ by $w$ a small neighborhood $g'(\cdot; x^0)$ extends holomorphically to $0 \in \mathbb{R}^3$. Since $\tilde{V}_{g'}(0; 0)$, we change of coordinates in $\tilde{U}$

if we choose $U$ and $V$ small, becomes the domain $\Omega$ in theorem, we can approximate polynomials and, thus, $w$ functions analytic in $\tilde{U}$. Then, a constant $K$ such that

\[
u(x^0)
\]

then it follows from (2.1)

(2.3) \[ \int_{\partial T_r} (N^1) \]

for every $u \in \mathcal{H}_0(U)$, on which branch of $t_i$ calculation shows that

\[\Phi \in \{w_3 = 0\} \text{ equ} \]
The sets \( I_x \cap \hat{\Gamma} \) and \( I_x' \cap \hat{\Gamma} \) coincide in \( \hat{U} \). By Lemma 1.2, we may choose the topological disks \( D_{x^0} \) and \( D_{x^0} \) to be the same. Let us denote this disk by \( D \). We make the change of coordinates \( w_3 = z_3 - \phi(z) \) and \( w' = z' \) to make \( \hat{\Gamma} \) the hyperplane \( \{ w_3 = 0 \} \). As we have seen above, the disk \( D \) is the real-analytic image in \( \{ w_3 = 0 \} \) under the mapping \( (t' = (t_1, t_2)) \)

\[
 w' = x^{0'} + f'(t'; x^0) + it' = g'(t'; x^0) + it'.
\]

of a simply connected domain \( \Omega \) with smooth boundary in the real \( t' \)-plane. Denote by \( \mathcal{W} \) a small neighborhood of \( 0 \) in the complex \( t' \)-space (i.e. in \( \mathbb{C}^2 \)) such that \( g'(\cdot; x^0) \) extends holomorphically to \( W \) for each \( x^0 \) in some neighborhood \( V \) of \( 0 \in \mathbb{R}^3 \). Since \( \nabla_v g'(0, 0) = 0 \), the equations (2.2) define, for each \( x^0 \in V \), a change of coordinates in \( \hat{U} \),

\[
 \hat{U} = \hat{U} \cap \hat{\Gamma},
\]

if we choose \( \hat{U} \) and \( V \) small enough. Under this change of coordinates the disk \( D \) becomes the domain \( \Omega \) in the real space \( \mathbb{R}^2 \) of \( \hat{\Gamma} \cong \mathbb{C}^2 \). By the Stone–Weierstrass theorem, we can approximate the continuous functions on \( \Omega \) uniformly by analytic polynomials and, thus, we may approximate the continuous functions on \( D \) by functions analytic in \( \hat{U} \). Now, in order to get a contradiction, assume that there is a constant \( K \) such that

\[
 u(x^0) + Ku(x^1) = 0, \quad \forall u \in \operatorname{Har}_0(U, \Gamma);
\]

then it follows from (2.1) that

\[
 \int_D (N(- x^0) \pm KN(- x^1)) \sum_{j=1}^{3} (-1)^j \frac{\partial u}{\partial z_j} \omega_j = 0
\]

for every \( u \in \operatorname{Har}_0(U, \Gamma) \). Which sign, \( + \) or \( - \), we should choose in (2.3) depends on which branch of the square root we choose in \( N(- x^1) \), \( j = 0, 1 \). A simple calculation shows that the restriction of the form

\[
 \sum_{j=1}^{3} (-1)^j \frac{\partial u}{\partial z_j} \omega_j
\]

to \( \hat{\Gamma} = \{ w_3 = 0 \} \) equals

\[
 \frac{\partial u}{\partial w_3} \left( 1 + \left\langle \nabla \phi, \nabla \phi \right\rangle \right) dw_1 \wedge dw_2.
\]
Note that the expression multiplying $\partial u/\partial w_2$ above vanishes at precisely the characteristic points (with respect to the Laplace operator) of $\hat{\Gamma}$, and, hence, not in a neighborhood of the origin (the characteristic points on a real-analytic hypersurface in $\mathbb{R}^n$ are precisely the singular ones). By the Cauchy–Kowalevskaya theorem (see [H1], ch. IX), there is, for every choice of analytic function $\psi$ in $\bar{U}$, a function $u$ in $\mathcal{H}^0_0(U, \Gamma)$, provided that we a priori choose $U$ small enough, that solves the complex Cauchy problem

$$\begin{cases}
\sum_{j=1}^{3} \frac{\partial^2 u}{\partial x_j^2} = 0, \\
u = 0 \\
\frac{\partial u}{\partial n} = \psi
\end{cases} \quad \text{on } \hat{\Gamma},$$

where $\partial u/\partial n$ denotes the derivative of $u$ in the conormal direction of $\hat{\Gamma}$, i.e. $\partial u/\partial w_3$ in the $w$ coordinates. Since the analytic functions in $\bar{U}$ approximate the continuous functions in $D$—note that we may subsequently have to shrink $\bar{U}$ even more, but that this property holds for the functions analytic in the domain where $z \rightarrow w$ is a change of coordinates—it follows that (2.3) holds only if $N(\cdot - x^0) \pm KN(\cdot - x^1)$ vanishes in $D$. Since $D$ is biholomorphic to $\Omega$ which has a non-empty interior in $\mathbb{R}^2$, it follows that $N(\cdot - x^0) \pm KN(\cdot - x^1)$ must vanish on all of $\hat{\Gamma}$. An explicit calculation with the Newtonian potentials shows that $\hat{\Gamma}$ must be contained in the algebraic hypersurface defined by the equation

$$\langle z - x^1, z - x^1 \rangle = K^2 \langle z - x^0, z - x^0 \rangle.$$

Consequently, $\Gamma$ is a portion of either a hyperplane or a sphere. The proof of the first case is complete.

(ii) The sets $I_{\infty} \cap \hat{\Gamma}$ and $I_x \cap \hat{\Gamma}$ do not coincide. We can think of the point evaluations $u(x^0)$ and $u(x^1)$ as analytic functionals, $T_0$ and $T_1$, on the space of analytic functions in $\bar{U}$, denoted by $\mathcal{O}(\bar{U})$, realized as the corresponding integrals over $D_0 = D_{x^0}$ and $D_1 = D_{x^1}$. The fact, established above, that there is a biholomorphic change of variables in $\bar{U}$ taking $D_j$ to a simply connected domain $\Omega_j$ in $\mathbb{R}^2$ proves, by the Stone–Weierstrass theorem, that the continuous functions in $D_j$ can be approximated uniformly by analytic functions in $\bar{U}$ for $j = 0, 1$. Consequently, since the measure realizing $T_j$ has support on all of $D_j$, no proper subset of $D_j$ can carry $T_j$, i.e., there is no measure supported on a proper subset of $D_j$ that can represent $T_j$. One says that $K_j = D_j$ is a support for $T_j$. Now assume, in order to get a contradiction, that there is a constant such that $T_0 + KT_1 = 0$. This implies that both $K_0$ and $K_1$ carry measure $T_0 = -KT_1$. By a theorem of C. O.

Kiselman (see corollary 2.6 in [I] the set

where $L$ is the $\mathcal{O}(\bar{U})$- hull of the in $K_0$ and a neighborhood $U', i$ then $L \setminus K_0 \cap U'$, does not meet $K_0 \cap U'$, is a proper subset of $K_0$ carries $T_0$ for $L \cap U' = K_0 \cap U'$ is to use (see e.g. [Sto]) that $L \cap U' = i$ through each point of $U' \setminus K_0$ set $K_0 \cup K_1$.

For the remainder of this case dimensions. Therefore, to $\Gamma$ we will avoid the vector not in the real $(t_1, t_2)$-plane, as in images of these domains ur

(iii.a) $\Omega_0 = \Omega_1$. We make in the real $(t_1, t_2)$-plane. V calculation shows that the

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Kiselman (see corollary 2.6 in [Ki]), the analytic functional $T_0$ must be carried by the set

$$K_0 \cap (L \setminus K_0 \cup K_1),$$

where $L$ is the $\mathcal{O}(\bar{U})$-hull of the union $K_0 \cup K_1$. If we can prove that there is a point in $K_0$ and a neighborhood $U'$, in $\mathbb{C}^2$, of that point such that $L \cap U' = K_0 \cap U'$ then $L \setminus K_0$ does not meet $K_0 \cap U''$ for some smaller neighborhood $U'' \subset U'$ and, thus, $K_0 \cap (L \setminus K_0 \cup K_1)$ is a proper subset of $K_0$. This contradicts the fact that no proper subset of $K_0$ carries $T_0$ and the proof would be finished. The way to prove that $L \cap U' = K_0 \cap U'$ is to use the Oka–Stolzenberg theorem. This theorem asserts (see e.g. [Sto]) that $L \cap U' = K_0 \cap U'$ if we are able to pass an analytic hypersurface through each point of $U' \setminus K_0$ and move it off to infinity without intersecting the set $K_0 \cup K_1$.

For the remainder of this proof, the arguments will essentially take place in two dimensions. Therefore, to help the reader better visualize the geometric picture, we will avoid the vector notation used above. We consider the domains $\Omega_0$ and $\Omega_1$ in the real $(t_1, t_2)$-plane, as in case (i) above, such that $D_0$ and $D_1$ are given as the images of these domains under the mappings (2.2). There are two possibilities:

(ii.a) $\Omega_0 = \Omega_1$. We make the change of coordinates taking $D_0$ to $\Omega = \Omega_0 = \Omega_1$ in the real $(t_1, t_2)$-plane. We denote the new variables by $\zeta = t_1 + i t_2$. A simple calculation shows that the disk $D_1$ is given by the mapping

$$\begin{align*}
\zeta_1 &= t_1 + i (g_1^1 - g_1^0), \\
\zeta_2 &= t_2 + i (g_2^1 - g_2^0),
\end{align*}$$

(2.4)

where we, for brevity, use the notation $g_j^k(t_1, t_2) = g_j(t_1, t_2; \mathbf{x}^k)$ for $k = 0, 1$ and $j = 1, 2$. Also, let $h(t_1, t_2)$ denote the function defining the domain $\Omega$, i.e. the real part of $\mathcal{g}(w_1, w_2; \mathbf{x}^0) = (w_1 - x_1^0)^2 + (w_2 - x_2^0)^2 + (\phi - x_3^0)^2$, evaluated at $(w_1, w_2) = (t_1 + ig_1^0, t_2 + ig_2^0)$, or equivalently $g(t_1, t_2; \mathbf{x}^1)$ evaluated at $(w_1, w_2) = (t_1 + ig_1^1, t_2 + ig_2^1)$. First, we claim that we can assume that

$$\frac{\partial h}{\partial t_1}(g_1^1 - g_1^0) + \frac{\partial h}{\partial t_2}(g_2^1 - g_2^0)$$

(2.5)

restricted to $\partial \Omega$ is not identically zero. To see this, note, as we noted in Lemma 1.2, that the disk $D_1$ and the curve $\gamma_1 = \gamma_1^k$ are not canonically defined. In the proof of Lemma 1.1 we could have continued to deform the surface $M$ to the surface $M'$, defined by the equations

$$\begin{align*}
x_1 &= g_1^1(y_1, y_2) + iy_2, \\
x_2 &= g_2^1(y_1, y_2) - iy_1, \\
x_3 &= \zeta(x_1 + iy_1, x_2 + iy_2).
\end{align*}$$

(2.6)
Here \( \varepsilon \) is a small number or a small function with small gradient. By choosing \( \hat{U} \) and \( V \) sufficiently small we can get a uniform, for \( x \in V \), allowed size on \( \varepsilon \) and its gradient. In this notation, which differs slightly from the one used in Lemma 1.1, \( M \) is \( M_0 \). The form of the perturbation is chosen such that the intersection \( I_x \cap M_\varepsilon \) is contained in \( \bar{\Gamma} \). Note that the perturbation is in the tangent space of a circle, centered at the origin, through the point \((y_1, y_2)\). By the same argument as in Lemma 1.1, the intersection \( I_x \cap M_\varepsilon \) is a smooth curve \( \bar{\gamma}_1 \) contained in \( \bar{\Gamma} \). Under this perturbation either \( \Omega_1 \), the projection of \( \bar{D}_1 \) (defined in the obvious way) as above, changes to become different from \( \Omega \) or it does not. If it does we proceed as in case (b) below. If it does not then we claim that the restriction to \( \partial \Omega \) of

\[
\frac{\partial h}{\partial t_1}(g_1^0 + \varepsilon t_2) + \frac{\partial h}{\partial t_2}(g_2^0 - \varepsilon t_1)
\]

changes. Assume, in order to get a contradiction, that it does not change; then it would follow from (2.5) that

\[
\frac{\partial h}{\partial t_1} t_2 - \frac{\partial h}{\partial t_2} t_1 = 0
\]

on \( \partial \Omega \). It is easy to see that this implies that \( \Omega \) is a ball. Hence, using equation (2.5) again we deduce that

\[
\begin{align*}
g_1^1(t_1, t_2) &= g_1^0(t_1, t_2) + \lambda(t_1, t_2) t_2, \\
g_2^1(t_1, t_2) &= g_2^0(t_1, t_2) - \lambda(t_1, t_2) t_1,
\end{align*}
\]

for some function \( \lambda \) which is small with small gradient, on the circle \( \partial \Omega \). In this case, we can modify \((g_1^1, g_2^1)\) in the same manner as above. We set

\[
\begin{align*}
g_1^1(t_1, t_2) &= g_1^0(t_1, t_2) - \tau \lambda(t_1, t_2) t_2, \\
g_2^1(t_1, t_2) &= g_2^0(t_1, t_2) + \tau \lambda(t_1, t_2) t_1.
\end{align*}
\]

Since we have assumed that \( \bar{\gamma}_1 \) does not change under such a perturbation, this deforms the disk \( D_1 \) to the disk \( D_0 \) as \( \tau \) runs from 0 to 1. This contradicts the fact that \( I_0 \cap \bar{\Gamma} \) does not equal \( I_x \cap \bar{\Gamma} \), because the common boundary of the disks \( D_0 \) and \( D_1 \) would be contained in the intersection \( I_0 \cap I_x \cap \bar{\Gamma} \). Consequently, we may assume that (2.5) is not identically zero on \( \partial \Omega \). By real-analyticity, it is then zero only at a finite number of points on \( \partial \Omega \). It follows that there is a point \( t^0 \in \partial \Omega \) such that the real line tangent to \( \partial \Omega \) at \( t^0 \),

\[at_1 + bt_2 = c,
\]

does not meet \( \Omega \) (let us assume that \( a(g_1^1 - g_1^0) + b(g_2^1 - g_2^0) \)) we cut off an open subset \( \Omega' \)

\[
(2.7)
\]

for some \( c' < c \), in which denote the domain

\[\{(\zeta_1, \zeta_2) = (t_1 \cdot \text{in } \mathbb{C}^2. \text{ It is easy to see the following curve of comp}
\]

\[(2.8)
\]

and that this curve of intersecting neither \( \Omega : U' \setminus \Omega \) does not meet \( I \)

proof of possibility (a')

(ii, b) \( \Omega_0 \neq \Omega_1 \). By \( \Omega_0 \) a finite number of poi

say on \( \partial \Omega_0 \), which is line

of \( \partial \Omega_0 \) at \( t^0 \) meets \( \{at_1 + bt_2 < c\} \). As 

denote the new vari

\( \Omega_0 \) and let \( \omega \) be the 

that \( \Omega_1 \) is container

Again, it is easy to curve of complex 
as \( r \to \infty \), with the 
of possibility (b') 

The proof of
does not meet $\Omega$ (let us assume that all points of $\Omega$ satisfy $at_1 + bt_2 < c$) and such that $a(g_1^1 - g_1^0) + b(g_2^1 - g_2^0) \neq 0$ at $t^0$. By moving this tangent line slightly into $\Omega$ we cut off an open subset of $\Omega$, namely

$$(2.7) \quad \omega = \Omega \cap \{at_1 + bt_2 > c\}$$

for some $c' < c$, in which $|a(g_1^1 - g_1^0) + b(g_2^1 - g_2^0)| > \delta$ for some $\delta > 0$. Let $U'$ denote the domain

$$\{(\zeta_1, \zeta_2) = (t_1 + is_1, t_2 + is_2) : (t_1, t_2) \in \omega \text{ and } |as_1 + bs_2| < \delta\}$$

in $\mathbb{C}^2$. It is easy to see that through each point $(\zeta_1^0, \zeta_2^0)$ in $U' \setminus \Omega$ we can pass the following curve of complex lines,

$$(2.8) \quad L_r = \{a\zeta_1 + b\zeta_2 = a\zeta_1^0 + b\zeta_2^0 + r\},$$

and that this curve of complex lines moves off to infinity, as $r \to \infty$, without intersecting neither $\Omega = K_0$ nor $K_1$. By the Oka–Stolzenberg theorem, we get that $U' \setminus \Omega$ does not meet $L_r$ the $O(\bar{U})$-hull of $K_0 \cup K_1$ defined above. This finishes the proof of possibility (a), by the theorem of Kiselman mentioned above.

(ii.b) $\Omega_0 \neq \Omega_1$. By real-analyticity, the boundaries $\partial \Omega_0$ and $\partial \Omega_1$ intersect only at a finite number of points. It follows that there is a point $t^0$ on one of the boundaries, say on $\partial \Omega_0$, which is not on the other boundary $\partial \Omega_1$ and such that the real tangent line

$$at_1 + bt_2 = c$$

of $\partial \Omega_0$ at $t^0$ meets neither $\Omega_0$ nor $\Omega_1$; let us assume that $\Omega_0 \cup \Omega_1$ is contained in $\{at_1 + bt_2 < c\}$. As above, we make the change of coordinates taking $D_0$ to $\Omega_0$ and denote the new variables by $\zeta_j = t_j + is_j$. Move the tangent line at $t^0$ slightly into $\Omega_0$ and let $\omega$ be the open subset of $\Omega_0$ defined as in (2.7). We choose $c - c'$ so small that $\Omega_1$ is contained in $\{at_1 + bt_2 < c\}$. We can take $U'$ to be the cylinder

$$\{(\zeta_1, \zeta_2) : (t_1, t_2) \in \omega\}.$$  

Again, it is easy to see that through each point $(\zeta_1^0, \zeta_2^0)$ in $U' \setminus \Omega$ we can pass the curve of complex lines defined by (2.8) and that this curve moves off to infinity, as $r \to \infty$, without intersecting neither $\partial \Omega_0 = K_0$ nor $K_1$. This finishes the proof of possibility (b) by combining the Oka–Stolzenberg theorem and the Kiselman theorem exactly as in the proof of possibility (a).

The proof of case (ii) and, hence, of the theorem is complete. \qed
Remark As we already noted in the introduction, the argument in case (ii) of the proof above, i.e. the case when \( I_{x^0} \cap \tilde{\Gamma} \) and \( I_{x^1} \cap \tilde{\Gamma} \) do not coincide (the points are not SR points with respect to \( \Gamma \) according to Definition 3.1 below), could be shortened considerably if we were to consider a stronger point to point reflection law than the one considered in Theorem 2.1. Namely, if we were to demand that “reflection of singularities” should hold as well, i.e. that any harmonic function vanishing on \( \Gamma \) and with a pole at \( x^0 \), e.g. a Green’s function on one side of \( \Gamma \) with pole at \( x^0 \), should have a pole also at \( x^1 \) then it is clear that \( I_{x^0} \cap \tilde{\Gamma} \) and \( I_{x^1} \cap \tilde{\Gamma} \) must coincide. To see this, note that the singularity of such a harmonic function propagates along the isotropic cone emanating from the pole so in order for the function to vanish on \( \Gamma \) the singularities of the function must cancel on \( \Gamma \), i.e. the isotropic cones \( I_{x^0} \cap \tilde{\Gamma} \) and \( I_{x^1} \cap \tilde{\Gamma} \) coincide. This was noted in [KS].

3. The four-dimensional case

Before we state the theorem on point to point reflection of harmonic functions in \( \mathbb{R}^4 \) we need some definitions.

Definition 3.1 Two points \( x^0 \) and \( x^1 \) in \( V \setminus \Gamma \) are said to be SR points, Study reflection points, with respect to \( \Gamma \) if the intersection \( I_{x^0} \cap \tilde{\Gamma} \) and \( I_{x^1} \cap \tilde{\Gamma} \) are equal as sets in \( \tilde{U} \) (cf. [KS]).

It was shown in [KS] that, in \( \mathbb{R}^n \) with \( n > 2 \), the set of points for which there is a Study reflected point is an algebraic set of codimension at least 1, unless \( \Gamma \) is a sphere or a hyperplane in which case there is always a reflected point (the ordinary “mirror” reflection in a plane and the Kelvin transformation for a sphere). In \( \mathbb{R}^2 \), the Study reflection coincides with the well known Schwarz reflection.

In [Kh], it is shown that there is a point to point reflection of functions in \( \text{Har}_0(U, \Gamma) \), where \( \Gamma \) is an axially symmetric surface in \( \mathbb{R}^4 \), between SR points on the axis of symmetry, i.e. for each pair of SR points \( x^0 \) and \( x^1 \) on the axis of symmetry there is a constant \( K \) such that

\[
(3.1) \quad u(x^0) + Ku(x^1) = 0, \quad \forall u \in \text{Har}_0(U, \Gamma).
\]

It was shown in the previous section that nothing like this is true in \( \mathbb{R}^3 \), but one might hope that it is true in general in \( \mathbb{R}^4 \). However, this turns out to be false. We have to introduce a stronger reflection relation, namely

Definition 3.2 Two points \( x^0 \) and \( x^1 \) in \( V \setminus \Gamma \) are said to be SSR points, strong Study reflection points, with respect to \( \Gamma \) and with constant \( \lambda \) if they are SR points and if

\[
dg(\cdot - x^0)|_{\tilde{\Gamma}} = \lambda dg(\cdot - x^1)|_{\tilde{\Gamma}},
\]
where \( g \) is the defining function for \( I_0 \), i.e.,

\[
g(z) = \langle z, z \rangle
\]

and \( \lambda \) is a complex constant, on the common intersection \( I_\phi \cap \bar{\Gamma} = I_\psi \cap \bar{\Gamma} \).

Of course, the notion of SSR generalizes the Schwarz reflection in two variables as well as the "mirror" reflection in hyperplanes and the Kelvin transformation for spheres. Moreover, the reader can readily verify that when \( \Gamma \) is axially symmetric each pair of SR points that, in addition, lie on the axis of symmetry does indeed satisfy the SSR condition (cf. also Proposition 3.1 below). Thus, the results of [Kvh] mentioned above are simple corollaries of the following theorem, which is the main result in this section.

**Theorem 3.1** Let \( \Gamma \) be a nonsingular, real-analytic hypersurface in some neighborhood \( U \) of the origin in \( \mathbb{R}^4 \). Then there is a neighborhood \( V \) of the origin, with \( \overline{V} \subset U \), such that, given two points \( x^0 \) and \( x^1 \) in \( V \setminus \Gamma \), the following are equivalent:

(a) there is a constant \( K \) such that (3.1) holds;

(b) the points \( x^0 \) and \( x^1 \) are SSR points with respect to \( \Gamma \) and with the constant \( \lambda \) equal to \( 1/K \).

Before we prove this, we digress momentarily to discuss and motivate the notion of SSR. To provide an intuitive insight for the reader into the nature of the SSR condition as well as to motivate its definition, let us briefly discuss the problem heuristically in the perhaps more familiar context of the Huygens principle for the wave equation.

Suppose that our surface \( \bar{\Gamma} = \{ z \in \mathbb{C}^4 : f(z) = 0 \} \) appears as a real hypersurface \( \bar{\Gamma} \) in the real subspace

\[
W = i\mathbb{R}^3 \times \mathbb{R} = \{(ix', x_4) : x_j \in \mathbb{R}\}
\]

(in general, this intersection has codimension 2). Let \( x^0 = (0, 0, 0, r) \) and \( x^1 = (0, 0, 0, -r) \) be SR points. Obviously, we have \( I_\phi \cap I_\psi \subset \{ z_4 = 0 \} \). Any \( u \in \text{Har}_0(U, \Gamma) \), extended as a holomorphic function into \( \mathbb{C}^4 \), satisfies in \( W \) the wave equation

\[
\sum_{j=1}^{3} \frac{\partial^2 u}{\partial x_j^2} - \frac{\partial^2 u}{\partial x_4^2} = 0.
\]

Let \( \bar{\Gamma}_0 = I_\phi \cap W, \bar{\Gamma}_1 = I_\psi \cap W \) be the ordinary "light cones" emanating from \( x^0 \) and \( x^1 \) that meet on \( \bar{\Gamma} \) along the 2-dimensional sphere \( S_r = \{(ix', 0) : |x'| = r\} \). Now, by...
Kirchhoff's formula (see e.g. [J], chapter 5), the value of \( u \) at \( x^0 \) can be calculated in terms of the Cauchy data on the plane \( \{x_4 = 0\} \) along the sphere \( S_t \) only:

\[
u(x^0) = \frac{C}{t^2} \int_{S_t} \left( t \frac{\partial u}{\partial x_4} + \sum_{j=1}^{3} \frac{\partial u}{\partial x_j} x_j \right) dS,
\]

where \( C \) is some constant and \( dS \) denotes the Lebesgue measure on \( S_t \), taken with positive (with respect to the outward normal) orientation. Note that we have used the fact that \( u \) vanishes on \( S_t \). Since \( -\nabla g(-x^0) = 2(x', r) \) on the plane \( \{x_4 = 0\} \), we can rewrite the above equation in the form

\[
u(x^0) = -\frac{C}{2t^2} \int_{S_t} \langle \nabla u, \nabla g(-x^0) \rangle dS
= -\frac{C}{2t^2} \int_{S_t} \frac{\partial u}{\partial n} \langle n, \nabla g(-x^0) \rangle dS,
\]

where \( n \) denotes the unit normal to \( \bar{\Gamma} \). Similarly, we have

\[
u(x^1) = -\frac{C}{2t^2} \int_{S_t} \frac{\partial u}{\partial n} \langle n, \nabla g(-x^1) \rangle dS.
\]

Now, if (3.1) holds then, in view of equations above and since \( \partial u/\partial n \) runs over a dense set of the continuous functions on \( S_t \), as \( u \) runs over \( \text{Haro}(U, \Gamma) \), it follows that

\[
\langle \nabla g(-x^0) + K \nabla g(-x^1), n \rangle = 0
\]

on \( S_t \). To fix the ideas, assume that \( \bar{\Gamma} \) can be written as \( \{x_4 = \phi(x')\} \). Then, since the intersection of \( \bar{\Gamma} \) with the plane \( \{x_4 = 0\} \) contains the sphere \( S_t \), it follows that \( \nabla \phi = \psi(x')x' \) on \( S_t \). Combining this with the equation above and the definition of \( S_t \), we find that \( \psi \) is constant on \( S_t \),

\[
\psi(x') = \frac{1}{t} \frac{1 - K}{1 + K}.
\]

Noting that

\[
\begin{align*}
- \left. dg(-x^0) \right|_{\bar{\Gamma}} &= 2 \langle x' + i\nabla \phi, dx' \rangle \\
\left. dg(-x^1) \right|_{\bar{\Gamma}} &= 2 \langle x' - i\nabla \phi, dx' \rangle
\end{align*}
\]

we obtain that

\[
\left. dg(-x^0) \right|_{\bar{\Gamma}} = \lambda \left. dg(-x^1) \right|_{\bar{\Gamma}},
\]

where \( \lambda = 1/K \), on \( S_t \) and, therefore, everywhere on \( I_{40} \cap \bar{\Gamma} = I_{41} \cap \bar{\Gamma} \). This is precisely the SSR condition in Definition 3.2.

Before we proceed with the proof geometric description of SSR points to be half the distance between the translation and rotation if necessary \( x^1 = (0, 0, 0, -a) \); let us denote the them from the fixed coordinates \( z \).

**Proposition 3.1** The two pair and with constant \( \lambda \) if and only if \( \bar{\Gamma} \)

\[
\Sigma = \{ (s', s') \}
\]

along the set \( \Lambda = \{ s_4 = \}

If \( \lambda = 1 \) then \( c = 0 \) and the set \( \Sigma \)

**Remark** The set \( \Lambda \) is easily cones \( I_0 \) and \( I_1 \).

**Proof** First assume that \( x^0 \)

with respect to \( \Gamma \) and with cones \( I_0 \cap I_1 \), where \( I_0 = I_1 \), is the irreducible SR points it is clear that \( \bar{\Gamma} \cap I_0 \) set is irreducible we have equal \( s_4 = \psi(s') \) near each point on \( \Lambda \), denote the function \( g(-x^0) \) at \( 1/2 \)

Taking the restriction to \( \bar{\Gamma} \) and \( 1/2 \)

From this equation and Defin

\[
\text{on } \Lambda, \text{ where}
\]
Before we proceed with the proof of the main result let us give an equivalent
description of SSR points. Fix two points $x^0$ and $x^1$ in $V \setminus \Gamma$ and define $a$
to be half the distance between the points. Without loss of generality, performing
translation and rotation if necessary, we can assume that $x^0 = (0, 0, 0, a)$ and
$x^1 = (0, 0, 0, -a)$; let us denote the coordinates by $s = (s_1, s_2, s_3, s_4)$ to distinguish
them from the fixed coordinates $z$.

**Proposition 3.1** The two points $x^0$ and $x^1$ are SSR points with respect to $\Gamma$
and with constant $\lambda$ if and only if $\hat{\Gamma}$ is tangent to the complex sphere

$$\Sigma = \left\{ \langle s', s' \rangle + \left( s_4 + \frac{1}{c} \right)^2 + a^2 = \frac{1}{c^2} \right\},$$

where

$$c = \frac{1 - \lambda}{a + a\lambda},$$

along the set

$$\Lambda = \{ s_4 = 0 \} \cap \{ \langle s', s' \rangle + a^2 = 0 \}.$$

If $\lambda = 1$ then $c = 0$ and the set $\Sigma$ becomes the hyperplane $\{ s_4 = 0 \}$.

**Remark** The set $\Lambda$ is easily seen to be the intersection between the isotropic
cones $I_{x^0}$ and $I_{x^1}$.

**Proof** First assume that $x^0 = (0, 0, 0, a)$ and $x^1 = (0, 0, 0, -a)$ are SSR points
with respect to $\Gamma$ and with constant $\lambda$. It is easy to check that the intersection
$I_0 \cap I_1$, where $I_j = I_{x^j}$, is the irreducible set $\Lambda$. Since the points, in particular, are
SR points it is clear that $\hat{\Gamma} \cap I_j$, for $j = 0, 1$, must be contained in $\Lambda$ and since this
set is irreducible we have equality. To fix the ideas, assume that $\hat{\Gamma}$ can be written
$s_4 = \psi(s')$ near each point on $\Lambda$ (the general case being similar). Also, let $g_-$
denote the function $g(\cdot - x^0)$ and $g_+$ the function $g(\cdot - x^1)$. It follows that

$$\frac{1}{2} dg_{\pm} = \langle s', ds' \rangle + (s_4 \mp a) ds_4.$$

Taking the restriction to $\hat{\Gamma}$ and using the coordinates $s'$ there we obtain

$$\frac{1}{2} dg_{\pm} \big|_{\hat{\Gamma}} = \langle s' + (s_4 \mp a) \nabla \psi, ds' \rangle.$$

From this equation and Definition 3.2 we see that if the points are SSR points then

$$d \psi = -c \langle s', ds' \rangle$$

on $\Lambda$, where

$$c = \frac{1 - \lambda}{a + a\lambda}.$$
This is equivalent to \( \tilde{\Gamma} \) being tangent to \( \Sigma \) along \( \Lambda \). This proves the "only if" part of the proposition. The other part follows from observing that all the steps above are reversible and using the fact, established in Lemma 1.3, that \( \tilde{\Gamma} \cap I \) is irreducible in \( \tilde{U} \) to deduce that \( \tilde{\Gamma} \cap I \) must equal \( \Lambda \) if \( \Sigma \) is tangent to \( \tilde{\Gamma} \) along \( \Lambda \).

Loosely speaking, Proposition 3.1 states that the condition for two points to be SSR points with respect to \( \Gamma \) means that \( \tilde{\Gamma} \) has an infinitesimal axial symmetry near the set \( \Lambda \) about the line connecting the two points.

Let us now return to Theorem 3.1. Before we actually enter the proof of it, we prove the following lemma. We state and prove it in the general even dimensional space, because we will need this result in Section 4. Thus, the setting for Lemma 3.1 below is \( \mathbb{R}^n \) with \( n \) even. Recall from Section 1 that \( \tilde{\Gamma} = \{ z_n = \phi(z') \} \) and that the coordinates \( w \) are defined by \( w' = z' \) and \( w_n = z_n - \phi(z') \).

**Lemma 3.1** For any \( u \in \text{Har}_0(U, \Gamma) \) and \( x^0 \in V \setminus \Gamma \), we have

\[
(3.2) \quad u(x^0) = \int_{\gamma_0} \omega_{x^0},
\]

where \( \omega_{x^0} \) is a \( n-2 \) form on \( I_{\phi} \cap \tilde{\Gamma} \) and the orientation of \( \gamma_{x^0} \) is chosen accordingly to the one in (1.1). Near each point on \( I_{\phi} \cap \tilde{\Gamma} \) at which

\[
\frac{1}{2} \frac{\partial g}{\partial w_k} \left( - x^0 \right) \bigg|_{\tilde{\Gamma}} = w_k - x_k^0 + \frac{\partial \phi}{\partial w_k} (\phi - x_n^0) \neq 0,
\]

we have

\[
(3.3) \quad \omega_{x^0} = \frac{2\pi c_n}{(n-1)!} \left[ \frac{\partial u}{\partial w_k} \right]^{p-1} \left( \frac{\partial u}{\partial w_n} \right) \left( \frac{\partial \phi}{\partial g} \frac{\partial w_n}{\partial w_k} \right) \omega_k,
\]

where \( p = (n-2)/2 \), \( \omega_k = (-1)^{k+1} dw_1 \wedge \cdots \wedge \hat{w}_k \wedge \cdots \wedge dw_{n-1} \) and where we, by a slight abuse of notation, write \( g \) instead of \( g(\cdot - x^0) \).

**Proof** We know from Lemma 1.1 that \( u(x^0) \) can be obtained as an integral over \( C = C_{x^0} \) of the form \( \alpha \) in (1.1). Since the number of dimensions is even, this form is single-valued and, hence, we may replace \( C \) by any cycle homologous to \( C \) in \( M \setminus I \), where \( I = I_{\phi} \). As in the proof of Lemma 2.1, we let \( T \) be a tube in \( M \) around \( \gamma = \gamma_{x^0} \). Then we may integrate over \( \partial T \) instead of over \( C \). Also, \( \partial T = \delta \gamma \), where \( \delta \) denotes the Leray coboundary operator (see e.g. [AY], Chapter III.16). Consequently, we get

\[
(3.4) \quad u(x^0) = 2\pi i \int_{\gamma} \text{Res} \alpha,
\]

where \( \text{Res} \) means the Leray resi-

then the proof is finished once

\[
\alpha = c_n \sum_{j=1}^{n} (-1)^j \left( \begin{array}{c} 2p(j) \\ n \end{array} \right) u^{2p(j)}
\]

where we, as above, write \( g \) j

We can write this as

\[
(3.5)
\]

for some regular forms \( \beta \) and \( \gamma \) first taking the restriction (j

an easy exercise to show th

Since \( u \) vanishes on \( \tilde{\Gamma} \), the to \( \tilde{\Gamma} \). The second term bec

\[
\frac{\delta t}{c_n \frac{\partial \phi}{\partial g}}
\]

on \( \tilde{\Gamma} \). Now, we can reduce 114, by noting that near |

\[
dw_1 \wedge \cdots \wedge dw_{n-1}
\]

In the end, we have a f

maximal degree on \( \tilde{\Gamma} \),

The result is (3.3).

Note that when \( n = \)

are now ready to prove

**Proof of Theorem**

there can be rio const

The proof of this is p

The objects correspond |

M_1, the manifolds du
where $\text{Res}$ means the Leray residue class (see [AY], Chapter III.16). If we set

$$\omega_{x_0} = 2\pi i \text{Res} \alpha$$

then the proof is finished once we show that $\omega_{x_0}$ has the form (3.3). Note that $\alpha$ is

$$\alpha = c_n \sum_{j=1}^{n} (-1)^j \left( \frac{2p(c_j - x_0^j)}{g^{p+1}} + \frac{1}{g^p} \frac{\partial u}{\partial z_j} \right) dz_1 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_n,$$

where we, as above, write $g$ instead of $g(-x^0)$. This should cause no confusion. We can write this as

(3.5)

$$\alpha = \frac{u\beta}{g^{p+1}} + \frac{\beta'}{g^p}$$

for some regular forms $\beta$ and $\beta'$. Since $\gamma$ is contained in $I \cap \tilde{\Gamma}$ we may calculate $\omega$ by first taking the restriction (pullback) on $\tilde{\Gamma}$ and then taking the residue at $I \cap \tilde{\Gamma}$. It is an easy exercise to show that taking the residue commutes with taking restrictions. Since $u$ vanishes on $\tilde{\Gamma}$, the first term in (3.5) vanishes when we take the restriction to $\tilde{\Gamma}$. The second term becomes, in the coordinates $w$,

$$\frac{\partial u}{c_n \partial w_n} \left( 1 + \frac{\nabla \phi, \nabla \psi}{g^p} \right) dw_1 \wedge \cdots \wedge dw_{n-1}$$

on $\tilde{\Gamma}$. Now, we can reduce the order of the pole on $I \cap \tilde{\Gamma}$ as described in [AY], p. 114, by noting that near points on $I \cap \tilde{\Gamma}$ at which $\partial g/\partial w_k \neq 0$ we have

$$dw_1 \wedge \cdots \wedge dw_{n-1} = (-1)^{k+1} d g|_{\tilde{\Gamma}} \wedge \frac{dw_1 \wedge \cdots \wedge \widehat{dw_k} \wedge \cdots \wedge dw_{n-1}}{\partial g/\partial w_k}.$$ 

In the end, we have a form with a simple pole on $I \cap \tilde{\Gamma}$ and, since this form has maximal degree on $\tilde{\Gamma}$, the residue can be calculated using the Poincaré formula. The result is (3.3).

Note that when $n = 4$, the differential operator in (3.3) is just the constant 1. We are now ready to prove the theorem.

**Proof of Theorem 3.1** Fix two points $x^0$ and $x^1$ in $V \setminus \Gamma$. Let us first note that there can be no constant $K$ such that (3.1) holds if $x^0$ and $x^1$ are not SR points. The proof of this is practically the same as case (ii) in the proof of Theorem 2.1. Hence, the objects corresponding to the disks $D_0$ and $D_1$ are the domains in $M_0$ and $M_1$, the manifolds denoted by $M$ in Lemma 1.1 defined for the points $x^0$ and $x^1$.\[\square\]
respectively, bounded by $\gamma_{x^0}$ and $\gamma_{x^1}$. Let us denote these domains by $B_0$ and $B_1$. Just note that in this situation the analytic functionals $T_0$ and $T_1$ are realized by measures supported on $\gamma_{x^0}$ and $\gamma_{x^1}$ instead of by measures supported on $B_0$ and $B_1$. However, the argument of case (ii) carries through with this modification also. We leave the details to the reader.

We are left with the case when the points $x^0$ and $x^1$ are SR points. By Lemma 1.2, we may assume that $\gamma_{x^0} = \gamma_{x^1} = \gamma$. Assume that there is a constant $K$ such that (3.1) holds. The continuous functions on $\gamma$ can be uniformly approximated, on $\gamma$, by analytic functions in $\hat{U}$ (by the same argument as in the proof of Theorem 2.1). By the Cauchy–Kowalevskaya theorem, exactly as in the proof of Theorem 2.1, it follows from (3.2) and (3.3), since $n = 4$ and hence $p = 1$, that

$$\omega'_{x^0} - K \omega'_{x^1} = 0$$

(3.6)

on $\gamma$, where

$$\omega'_{x^j} = \frac{\omega_{x^j}}{\partial u / \partial w_4}.$$  

The minus sign in the equation (3.6) is due to the fact that the cycles $\gamma_{x^0}$ and $\gamma_{x^1}$ have opposite orientation. To see this, note that the intersection between the isotropic cones $I_{x^0}$ and $I_{x^1}$ is contained in the (complexified) hyperplane $\Pi$ perpendicular to and bisecting the line connecting the two points. For simplicity, let us just consider the case when $\Gamma$ is the hyperplane $\{x_4 = 0\}$ (the general case is similar). In this case, the deformation of the spheres $S^3(x^0, \epsilon)$ and $S^3(x^1, \epsilon)$ carried out in Section 1 takes place in a 5 (real) dimensional subspace $W$ of $\mathbb{C}^4$. The intersection $W \cap \Pi$ (in this case, when $\Gamma$ is a hyperplane, $\Pi = \hat{\Gamma}$ of course) is a hyperplane in $W$ of which the two points $x^0$ and $x^1$ are on opposite sides. Since the two spheres $S^3(x^0, \epsilon)$ and $S^3(x^1, \epsilon)$ have positive orientation with respect to the cones $I_{x^0}$ and $I_{x^1}$, respectively, it is clear that the deformed spheres surrounding the intersection $W \cap I_{x^0} \cap I_{x^1}$ have opposite orientation with respect to that intersection. Hence, the cycles $\gamma_{x^0}$ and $\gamma_{x^1}$ have opposite orientation.

It follows from (3.3) that

$$(3.7) \quad dg(-x^0)|_{\hat{\Gamma}} - \frac{1}{K} dg(-x^1)|_{\hat{\Gamma}} = 0$$

on $\gamma$. Since there is a biholomorphic change of coordinates in $\hat{\Gamma} = \{w_4 = 0\}$ taking $\gamma$ to a set of real codimension one in the real subspace of $\{w_4 = 0\}$ and since the intersection $I_{x^0} \cap \hat{\Gamma}$ is irreducible, it follows that (3.7) holds on all of $I_{x^0} \cap \hat{\Gamma}$. This proves the implication (a) $\Rightarrow$ (b). The opposite implication is obvious. □

4. The general even-dime

Let us see what we have to do. Let us first look at Lemma 3.1. It is even but greater than 4. We namely that $\partial u / \partial w_4$ does not act on $\gamma$. This causes some prob and have the corresponding $\gamma_{x^0}$ forms with merely continu functionals $T_0$ and $T_1$, used in the proof of Theorem 2.1, it follows from (3.2) and (3.3), since $n = 4$ and hence $p = 1$, that

The next question is what should be. It is clear from the case is not sufficient. If we forward to check that the implication (b) $\Rightarrow$ (a) in Theorem 1.1 immediately extends to $\lambda$ such that

$$g(-x^0)|_{\hat{\Gamma}} = 0$$

where $g$ is as in Lemma 3.1.

It is clear that this definition is similar to the situation complex sphere (or hype the points $x^0$ and $x^1$ with isotropic cones $I_{x^0}$ and $I_{x^1}$) and condition by giving a $\gamma_\lambda$.

Now, if we would like to have the implication in Theorem 1.1, it will read

**Definition 4.1** Two points $x^0$ and $x^1$ are said to have isotropic cones $I_{x^0}$ and $I_{x^1}$, respectively, if there is a biholomorphic change of coordinates in $\hat{\Gamma} = \{w_4 = 0\}$ taking $\gamma$ to a set of real codimension one in the real subspace of $\{w_4 = 0\}$ and since the intersection $I_{x^0} \cap \hat{\Gamma}$ is irreducible, it follows that (3.7) holds on all of $I_{x^0} \cap \hat{\Gamma}$. This proves the implication (a) $\Rightarrow$ (b). The opposite implication is obvious. □
4. The general even-dimensional case

Let us see what we have to do to generalize Theorem 3.1 to higher even dimensions. Let us first look at Lemma 3.1 in the case where the number of dimensions is even but greater than 4. We notice a difference from the four dimensional case, namely that \( \partial u / \partial \omega_n \) does not appear as a factor in front of the form we are integrating. This causes some problems, because we want to vary the function \( \partial u / \partial \omega_n \) and have the corresponding form \( \omega_n \) vary over a sufficiently large subspace of the forms with merely continuous coefficients in order to deduce that the analytic functionals \( T_0 \) and \( T_1 \), used in the proofs of Theorems 2.1 and 3.1, are not supported by any proper subset of \( \gamma^c \). In this case the functionals are represented by distributions on \( \gamma^c \) instead of by measures as in the four dimensional case, but this makes no difference. Because of the rather complicated dependence on \( \partial u / \partial \omega_n \), we cannot use the Stone-Weierstrass theorem directly as in the proof of Theorem 3.1. However, once we show that we can make the forms \( \omega_j \), for \( j = 0, 1 \), range over a sufficiently large subspace of the forms with continuous coefficients as we let \( \partial u / \partial \omega_n \) range over the analytic functions in \( \tilde{U} \) (recall \( \tilde{U} = \tilde{U} \cap \tilde{\Gamma} \)), the proof that the existence of a constant \( K \) such that (3.1) holds implies that \( x^0 \) and \( x^1 \) are SR points (the Definition 3.1 immediately generalizes to higher dimensions) can be carried out exactly as in the proof of Theorem 3.1.

The next question is what the generalization of SSR points to higher dimensions should be. It is clear from (3.3) that the same definition as in the four dimensional case is not sufficient. If we expand (3.3) using the Leibnitz formula, it is straightforward to check that the following definition of SSR points is sufficient for the implication (b) \( \Rightarrow \) (a) in Theorem 3.1 to hold in the higher dimensional case.

**Definition 4.1** Two points \( x^0 \) and \( x^1 \) in \( V \setminus \Gamma \) are said to be SSR points, strong study reflection points, with respect to \( \Gamma \) in \( \mathbb{R}^n_+ \) if they are SR points (the Definition 3.1 immediately extends to arbitrary dimensions) and if there is a complex constant \( \lambda \) such that

\[
g(\cdot - x^0)|_{\tilde{\Gamma}} = \left( (\lambda)^{1/p} + h' \left( g(\cdot - x^1)|_{\tilde{\Gamma}} \right) \right)^p g(\cdot - x^1)|_{\tilde{\Gamma}},
\]

where \( p \) is as in Lemma 3.1 and \( h' \) is some analytic function in \( \tilde{U} \).

It is clear that this definition coincides with Definition 3.2 in the case \( n = 4 \). Similarly to the situation in \( \mathbb{R}^4 \), Definition 4.1 is equivalent to \( \tilde{\Gamma} \) meeting a certain complex sphere (or hyperplane, cf. Proposition 3.1) with center on the line through the points \( x^0 \) and \( x^1 \) with \( p \)-th order tangency along the intersection between the isotropic cones \( I_{x^0} \) and \( I_{x^1} \). At the end of this section, we illustrate the SSR condition by giving a couple of examples.

Now, if we would like to prove that this definition also gives the full opposite implication in Theorem 3.1, i.e. (a) \( \Rightarrow \) (b), then we have to know (similar to the
situation described above) that as we let $\partial u/\partial w_n$ range over the analytic functions in $\bar{U}$ the form $\omega_\phi - K\omega_\phi$ ranges over such a large subspace of the forms with continuous coefficients that (a) implies that the differential operator in the expression for $\omega_\phi - K\omega_\phi$ must be identically zero. Given this, the proof of the implication (a) $\Rightarrow$ (b) in Theorem 3.1 for general even dimensions would consist of verifying that the differential operator vanishes identically only if $x^0$ and $x^1$ are SSR points. This is easy to see if we write $\omega_\phi - K\omega_\phi$ in the form (4.1) below.

The desired properties of $\omega_\phi$ and $\omega_\phi - K\omega_\phi$, mentioned above follow from Assertion 4.1 below and the fact that the analytic functions in $\bar{U}$, and hence the analytic functions in $\bar{U} \cap I_\gamma$ for $j = 0, 1$, approximate the continuous functions on $\gamma_\phi$. Hence, Assertion 4.1 in combination with the proof of Theorem 3.1 proves Theorem 3.1 in the general even dimensional case with the Definition 4.1 of SSR points.

In order to state and prove Assertion 4.1 we need to make some simplifications and introduce some more notation. As we have noted before, the function

$$1 + \left\langle \nabla \phi, \nabla \phi \right\rangle$$

does not vanish in $\bar{U}$. Consequently, there is a one-to-one correspondence between choices of $\partial u/\partial w$ and $\partial u/\partial w_n(1 + \left\langle \nabla \phi, \nabla \phi \right\rangle)$ by $\psi$. Let $\omega_\phi$ be the mapping taking an analytic function $\psi$ in $\bar{U}$ to the analytic $n - 2$ form $\omega_\phi$ on $\bar{U} \cap I_\gamma$ for $j = 0, 1$. We write $\mathcal{O}(\bar{U})$ for the space of analytic functions in $\bar{U}$ and $\Omega^{n-2}_\gamma(\bar{U} \cap I_\gamma)$ for the space of analytic $n - 2$ forms on $I_\gamma \cap \bar{U}$. The mapping $\omega_\phi$ can be described as follows:

$$\omega_\phi(\psi) = \text{Res}_{I_\gamma \cap \hat{U}} \frac{\psi}{(1 + \left\langle \nabla \phi, \nabla \phi \right\rangle)^p} dw_1 \wedge \cdots \wedge dw_{n-1}.$$ 

Assume that $x^0$ and $x^1$ are SSR points, that $\gamma_\phi = \gamma_\phi = \gamma$ and let $Y$ denote the common intersection $I_\phi \cap \hat{Y} = I_\phi \cap \hat{Y}$. The fact that $x^0$ and $x^1$ are SSR points implies that the spaces $\Omega^{n-2}_\gamma(\bar{U} \cap I_\gamma)$ coincide for $j = 0, 1$. We denote this space by $\Omega^{n-2}_\gamma(\bar{U} \cap Y)$. We denote the space of analytic functions in $\bar{U} \cap Y$ by $\mathcal{O}_\gamma(\bar{U} \cap Y)$. 

**Assertion 4.1** For every constant $K$, the image of the mapping $\varphi = \varphi_0 - K\varphi_1$ is an $\mathcal{O}_\gamma(\bar{U} \cap Y)$-submodule of $\Omega^{n-2}_\gamma(\bar{U} \cap Y)$; in particular, if $\eta \in \text{Im} \varphi$ then $t\eta \in \text{Im} \varphi$ for every $t \in \mathcal{O}_\gamma(\bar{U} \cap Y)$. The image is not trivial, i.e., not equal to $\{0\}$, unless the differential operator in the local expression for $\varphi$ (see (4.1) below) is identically zero.

**Proof** Let us write the formula (3.3) in a more canonical form. At each point $A$ on $\bar{U} \cap Y$, we make a change of coordinates $s = s(w)$ on $\hat{Y}$ such that

$$g(w'(s) - x^1) = s_{n-1}.$$ 

Since

$$g(.),$$

for some non-vanishing analytic $\varphi$ at $A$:

$$\varphi(\psi) = \frac{ds_1 \wedge \cdots \wedge ds_{n-2}}{(p-1)!} \left( \frac{\partial/\partial s_{n-1}}{h + s_{n-1} \partial h/\partial s_{n-1}} \right)$$

where $D$ is the determinant of $\psi$ in the submanifold $Y$ as a function from $r \in \mathcal{O}_\gamma(\bar{U} \cap Y)$ only on $(s_1, ..., s_{n-2})$. Hence, $s$ derivatives with respect to the any $r \in \mathcal{O}_\gamma(\bar{U} \cap Y)$ we have

$$s^j$$

near $A$, where $r \psi$ means, by a the submanifold $Y$ as a function conclude that $\varphi(\psi)$ belongs (4.2), as defined above, is or globally from $Y$ such that (and we leave this alternating the different spaces introduc ch. IV, or [B], Appendix A linear sheaf map induced by

where $\bar{\Omega}^{n-2}_\gamma$ denotes the tr $A \in \bar{U}$ equals the stalk of $I$ 145). Then $\text{Im} \varphi$ is a subs $A$ of $\bar{U} \cap Y$ belongs to $\text{Im}$ germ is zero and, hence, $\bar{\Omega}^{n-2}_\gamma$ that $t \varphi(\psi)$ belongs of $\bar{\Omega}^{n-2}_\gamma$, where $\bar{\Omega} = \text{Im} \varphi$ is trivial, unless the difference the differential operator is identically zero). Thi
Since
\[ g(\cdot - x^0) = h g(\cdot - x^1) \]
for some non-vanishing analytic function \( h \) in \( \bar{U} \), we get the following formula for \( \varphi \) at \( A \):
\[
(4.1) \quad \varphi(\psi) = \frac{d s_1 \wedge \cdots \wedge d s_{n-2}}{(p-1)!} \times \left[ \left( \frac{\partial / \partial s_{n-1}}{h + s_{n-1} \partial h / \partial s_{n-1}} \right)^{p-1} \left( \frac{\psi D}{h + s_{n-1} \partial h / \partial s_{n-1}} \right) - K \left( \frac{\partial^{p-1} (\psi D)}{\partial s_{n-1}^{p-1}} \right) \right]_{\bar{Y}}.
\]
where \( D \) is the determinant of the Jacobian of the coordinate change. Now, note that any function \( t \in \mathcal{O}_Y(\bar{U} \cap Y) \) expressed near \( A \) in the coordinates \( s \) depends only on \( (s_1, \ldots, s_{n-2}) \). Hence, since the differential operator in (4.1) only involves derivatives with respect to the \( s_{n-1} \) variable, it is clear that for any \( \psi \in \mathcal{O}(\bar{U}) \) and any \( t \in \mathcal{O}_Y(\bar{U} \cap Y) \) we have
\[
(4.2) \quad \varphi(t \psi) = t \varphi(\psi)
\]
near \( A \), where \( t \psi \) means, by a slight abuse of notation, the function \( t \) extended from the submanifold \( Y \) as a function independent of \( s_{n-1} \) multiplied by \( \psi \). In order to conclude that \( t \varphi(\psi) \) belongs to \( \text{Im} \varphi \) in \( \Omega^{n-2}_Y(\bar{U} \cap Y) \) (note that the function \( t \psi \) in (4.2), as defined above, is only defined locally, although it is possible to extend \( t \) globally from \( Y \) such that (4.2) holds; this, however, requires some calculations and we leave this alternative conclusion of the proof to the reader), we consider the different spaces introduced above as sheaves. We refer the reader e.g. to [GR], ch. IV, or [B], Appendix A:II, for the basics of sheaf theory. We let \( \tilde{\varphi} \) denote the linear sheaf map induced by \( \varphi \), mapping
\[
\tilde{\varphi}: \mathcal{O} \mapsto \tilde{\Omega}^{n-2},
\]
where \( \tilde{\Omega}^{n-2}_Y \) denotes the trivial extension of \( \Omega^{n-2}_Y \), i.e. the stalk of \( \tilde{\Omega}^{n-2}_Y \) at a point \( A \in \bar{U} \) equals the stalk of \( \Omega^{n-2}_Y \) at \( A \) if \( A \in \bar{U} \cap Y \) and \( \{0\} \) if \( A \not\in \bar{U} \cap Y \) (see [GR], p. 145). Then \( \text{Im} \tilde{\varphi} \) is a subsheaf of \( \tilde{\Omega}^{n-2}_Y \). By (4.2), the germ of \( t \varphi(\psi) \) at each point \( A \) of \( \bar{U} \cap Y \) belongs to \( \text{Im} \tilde{\varphi}(A) \) (the stalk of \( \text{Im} \tilde{\varphi} \) at \( A \)). For \( A \) not on \( \bar{U} \cap Y \), the germ is zero and, hence, also in \( \text{Im} \tilde{\varphi}(A) \). This means, since \( \text{Im} \tilde{\varphi} \) is a subsheaf of \( \tilde{\Omega}^{n-2}_Y \), that \( t \varphi(\psi) \) belongs to \( \text{Im} \tilde{\varphi}(\bar{U}) \). It also follows that \( \text{Im} \tilde{\varphi} \) is an \( \tilde{\Omega}_Y \)-submodule of \( \tilde{\Omega}^{n-2}_Y \), where \( \tilde{\Omega}_Y \) is the trivial extension of \( \mathcal{O}_Y \). Thus, \( t \varphi(\psi) \) belongs to \( \text{Im} \tilde{\varphi} \) in \( \tilde{\Omega}^{n-2}_Y(\bar{U} \cap Y) \), and \( \text{Im} \varphi \) is an \( \tilde{\Omega}_Y(\bar{U} \cap Y) \)-submodule of \( \tilde{\Omega}^{n-2}_Y \). Obviously, \( \text{Im} \varphi \) is not trivial, unless the differential operator in (4.1) is identically zero (if it is trivial then the differential operator must, in particular, annihilate all polynomials and, thus, it is identically zero). This completes the proof. \( \square \)
Let us summarize the results of Section 3 and Section 4 by restating Theorem 3.1 in the general case:

**Theorem 4.1** Let $\Gamma$ be a a nonsingular, real-analytic hypersurface in some neighborhood $U$ of the origin in $\mathbb{R}^n$, with $n$ even. Then there is a neighborhood $V$ of the origin, with $\overline{V} \subset U$, such that, given two points $x^0$ and $x^1$ in $V \setminus \Gamma$, the following are equivalent:

(a) there is a constant $K$ such that (3.1) holds;

(b) the points $x^0$ and $x^1$ are SSR points with respect to $\Gamma$ with the constant $\lambda$ in the Definition 4.1 equal to $1/K$; the number $p$ is as in Lemma 4.1.

In view of Proposition 0.1, we have the following:

**Corollary 4.1** Assume that $\Gamma$ is neither part of a hyperplane nor a sphere. Then the functions from $\text{H}_{\text{an}}(U, \Gamma)$ separate the points $x^0, x^1 \in V \setminus \Gamma$ if and only if they are not SSR points with respect to $\Gamma$.

We conclude this section with the following examples.

**Example 4.1** Let us consider axially symmetric hypersurfaces in $\mathbb{R}^n$, with $n \geq 4$ even. By choosing coordinates appropriately, such a surface $\Gamma$ can be expressed by an equation of the form $f(x_1, \rho) = 0$, where $f$ is a real analytic function of two variables with nonvanishing gradient along its zero locus and where $\rho^2 = x_2^2 + \cdots + x_n^2$. Geometrically, we can think of $\Gamma$ as being obtained by revolving a curve

$$\gamma = \{(x_1, \rho) \in \mathbb{R}^2 : f(x_1, \rho) = 0\},$$

symmetric with respect to the $x_1$-axis, around the $x_1$-axis in $\mathbb{R}^n$. (We imbed $\mathbb{R}^2$ as a two-plane in $\mathbb{R}^n$ such that the $x_1$-axis in $\mathbb{R}^2$ coincides with the $x_1$-axis in $\mathbb{R}^n$.) We call $\gamma$ the meridian curve and $\mathbb{R}^2$ with coordinates $(x_1, \rho)$ the meridian plane. As mentioned earlier, this situation with $n = 4$ was studied in [Kh]. The result in that paper is that for each point $x^0$ on the $x_1$-axis there is a point $x^1$, also on the $x_1$-axis, such that (3.1) holds. If we identify $x^0$ with its corresponding point $(t, 0) \in \mathbb{R}^2$ (where $t = x_2^0$) then the point $x^1$ corresponds to $(S(t), 0) \in \mathbb{R}^2$, where $S(x_1 + i\rho)$ denotes the Schwarz function of the meridian curve $\gamma$, and the constant $K$ in (3.1) equals $-S'(t)$. Let us verify that $x^0$ and $x^1$ so defined satisfy the SSR condition with respect to $\Gamma$ in $\mathbb{R}^4$. Introducing coordinates $\zeta = z_1 + i\rho, \zeta^* = z_1 - i\rho$ in the complexified meridian plane $(z_1, \rho)$ (we allow $\rho$ to be complex here), we can write the equations of the isotropic cones $I_{x^0}, I_{x^1}$ ($t \in \mathbb{R}$) as follows:

$$g( - x^0) = (z_1 - t)^2 + \rho^2 = (\zeta - t)(\zeta^* - t) = 0,$$

$$g( - x^1) = (z_1 - S(t))^2 + \rho^2 = (\zeta - S(t))(\zeta^* - S(t)) = 0.$$

where $a^2 \neq 1$, and let $\Gamma$ be the ellipsoid $x^2/n + y^2/m + z^2/l = 1$, with $a, b, c$ satisfying $a^2 + b^2 + c^2 = 1$. Let $\alpha \in \mathbb{R}$ be the angle between the planes normal to $x^0$ and $x^1$ at $t$. Then we have

$$I_{x^0} \cap \mathbb{R}^2 = \{(x, y, z) : g(x^0) = 0, g(x^1) = 0, h(x^0) = 0, h(x^1) = 0\}.$$

Thus, the SSR condition in $\mathbb{R}^4$ and $x^1$ at $(t, S(t))$ and $(S^{-1}(t), t)$ (for the second point we must fact that $\gamma$ is symmetric with result in [Kh] is seen to agree Recall that, geometrically, at the appropriate points, to dimensions, $n \geq 6$, the SSR condition of contact between $\Gamma$ and the symmetric hypersurfaces is satisfied.

(i) Let $\gamma \subset \mathbb{R}^2$ be the elliptic cone $x^2/a^2 + y^2/b^2 = 1$, and let $\Gamma$ be the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, with $a, b, c$ satisfying $a^2 + b^2 + c^2 = 1$. Let $\alpha \in \mathbb{R}$ be the angle between the planes normal to $x^0$ and $x^1$ at $t$. Then we have

$$I_{x^0} \cap \mathbb{R}^2 = \{(x, y, z) : g(x^0) = 0, g(x^1) = 0, h(x^0) = 0, h(x^1) = 0\}.$$

Thus, the SSR condition in $\mathbb{R}^4$ and $x^1$ at $(t, S(t))$ and $(S^{-1}(t), t)$ (for the second point we must fact that $\gamma$ is symmetric with result in [Kh] is seen to agree Recall that, geometrically, at the appropriate points, to dimensions, $n \geq 6$, the SSR condition of contact between $\Gamma$ and the symmetric hypersurfaces is satisfied.

(ii) Let $\gamma \subset \mathbb{R}^2$ be the contour ellipsoid $x^2/a^2 + y^2/b^2 = 1$, and let $\Gamma$ be the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, with $a, b, c$ satisfying $a^2 + b^2 + c^2 = 1$. Let $\alpha \in \mathbb{R}$ be the angle between the planes normal to $x^0$ and $x^1$ at $t$. Then we have

$$I_{x^0} \cap \mathbb{R}^2 = \{(x, y, z) : g(x^0) = 0, g(x^1) = 0, h(x^0) = 0, h(x^1) = 0\}.$$
I_0 and I_\ast meet on the plane \( z_1 = (t + S(t))/2 \), and the intersections of \( I_0 \) with \( \hat{\Gamma} \) meet the complexified meridian plane \((\zeta, \zeta^*)\) at the two points \((S^{-1}(t), t)\) and \((t, S(t))\); note that \( \hat{\gamma} \subset \mathbb{C}^2 := \{(\zeta, \zeta^*)\} \) is given by the equation \( \zeta^* = S(\zeta) \). Hence, in view of (4.3) we have on \( I_0 \cap \hat{\Gamma} = I_\ast \cap \hat{\Gamma} \)

\[
\begin{align*}
\left\{ \begin{array}{l}
g_d(\cdot - x^0)\Big|_{\hat{\Gamma}} = ((S(\zeta) - t) + (\zeta - t)S'(\zeta))d\zeta, \\
g_d(\cdot - x^1)\Big|_{\hat{\Gamma}} = ((S(\zeta) - S(t)) + (\zeta - S(t))S'(\zeta))d\zeta.
\end{array} \right.
\]

Thus, the SSR condition in \( \mathbb{R}^4 \) (in its equivalent formulation Definition 3.2) for \( x^0 \) and \( x^1 \) at \((t, S(t))\) and \((S^{-1}(t), t)\) is fulfilled with

\[
\lambda = -\frac{1}{S'(t)}
\]

(for the second point we must keep in mind that \( S = S^{-1} \), which follows from the fact that \( \gamma \) is symmetric with respect to the \( x_1 \)-axis), as claimed above, and the result in [Kh] is seen to agree with Theorems 3.1 and 4.1 of the present paper.

Recall that, geometrically, the SSR condition in \( \mathbb{R}^4 \) means that \( \hat{\Gamma} \) is tangent, at the appropriate points, to the sphere given in Proposition 3.1. In higher even dimensions, \( n \geq 6 \), the SSR condition is more restrictive (it calls for higher order of contact between \( \hat{\Gamma} \) and the sphere), and the corresponding statement for axially symmetric hypersurfaces is not true in general as the following examples illustrate.

(i) Let \( \gamma \subset \mathbb{R}^2 \) be the ellipse

\[
x_1^2 + \frac{x_2^2}{a^2} = 1,
\]

where \( a^2 \neq 1 \), and let \( \Gamma \) be the corresponding axially symmetric ellipsoidal surface in \( \mathbb{R}^n \), \( n \geq 4 \) even. Since \( \hat{\Gamma} \) is a quadric, it cannot have an order of contact greater than one with another quadric unless the two quadrics are the same. Hence, since \( a^2 \neq 1 \), no two points are SSR with respect to \( \Gamma \) in \( \mathbb{R}^n \), with \( n \geq 6 \) even.

(ii) Let \( \gamma \subset \mathbb{R}^2 \) be the curve

\[
x_1 = c(r^2 + 1)^k,
\]

where \( k \geq 2 \) is an integer and \( c > 0 \) is small, and let \( \Gamma \) be the corresponding surface of revolution in \( \mathbb{R}^n \), with \( n \geq 4 \) even. We claim that the points \( x^0 = (1, 0, \ldots, 0) \) and \( x^1 = (-1, 0, \ldots, 0) \) are SSR points with respect to \( \Gamma \) and with constant \( \lambda = 1 \) if and only if \( n \leq 2k \). To prove this, we shall verify that the complex hyperplane \( \{z_1 = 0\} \subset \mathbb{C}^* \) meets \( \hat{\Gamma} \) with order of contact \( k - 1 \) along the intersection with the isotropic cones \( I_0 \) and \( I_\ast \), in accordance with the geometric interpretation of
the SSR condition. By axial symmetry, it suffices to check that \( \tilde{\gamma} \) in the complex meridian plane \( \mathbb{C}^2 \) meets the two points \((0, i)\) and \((0, -i)\) (the points in \( \mathbb{C}^2 \) that generate the intersection \( I_\varphi \cap I_k \) in \( \mathbb{C}^n \) by revolution) and, moreover, that \( \tilde{\gamma} \) meets the hyperplane \( \{z_1 = 0\} \) with order of contact \( k - 1 \) at \((0, i)\) and \((0, -i)\). If we denote the variables in \( \mathbb{C}^2 \) by \((z_1, \rho)\) (as above, we allow \( \rho \) to be complex here) then \( \tilde{\gamma} \) is given by

\[
z_1 = c(\rho^2 + 1)^k.
\]

Obviously, the two points \((0, i)\) and \((0, -i)\) are on \( \tilde{\gamma} \) and the order of contact between \( \gamma \) and \( \{z_1 = 0\} \) at each of the two points is, as claimed, \( k - 1 \). This proves the claim above, since \( \mathbf{x}^0 \) and \( \mathbf{x}^1 \) are SSR with respect to \( \Gamma \) and with constant \( \lambda = 1 \) if and only if \((n - 2)/2 \leq k - 1 \).

5. The general odd-dimensional case

Let us finally consider the general odd dimensional case. First we need a representation lemma similar to Lemmas 2.1 and 3.1. Recall, as in the previous sections, that \( \tilde{\Gamma} = \{z_n = \phi(x')\} \) and that the coordinates \( \mathbf{w} \) are defined by \( w_j = z_j \)

\[
\text{for } j = 1, \ldots, n - 1 \text{ and } w_n = z_n - \phi(x').
\]

**Lemma 5.1** Let \( D_\varphi \) be the relatively compact component of \( (\tilde{\Gamma} \cap M) \setminus \gamma_\varphi \), i.e. the domain in \( \tilde{\Gamma} \cap M \) bounded by the deformed sphere \( \gamma_\varphi \). Then, for any \( u \in \text{Har}_0(U, \Gamma) \) and \( \mathbf{x}^0 \in V \setminus \Gamma \), we have

\[
u(x^0) = 2 \int_{D_\varphi} \omega_\varphi,
\]

where \( \omega_\varphi \) is a \( n - 1 \) form in \( D_\varphi \) and the orientation of \( D_\varphi \) is chosen according to the one in (1.1). The form \( \omega_\varphi \) can be written as

\[
\omega_\varphi = c_n \frac{\partial u}{\partial w_n} \frac{1}{g(\cdot - x^0)^{(n-2)/2}} \, dw_1 \wedge \cdots \wedge dw_{n-1},
\]

where \( g(\cdot - x^0) \) is the defining function for \( I_\varphi \) as before.

**Remark** Note that the form \( \omega_\varphi \) has a singularity on the boundary \( \gamma_\varphi \). However, this singularity is integrable.

**Proof** If we would try to prove this directly using a shrinking tube as in the proof of Lemma 2.1 we would run into difficulties. It is not even clear from the rough estimates used in that proof that the two integrals appearing there converge, for \( n \geq 5 \), as the radius of the tube tends to zero; much less that the integral over the torus tends to zero. We use another approach. Since we have calculated the representation for the general even dimensional case we can use Hadamard's method of descent (cf. e.g. [J], denote the variables in \( \mathbb{R}^{n+1} \) by \( s = s' + is'' \), the function \( \nu \) that does not depend on the variable cylinder in \( \mathbb{R}^{n+1} \). The fact that we in the higher dimensional space is exactly the one in the previous section point \((0, x^0) \) in \( \mathbb{R}^{n+1} \). By Lemma

\[
h_{\pm} = \pm i \sqrt{\lambda},
\]

\((y \text{ is the imaginary part of } z)\),

where \( h^*_\varphi \) denotes the pullback \( D_\varphi \) is chosen in accordance with which we have used previous case though, it is convenient coordinates. Recall that the \( 1, \ldots, n - 1 \) and \( w_n = z_n - \phi(x') \)

equation (3.1) using the fact

which is never zero on \( \gamma_\varphi \), \( D_\varphi \), and the facts that \( u \) at the function under the square \( D_\varphi \). This calculation show

\[
\omega_\varphi = h^*_\varphi \omega_\varphi = \frac{2\pi ic_n}{p} \times
\]

method of descent (cf. e.g. [J], chapter 5). We consider the space $\mathbb{R}^{n+1}$. We denote the variables in $\mathbb{R}^{n+1}$ by $'x = (s', x)$ and the variables in $\mathbb{C}^{n+1}$ by $'z = (s, z)$ with $s = s' + is''$. The function $u$ can be extended as a harmonic function in $\mathbb{R}^{n+1}$ that does not depend on the variable $s'$. We consider also the hypersurface $\Gamma$ as a cylinder in $\mathbb{R}^{n+1}$. The fact that we use the same notation for the objects considered in the higher dimensional space should cause no confusion. Now, the situation is exactly the one in the previous section because $n + 1$ is even. Let $'x^0$ denote the point $(0, x^0)$ in $\mathbb{R}^{n+1}$. By Lemma 3.1, we have

$$u(x^0) = \int_{\gamma_{x^0}} \omega_{x^0}.$$ 

Since $\hat{\Gamma}$ is a cylinder in $\mathbb{C}^{n+1}$ it is defined by the equation $\{z_n = \phi(z')\}$, where $\phi$ does not depend on $s$. It follows, from the calculations in Section 1, that $\gamma_{x^0}$ can be written as the union $\gamma_+ \cup \gamma_-$ where $\gamma_{\pm}$ can be written as a graph over $D_{x^0}$, considered as lying in the subspace $\{s = 0\}$, given by the equation $s = h_{\pm}(z')$ and

$$h_{\pm} = \pm i \sqrt{(\|h\|^2 + 1)(\xi - x_n^2) - (\|\xi\|^2 + n^2)}$$

(5.2)

(y is the imaginary part of $z$). It follows that

$$u(x^0) = 2 \int_{D_{x^0}} h_{+}^* \omega_{x^0},$$

where $h_{+}^*$ denotes the pullback by the mapping induced by $h_+$ and the orientation of $D_{x^0}$ is chosen in accordance with (1.1). Let us change to another set of coordinates which we have used previously, the coordinates we have denoted by $w$. In this case though, it is convenient to use the notation $(t, w) = (t, w_1, ..., w_n)$ for these coordinates. Recall that these coordinates are given by $t = s$, $w_j = z_j$ for $j = 1, ..., n - 1$ and $w_n = z_n - \phi(z')$. The form $h_{+}^* \omega_{x^0}$ is now readily calculated from equation (3.1) using the fact, notation as in Lemma 3.1, that

$$\frac{\partial g(-'x^0)}{\partial t} = 2t$$

which is never zero on $\gamma_{x^0}$ except on the intersection with $\gamma_{x^0}$, the boundary of $D_{x^0}$, and the facts that $u$ and $\phi$ are independent of the variable $t$. Also, note that the function under the square root sign in (5.2) equals the restriction of $g(-'x^0)$ to $D_{x^0}$. This calculation shows that

$$\omega_{x^0} = h_{+}^* \omega_{x^0} = \frac{2\pi i c_{n+1}}{(p - 1)!} \frac{\partial u}{\partial w_n} \left(1 + \left< \nabla \phi, \nabla \phi \right> \right)$$

$$\times \left[ \left( \frac{\partial}{\partial t} \right)^{p-1} \left( \frac{1}{2t} \right) \right]_{t = \sqrt{g(-'x^0)}} dw_1 \wedge \cdots \wedge dw_{n-1},$$

\[\text{where, in the \textit{vege,\textit{egral lated ard's}}}.\]
where \( p = [(n + 1) - 2]/2 = (n - 1)/2 \) is the integer appearing in Lemma 3.1. It is straightforward to verify that
\[
\left( \frac{\partial}{\partial t} \right)^{p-1} \left( \frac{1}{2t} \right) (-1)^{p+2}(2p-3)!! \frac{2n!}{2^{2p-1}}.
\]
If we put this into the expression for \( \omega_x \) above, use the definition of the integer \( p \) and the fact (cf. [1], page 97) that
\[
c_{n+1} = \frac{2^{(n-1)/2}((n - 3)/2)!!}{2n(n-4)!!} c_n
\]
we obtain equation (5.1).

Now, we formulate and prove the last theorem in this paper. This theorem, together with Theorem 4.1, completes the investigation of reflection properties of harmonic functions in \( \mathbb{R}^n \).

**Theorem 5.1** Let \( \Gamma \) be a nonsingular, real-analytic hypersurface in some neighborhood \( U \) of the origin in \( \mathbb{R}^n \), with \( n \) odd. Suppose that \( \Gamma \) is neither part of a hyperplane nor a sphere. Then there is a neighborhood \( V, \overline{V} \subset U \), such that for no pair of points \( x^0, x^1 \) in \( V \setminus \Gamma \) is there a constant \( K \) satisfying
\[
u(x^0) + Ku(x^1) = 0, \quad \forall u \in \text{Har}_0(U, \Gamma),
\]
where \( \text{Har}_0(U, \Gamma) \) denotes the class of harmonic functions in \( U \) vanishing on \( \Gamma \).

**Proof** Note that the representation of a harmonic function in \( \mathbb{R}^n \), with \( n \) odd, given by Lemma 5.1 is of the same form as in \( \mathbb{R}^3 \), i.e. the normal derivative of the function at \( \bar{f} \) appears as a factor in the form that we are integrating. Hence, the proof of Theorem 2.1 immediately carries over to this case to prove that if there is a constant such that (5.3) holds then we must have
\[
\frac{1}{g(-x^0)(n-3)/2} - K \frac{1}{g(-x^1)(n-3)/2} = 0
\]
on \( \Gamma \). Consequently, \( \Gamma \) is contained in the real-algebraic hypersurface
\[
\langle z - x^0, z - x^0 \rangle = K^{2/(n-2)} \langle z - x^0, z - x^0 \rangle,
\]
i.e. \( \Gamma \) is contained in either a hyperplane or a sphere.

As before, combining Theorem 5.1 with Proposition 0.1 we obtain:

**Corollary 5.1** If, for \( x^0, x^1 \in V \setminus \Gamma \), the functions from \( \text{Har}_0(U, \Gamma) \) do not separate those points then \( \Gamma \) is either part of a sphere or a hyperplane.

**6. A final remark**

Thus, Theorems 4.1 and 5.1 reflection of harmonic function the Huygens principle (the \( H_0 \) for general hyperbolic equation general elliptic equations almost near a real-analytic curve \( \Gamma \subset I \) Helmholtz equation vanishing the proof of this). On the other principle in [G]. Namely, poi solutions of rather general part each point \( x^0 \) there is a compact hypersurface \( \Gamma \) and a measure on \( K \) such that

for all solutions of \( P(x, D)u = 0 \), we conjecture for the He in [SSS].

Part of this work was carried out in the spring of 1993, which the Foundation and the Royal Inst
6. A final remark

Thus, Theorems 4.1 and 5.1 completely solve the problem of point to point reflection of harmonic functions in higher dimensions. As in the situation with the Huygens principle (the Huygens principle in the strong form does not hold for general hyperbolic equations), point to point reflection for solutions of more general elliptic equations almost always fails. For example if, for a pair of points near a real-analytic curve $\Gamma \subset \mathbb{R}^2$, the equation (0.1) holds for all solutions of the Helmholtz equation vanishing on $\Gamma$ then $\Gamma$ must be a straight line (see [KS] for the proof of this). On the other hand, Garabedian suggested a weaker reflection principle in [G]. Namely, point to compact set reflection, which may hold for solutions of rather general partial differential equations $P(x,D)u = 0$, i.e. that for each point $x^0$ there is a compact set $K = K(x^0, \Gamma, P(x,D))$ "on the other side" of the hypersurface $\Gamma$ and a measure (or a distribution) $\mu = \mu(x^0, \Gamma, P(x,D))$ supported on $K$ such that

$$u(x^0) = \int_K u(x)d\mu$$

for all solutions of $P(x,D)u = 0$ with $u|_\Gamma = 0$. An explicit calculation confirming this conjecture for the Helmholtz operator in two dimensions has been done in [SSS].

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