# AN OVERDETERMINED PROBLEM IN POTENTIAL THEORY

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Abstract. We investigate a problem posed by L. Hauswirth, F. Hélein, and F. Pacard [10], namely, to characterize all the domains in the plane that admit a "roof function", i.e., a positive harmonic function which solves simultaneously a Dirichlet problem with null boundary data, and a Neumann problem with constant boundary data. As they suggested, we show, under some a priori assumptions, that there are only three exceptional domains: the exterior of a disk, a halfplane, and a nontrivial example found in [10] that is the image of the strip $|\Im\zeta| < \pi/2$ under $\zeta \to \zeta + \sinh(\zeta)$. We show however that one cannot obtain any axially symmetric analogues of this example in $\mathbb{R}^4$.

## 1. Introduction

In [10], the authors have posed the following problem: find a smooth bounded domain $\Omega$ in a Riemannian manifold $\mathcal{M}_g$ with metric $g$, such that the first eigenvalue $\lambda_1$ of the Laplace-Beltrami operator on $\Omega$ has a corresponding real, positive eigenfunction $u_1$ satisfying $u_1 = 0, \frac{\partial u_1}{\partial n} = 1$ on the boundary of $\Omega$. Any such domain is called *extremal* because it provides a local minimum for the first eigenvalue $\lambda_1$ of the Laplace-Beltrami operator, under the constraint of fixed total volume of $\Omega$ (see [10] and references therein).

In special cases one can find a sequence of extremal domains $\{\Omega_t\}$ with increasing volumes, such that the limit domain $\Omega = \Omega_{t\to\infty}$ is unbounded, and its first eigenvalue vanishes as $t \to \infty$. This limit extremal domain is then called *exceptional*, and the corresponding limit function $(u_{1,t})_{t\to\infty} \to u$ is a positive, harmonic function on $\Omega$ which solves simultaneously the overdetermined boundary value problem with null Dirichlet data and constant Neumann data.

The problem of finding exceptional domains in $\mathbb{R}^n$ and their corresponding functions $u$ (called "roof" functions by the authors of [10]) is nontrivial as a problem of potential theory. There is no obvious variational principle to use, on the one hand because $\Omega$ is unbounded (so the Dirichlet energy of $u$ [2, Ch. 1] will diverge), and, on the other hand, because the constant Neumann data constraint is not conformally invariant.

In the absence of a suitable variational formulation, we may interpret the scaling $t \to \infty$ described above as a dynamical process, in which the pair $(\Omega_t, u_t)$ evolves so that the limit $t \to \infty$ solves the overdetermined problem. In other words, we can turn this observation into a constructive method for finding (building) exceptional domains. In order to do this, it is helpful to note that, upon compactification of the boundary $\partial\Omega$ (with metric $d\sigma^2$), the pair $(\Omega, u)$ with flat metric becomes conformal to the half-cylinder $\mathcal{N} := \mathbb{R}_+ \times \overline{\partial\Omega}$, with metric

$$ds^2 = e^{-2u}(du^2 + d\sigma^2).$$

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Under this reformulation, scaling of $(\Omega_t, u_t)_{t\to\infty}$ becomes equivalent to scaling of the metric structure given above, defined over the fixed space $\mathcal{N}$. This is reminiscent of the Ricci flow, in which the metric structure $g$ evolves with respect to a deformation parameter $t \in \mathbb{R}$ according to the equation

$$\frac{dg_{ij}}{dt} = -2R_{ij},$$

with the right side of the equation given by the covariant Ricci tensor. It is known [21] that for the case of a two-dimensional manifold, with metric given by $ds^2 = e^{-2u}(dx^2 + dy^2)$, the Ricci flow equations reduce to a single nonlinear equation

$$\frac{\partial u}{\partial t} = \nabla_g^2 u$$

(since in two dimensions the Riemann tensor has only one independent component). This is a heat equation with the generator given by the Laplace-Beltrami operator corresponding to the metric $ds^2$. Therefore, if there is a stationary solution $\frac{\partial u}{\partial t} \to 0$ as $t \to \infty$, it will correspond to the scaling of the first eigenvalue $\lambda_1(t) \to 0$ and, by conformally mapping back $\mathcal{N}$ using the solution $u(t \to \infty)$, we will obtain the solution $(\Omega, u)$.

In other words, we can summarize this constructive method for finding exceptional domains in $\mathbb{R}^2$ as follows: starting from a 2-dimensional Riemannian manifold with finite volume and metric encoded through the positive real function $u$, and boundary set defined via $u = 0$, consider the time evolution given by the Ricci flow, without volume renormalization. Then [21] the manifold will remain Riemannian at all times, and in the $t \to \infty$ limit the function $u$ will become a solution of the nonlinear Laplace-Beltrami equation. Furthermore, if $u$ remains finite everywhere in the domain, then it is harmonic and satisfies both Dirichlet and Neumann conditions at all finite boundary components, so it is a solution for the overdetermined potential problem. Considered together with the (boundary) point at infinity, the manifold is equivalent [16] to a pseudosphere (flat everywhere except at the infinity point, with overall positive curvature). (We wish to emphasize that there is no reason to assume that such constructive methods would be exhaustive.)

Thus, so motivated, it is natural to try to characterize exceptional domains in flat Euclidean spaces. The authors in [10] suggested that in two dimensions there are only three examples: a disk, a halfplane, and a nontrivial example obtained as the image of the strip $|\Im\zeta| \leq \pi/2$ under the mapping $\zeta \to \zeta + \sinh(\zeta)$. They posed as an open problem to determine if these are the only examples [10, Section 7]. (They gave some evidence by characterizing the halfplane under a global assumption on the gradient of the roof function [10, Prop. 6.1].) They also posed the problem of finding nontrivial examples in higher dimensions and suggested the possibility of axially symmetric examples similar to the nontrivial example in the plane [10, Remark 2.1].

We address both of these problems. The paper is organized as follows. In Section 2, we review the theory of Hardy spaces in order to address a subtlety that arises in connection with the regularity of the boundary of an exceptional domain. In Section 3, we characterize exteriors of disks as being the only exceptional domain whose complement is bounded and connected. In Section 4, we establish a connection between the "roof function" of an exceptional domain and the so-called *Schwarz function* of its boundary. We use this to show that, under some smoothness assumptions, the boundary can pass at most twice through infinity. In Section 5, we show that with some smoothness assumptions on the boundary, if an exceptional domain is simply connected and the boundary passes once through infinity, then the domain must be a halfplane. In Section 6, we similarly characterize the nontrivial example found in [10, Section 2].

Sections 3 through 6 together show that with some assumptions on the topology, the smoothness of the boundary, and its behavior at infinity, there are only three exceptional domains: the exterior of a disk, a halfplane, and the image of the strip $|\Im\zeta| \leq \pi/2$ under the conformal map $\zeta \to \zeta + \sinh(\zeta)$. Under these additional assumptions, this confirms what was suggested in [10, Section 7].

In Section 7, we extend the result of Section 3 to higher dimensions. In Section 8, we show that the nontrivial example from Section 6 does not allow an extension to axially symmetric domains in four dimensions, contrary to what was suggested in [10, Remark 2.1] (and we conjecture that this example has no analogues in any number of dimensions greater than two). In Section 9, we discuss an alternative approach to the problems discussed in this paper.

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## 2. Preliminaries: Classical vs. Weak Solutions, Regularity of the Boundary, Hardy Spaces

From the rigidity of the Cauchy problem, one might expect to obtain, "for free", regularity of the boundary of an exceptional domain (as is often the case for solutions of free boundary problems). Unfortunately, the problem at hand is complicated by a remarkable family of examples with rectifiable but non-smooth boundaries, a.k.a. non-Smirnov domains - cf. [6, Ch. 10]. This results in adding a Smirnov condition to the assumptions on the domains if we desire to consider "weak solutions", i.e., harmonic "roof functions" satisfying the Dirichlet and Neumann boundary conditions almost everywhere with respect to the Lebesgue measure.

In order to address this subtlety, we first give some background from $H^p$ theory - cf. [6] for details.

An analytic function $f : \mathbb{D} \to \mathbb{C}$ is said to belong to the Hardy class $H^p$, $0 < p < \infty$, if the integrals:

$$\int_0^{2\pi} |f(re^{i\theta})|^p d\theta$$

remain bounded as $r \to 1$.

Recall that a *Blaschke product* is a function of the form

$$B(z) = z^m \prod_n \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a_n} z},$$

where $m$ is a nonnegative integer and $\sum(1 - |a_n|) < \infty$. The latter condition ensures convergence of the product (See Theorem 2.4 in [6]).

A function analytic in $\mathbb{D}$ is called an *inner function* if its modulus is bounded by 1 and its modulus has radial limit 1 almost everywhere on the boundary. If $S(z)$ is an inner function without zeros, then $S(z)$ is called a *singular inner function.*

An *outer function* for the class $H^p$ is a function of the form

$$F(z) = e^{i\gamma} \exp\left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log \psi(t) dt \right\},$$

where $\gamma$ is a real number, $\psi(t) \geq 0$, $\log \psi(t) \in L^1$, and $\psi(t) \in L^p$.

The following theorem [6, Ch. 2, Ch. 5] (also cf. [9]) provides the parametrization of functions in Hardy classes by their zero sets, associated singular measures, and moduli of their boundary values.

**Theorem 2.1.** *Every function $f(z)$ of class $H^p$ ($p > 0$) has a unique (up to a unimodular constant factor) factorization of the form $f(z) = B(z)S(z)F(z)$, where $B(z)$ is a Blaschke product, $S(z)$ is a singular inner function, and $F(z)$ is an outer function for the class $H^p$.*

Suppose $\Omega$ is a Jordan domain with rectifiable boundary and $f : \mathbb{D} \to \Omega$ is a conformal map. Then $f' \in H^1$ by Theorem 3.12 in [6]. By Theorem 2.1, $f'$ has a canonical factorization $f'(z) = B(z)S(z)F(z)$, and since $f$ is a conformal map $f'$ does not vanish, so $f'(z) = S(z)F(z)$. Then $\Omega$ is called a *Smirnov domain* if $S(z) \equiv 1$ so that $f'(z) = F(z)$ is purely an outer function. This definition is independent of the choice of conformal map.

There are examples of non-Smirnov domains with, as above, $f'(z) = S(z)F(z)$, but now $F(z) \equiv 1$ and the singular inner function $S(z)$ is not constant. Such examples were first constructed by M. Keldysh and M. Lavrentiev [13] using complicated geometric arguments. Their existence was somewhat demystified by an analytic proof provided by P. Duren, H. S. Shapiro, and A. L. Shields [7]. Like the disk, such a domain has harmonic measure proportional to arc-length. Mapping by $z \to \frac{1}{z}$ gives an example of an exceptional domain where the roof function $u$ is the Green's function with singularity at infinity. This is a weak solution satisfying the boundary conditions almost everywhere.

If we require $u$ to be a "classical solution" that satisfies the boundary condition everywhere (and not just almost everywhere), then non-Smirnov domains are ruled out. Moreover, real-analyticity of the boundary then follows automatically. To be precise, we have the following Lemma.

**Lemma 2.2.** *If $\Omega \subset \mathbb{R}^2$ is exceptional and the roof function $u$ is a "classical solution" in $C^1(\overline{\Omega})$, then $\partial\Omega$ is locally real-analytic.*

*Proof.* The analytic completion $f(z) = u + iv$ (possibly multivalued) maps $\Omega$ into the right halfplane, since $u$ is positive. The Neumann condition for $u$ implies that $|f'(z)| = 1$ on $\partial\Omega$. Also, $u \in C^1(\overline{\Omega})$ implies that $f' \in C(\overline{\Omega})$.

Choose a point $z_0 \in \partial\Omega$, and let $\zeta_0 = f(z_0)$. Let $g(\zeta) = f^{-1}(\zeta)$ denote the local inverse of $f(z)$. Choose a neighborhood $U$ of $\zeta$ and let $F := \overline{U \cap \{\Re(\zeta) \geq 0\}}$. Choose $U$ small enough so that $g \in C(F)$.

Since $|g'(\zeta)| = 1$ on $\partial\Omega$, we can also choose $U$ small enough that $g'$ does not vanish in $F$. This implies that $h(\zeta) = \mathrm{Log}(g'(\zeta))$ is analytic in the interior of $F$ and continuous in $F$. We have $\Re\{h(\zeta)\}$ vanishes on the imaginary axis, since $|g'(\zeta)| = 1$ there. Thus $h(\zeta)$ extends to a neighborhood of $\zeta_0$ by the Schwarz reflection principle. This allows us to extend $g'(z)$ and therefore $g(z)$ and $f(z)$ extend analytically across $z_0$, since $u := \Re f = 0$ on $\partial\Omega$ and $|\nabla u| = 1$ on $\partial\Omega$ near $z_0$. The lemma is proved.

□

**Corollary 2.3.** *If $\partial\Omega$ is $C^2$-smooth and $\Omega$ is exceptional then $\partial\Omega$ is locally real-analytic.*

*Proof.* $C^2$-smoothness of $\partial\Omega$ implies that $u$ is in $C^1(\overline{\Omega})$. It remains now to refer to Lemma 2.2. □

Using Kellogg's theorem on regularity of conformal maps up to the boundary, cf. [17, Ch. 3], one easily extends the above corollary to $C^{1,\alpha}$, $\alpha > 0$, boundaries and even merely to $C^1$ boundaries. We shall not pursue these details here. It would be interesting to find sharp necessary and sufficient conditions for the a priori regularity of the boundary that would guarantee the conclusion of Corollary 2.3. As we have mentioned in the beginning of this section, it is necessary to assume that the domain is Smirnov, but it is not at all obvious that this is indeed sufficient - cf. a related discussion in [5] regarding nonconstant functions of class $E^p$ with real boundary values.

**Ansatz:** In order to streamline and clarify the arguments we shall always assume that the roof function is a classical solution so that Lemma 2.2 applies. Thus, from now on we include the assumption that $u$ is a classical solution in the definition of an exceptional domain. However, for two of the three exceptional domains (exteriors of disks and halfplanes) it is possible to relax the definition to allow weak solutions and extend the results merely adding the assumption that the domains are Smirnov. In these cases we indicate the appropriate changes in the remarks following the proof.

## 3. The Case When Infinity is an Isolated Boundary Point

**Theorem 3.1.** *Suppose $\Omega$ is an exceptional domain whose complement $\mathbb{C} \setminus \Omega$ is bounded and connected. Then $\Omega$ is the exterior of a disk.*

*Remark.* (i) The theorem is still true if the definition of exceptional domain is relaxed to allow the roof function to be a weak solution satisfying the boundary conditions a.e., but as discussed in Section 2 it is then necessary to assume that the inversion of $\Omega$ by $z \to 1/z$ satisfies the Smirnov condition (where, without loss of generality, the origin is not in $\Omega$). For details, see the comments after the proof of the theorem.

(ii) We have been unable to drop the assumption that the complement is connected.

*Proof.* Let $u$ be a roof function for $\Omega$. Positivity of $u$ implies, by Bôcher's Theorem [3, Ch. 3], $u(z) = u_0(z) + C \log|z|$ for some constant $C$, where $u_0(z)$ is harmonic in $\Omega \cup \{\infty\}$. We may assume that $C = 1$ (multiply $u(z)$ by $1/C$, if necessary), so the harmonic conjugate $v(z)$ has period $2\pi$ around $\partial\Omega$. Then define $f(z) = u(z) + iv(z) = f_0(z) + \mathrm{Log}(z)$, where $f_0(z)$ is analytic in $\Omega \cup \{\infty\}$. Note that (for $0 \in \mathbb{C} \setminus \Omega$) $f'(z) = f_0'(z) + \frac{1}{z}$ is then analytic in $\Omega \cup \{\infty\}$.

Let $g(z) = \exp\{f(z)\}$. We have

$$g'(z) = f'(z) \exp\{f(z)\}. \tag{3.1}$$

First notice from this equation that $g'(z) = zf'(z) \exp\{f_0(z)\}$ which implies that $g'(z)$ is analytic in $\Omega \cup \{\infty\}$. Indeed, $\infty$ is a removable singularity for $zf'(z)$ by virtue of the fact that $|f'(z)| = O(\frac{1}{|z|})$.

Let $\Gamma := \partial\Omega \setminus \{\infty\}$ denote the finite part of the boundary of $\Omega$. The Dirichlet data for $u$ implies that $|\exp\{f(z)\}| = \exp\{\Re f(z)\} = 1$ on $\Gamma$. The Neumann data for $u$ and the Cauchy-Riemann equations imply $|f'(z)| = 1$ on $\Gamma$. Thus, on $\Gamma$, Eq.

(3.1) gives $|g'(z)| = 1$. We analyze $|g'(z)|$ near infinity by inserting $f'(z) = f_0'(z) + \frac{1}{z}$ into Eq. (3.1):

$$g'(z) = \left(f_0'(z) + \frac{1}{z}\right) z \exp\{f_0(z)\}. \tag{3.2}$$

As $z \to \infty$, $|\exp\{f_0(z)\}|$ is bounded and $|f_0'(z)| = O(\frac{1}{|z|^2})$, so that $|g'(z)| \approx \exp\{\Re f_0(z)\} \to 1$.

Now $|g'(z)| = 1$ on the finite boundary of $\Omega$ and at $\infty$. Moreover, $\log |g'(z)|$ is harmonic in $\Omega \cup \{\infty\}$, since $g'(z)$ is free of zeros. To see why $g'(z)$ does not vanish, notice that $g'(z_0) = 0$ implies (by (3.1) evaluated at $z_0$) that $f'(z_0) = 0$ which implies that $z_0$ is a critical point of $u$. But $u$ is Green's function for $\Omega \cup \{\infty\}$ with the pole at $\infty$, so the number of critical points equals the Betti number (see [9]) which is zero in this case, since $\Omega \cup \{\infty\}$ is simply connected.

By the maximum principle, $\log |g'(z)| = 0$ is constant. Therefore $g'(z) = e^{i\alpha}$ is a unimodular constant, and $g(z) = e^{i\alpha} z + C$.

Now $u(z) = \log |e^{i\alpha} z + C|$, and thus $\Omega$ is the exterior of a disk. □

Let us now show how to adjust the proof to extend the result to allow weak solutions in the definition. Thus, assume merely in the hypothesis that the inversion of $\Omega$ is a Smirnov domain. In that case, we can no longer use the maximum principle, since $|g'(z)| = 1$ only holds a.e. Instead, we assert that $1/g(z)$ is still a conformal map to the disk, and taking the reciprocal of the inverse gives a conformal map $h(\zeta)$ from the disk to the inversion of $\Omega$, with $|h'(\zeta)| = 1$ a.e. on $\partial\mathbb{D}$. From the Smirnov condition we see that $h'(\zeta)$ is an outer function. Therefore, by the representation for outer functions described in Section 2, the boundary values $|h'(\zeta)| = 1$ a.e. determine $h'(\zeta)$ is a unimodular constant.

## 4. The Schwarz Function of an Exceptional Domain

The *Schwarz function* of a real-analytic curve $\Gamma$ is the (unique and guaranteed to exist near $\Gamma$) complex-analytic function that coincides with $\bar{z}$ on $\Gamma$.

We recall two basic facts needed in the proof of the next proposition.

(i): On $\Gamma$, $|S'(z)| = 1$.

(ii): The complex conjugate of $\sqrt{-S'(z)}$ is normal to $\Gamma$.

Statement (i) follows from the chain rule and the fact that the complex conjugate of the Schwarz function, $\overline{S(z)}$, is an involution (see [4, Ch. 7]). Statement (ii) follows from the formula for the complex unit tangent vector $T(z) = \frac{dz}{ds} = \frac{1}{\sqrt{S'(z)}}$ expressing the derivative of $z$ with respect to the arc-length along $\Gamma$ (see again [4, Ch. 7]).

**Proposition 4.1.** *If $\Omega$ is an exceptional domain, then $\Gamma = \partial\Omega$ is locally analytic, and the $z$-derivative of the roof function is given by $u_z(z) = c\sqrt{-S'(z)}$, where $c$ is a real constant and $S(z)$ is the Schwarz function of $\partial\Omega$. In particular, $S'(z)$ is analytic throughout $\Omega$.*

*Remark:* The sign of the constant $c$ depends on the orientation of the boundary.

*Proof.* Since $\Gamma$ is real-analytic by Lemma 2.2, it has a Schwarz function $S(z)$. The complex conjugate of $u_z$ is normal to $\Gamma$ (since $u$ has zero Dirichlet data). In light of the constant Neumann data, we then have $|u_z(z)| = \frac{1}{2}|u_x - iu_y| = \frac{1}{2}\sqrt{u_x^2 + u_y^2}$

is constant on $\Gamma$. This, along with the statements (i) and (ii) above, shows that on $\Gamma$ the vectors $u_z(z)$ and $\sqrt{-S'(z)}$ are parallel and each have constant length. Therefore, for some real constant $c$, the equation $u_z(z) = c\sqrt{-S'(z)}$ holds on $\Gamma$. But since $u_z$ and $\sqrt{-S'(z)}$ are both analytic, the equation is true everywhere that either side is defined. In particular, this guarantees analytic continuation of $S'(z)$ throughout $\Omega$. □

**Corollary 4.2.** *Suppose infinity is a point on the boundary of $\Omega$, and the tangent vector on each component of the boundary smoothly approaches a limit at infinity. Then, the angle at infinity between consecutive boundary components is $\pi$. In particular, there can be at most two boundary components passing through infinity.*

*Proof.* The derivative of the Schwarz functions of two arcs meeting at an angle different from 0, $\pi$, or $2\pi$ clash and have a branch cut singularity along an arc that propagates into the domain from the vertex. To see why this is the case, note that the Schwarz function of an arc can be approximated near a point by the Schwarz function of the tangent line (guaranteed to exist by the assumption that the tangent vector smoothly approaches a limit). Thus, to first order, the jump along the branch cut is linear, so to zeroth order, the jump of $S'$ is determined by the slopes of the tangent lines.

If the angle is 0 or $2\pi$ then the tangent line is the same for each arc, but the orientation changes, so there is still a jump due to the sign change. In the case of an angle of $\pi$ both the tangent line and the orientation are unchanged. Thus, for any angle other than $\pi$, $S'(z)$ has a jump across a branch cut between the two boundary boundary components, contradicting the condition that $u$ is a global solution throughout $\Omega$. □

## 5. The Case When the Boundary is Locally Jordan at Infinity

In this Section, we characterize the halfplane as the only simply connected exceptional domain having infinity as a single point on the boundary. This extends [10, Prop. 6.1] by removing the additional hypothesis $\partial_x u > 0$ in [10] on the roof function in $\Omega$.

**Theorem 5.1.** *Suppose $\partial\Omega$ is a curve that passes once through infinity, where it is locally Jordan. If $\Omega$ is exceptional, then $\Omega$ is a halfplane.*

*Remark.* (i) As with Theorem 3.1, the theorem is still true if the definition of exceptional domain is relaxed to allow the roof function to be a weak solution, but it is then necessary to assume that the inversion of $\Omega$ by $z \to 1/z$ is Smirnov (where, without loss of generality, the closure of $\Omega$ does not contain the origin). In that case, the same proof goes through, except the assertion below that $\Omega$ is a Smirnov domain now holds because of our assumption.

(ii) Implicit in the boundary condition is the assumption that $\Omega$ is simply connected. We have not been able to drop this condition.

*Proof.* Let $u$ be a roof function for $\Omega$, and $f(z) = u + iv$ its analytic completion. Since $u > 0$, $f(z)$ takes $\Omega$ into the right halfplane $H$. By Lemma 2.2, the boundary of $\Omega$ is locally real-analytic.

Let $\hat{\Omega}$ be the image of $\Omega$ under $z \to 1/z$, where without loss of generality the closure of $\Omega$ does not contain the origin.

Let $h(\zeta)$ be a conformal map from the disk to $\hat{\Omega}$ so that $h(1) = 0$. So $f(\frac{1}{h(\zeta)})$ takes the disk $\mathbb{D}$ into the right halfplane $H$. We may assume that $f(\frac{1}{h(0)}) = 1$. By Herglotz's Theorem (see [11, Ch. 3], [6, Ch. 1]),

$$f\left(\frac{1}{h(\zeta)}\right) = \int_{\partial\mathbb{D}} \frac{e^{i\theta}+\zeta}{e^{i\theta}-\zeta} d\mu(\theta)$$

for some finite positive measure $\mu$ on $\partial\mathbb{D}$.

Since $f(1/h(\zeta))$ is continuous up to the boundary of $\mathbb{D}$ everywhere except at $\zeta = 1$, and $\Re\{f(1/h(\zeta)\} = 0$ on $\partial\mathbb{D}$, then $d\mu = 0$ except at $\theta = 0$, so it is an atomic measure. Thus,

$$f\left(\frac{1}{h(\zeta)}\right) = C\frac{1+\zeta}{1-\zeta}. \tag{5.1}$$

Next we show that $f'(1/h(\zeta))$ is an outer function, as defined in Section 2. In order to avoid confusion, we emphasize that we are considering the pullback of $f'(z)$ by $1/h(\zeta)$.

Differentiating (5.1), $f'(1/h(\zeta))\frac{-h'(\zeta)}{h(\zeta)^2} = \frac{-2C}{(1-\zeta)^2}$. So,

$$f'(1/h(\zeta)) = 2C\frac{h(\zeta)^2}{(1-\zeta)^2 h'(\zeta)} \tag{5.2}$$

Since the boundary of $\hat{\Omega}$ is $C^1$-smooth (in fact, real-analytic), $\hat{\Omega}$ is Smirnov [6, Ch. 10]. Therefore, $h'(\zeta)$ is an outer function. Since $h(\zeta)$ is a conformal map to a bounded domain not containing 0, $h(\zeta)$ is an outer function. Finally, $(1-\zeta)^2$ is an outer function. Thus, (5.2) is a quotient of products of outer functions which must be outer.

Since $f'(1/h(\zeta))$ is an outer function, it is the exponential of the Poisson integral of the log of its own modulus. But $|f'(1/h(\zeta))| = 1$ almost everywhere on $\partial\mathbb{D}$. Hence, $f'(1/h(\zeta))$ is a unimodular constant. Therefore, $\Omega$ is a halfplane. □

## 6. The Case When Infinity is a Double Point of the Boundary

**Theorem 6.1.** *Suppose the boundary of $\Omega$ is a curve with a double point at infinity, and the tangent vector smoothly approaches a limit on each side of this point. If $\Omega$ is exceptional, then (up to Möbius transformations) $\Omega$ is the image of the strip $|\Im\zeta| \leq \pi/2$ under the conformal map $g(\zeta) = \zeta + \sinh(\zeta)$, while the analytic completion of the function $u(g(\zeta))$ is the function $f(g(\zeta)) = \cosh(\zeta)$.*

*Remark:* The exceptional domain $\Omega$ that is the image of the strip under the conformal map $\zeta \to \zeta + \sinh\zeta$ is precisely the exceptional domain found by the authors in [10]. Theorem 6.1 together with Theorems 3.1 and 5.1 and Corollary 4.2 show that this $\Omega$ is, under an assumption on the topology of $\Omega$ and some mild smoothness assumptions at $\infty$, essentially the only nontrivial example of an exceptional domain in $\mathbb{R}^2$.

*Proof.* We will use the same notation for the mapping from $\mathbb{D}$ into $\Omega$ as in Theorem 5.1. Starting from the formula which relates the tangent vector on $\partial\Omega$ and the derivative of the analytic completion $f$ of $u(z)$,

$$\tau(z) = \frac{dz}{ds} = \frac{-i}{f'(z)} = \frac{1}{\sqrt{S'(z)}},$$

we obtain from the continuity of $\tau(z)$ through the double point at infinity (see Figure 1), that

$$\oint_{\partial\Omega} d\log f'(z) = 2\pi i.$$

Note that continuity of $\tau(z)$ through the double point is guaranteed by Corollary 4.2. Since $f'(z)$ is analytic in $\Omega$ and, by our smoothness assumptions on $\partial\Omega$, is continuous up to $\overline{\Omega}$ (at $\infty$ as well), we conclude that it is a single-cover from the domain $\Omega$ to the unit disk, and that it has only one zero, at some point $z_0 \in \Omega$. Up to a Möebius transformation, this point can be chosen such that $f(z_0) = 1$. Consider now the function defined on $\Omega$, taking values in the unit disk $\mathbb{D}$,

$$g(z) := \sqrt{\frac{f(z)-1}{f(z)+1}},$$

which is also a univalent map from $\Omega$ into $\mathbb{D}$, since $\frac{f(z)-1}{f(z)+1}$ is a double cover by argument principle, mapping each of the two boundary components shown in Figure 1 onto $\mathbb{T}$. But then the ratio

$$r(z) \equiv \frac{f'(z)}{g(z)}$$

has no zeros or poles in $\Omega$ and its modulus equals 1 on $\partial\Omega$, so we conclude that $r(h(\zeta))$ is a (unimodular) constant. With no loss of generality, we can choose $r = 1$, so we conclude that

$$f' = \sqrt{\frac{f-1}{f+1}}, \quad z \in \Omega. \tag{6.1}$$

The solution of this equation follows from the identity

$$\int \sqrt{\frac{f+1}{f-1}} df = z + C,$$

solved by the substitution $f = \cosh(\zeta), z = \zeta + \sinh(\zeta)$ (fixing the constant of integration to equal zero). Now using the conditions

$$\Re f(z(\zeta)) = 0 \text{ for } \zeta \in \partial\Omega, \text{ and } \Re\mathrm{f(z(\zeta))} > 0 \text{ for } \zeta \in \Omega,$$

and the identity $\Re\cosh(x+iy) = \cosh(x)\cos(y)$, we find that the pre-image of the domain in the $\zeta$-plane is the strip $|\Im\zeta| \leq \pi/2$. Therefore, $\Omega$ can be described as the image of the strip under the map $z(\zeta) = \zeta + \sinh(\zeta)$.

□

*Remark.* (i) The differential equation (6.1) can be solved by a more general substitution using Jacobi elliptic functions [1, p. 567, §16]: $f(\zeta,k) \equiv \cos(\theta)\mathrm{cn}(\zeta,k) + \sin(\theta)\mathrm{sn}(\zeta,k)$ and $z(\zeta) = \phi(\zeta) + \cos(\theta)\mathrm{sn}(\zeta,k) - \sin(\theta)\mathrm{cn}(\zeta,k)$, where $\phi(\zeta) = \int^{\zeta} \mathrm{dn}(\xi,k)d\xi$ and $\theta$ is an arbitrary phase, $\theta \in [0,2\pi]$.

For a given value of the elliptic modulus $k \in [0,1]$, we define the corresponding domain $\mathbb{F}$ through its fundamental periods $T_1(k) = 4F(\pi/2, \sqrt{1-k^2})$ and $T_2(k) = 4F(\pi/2,k)$, where $F(\pi/2,k) = K(k)$ is the complete elliptic integral of the first kind [1, p. 590, §17.3]:

$$K(k) \equiv \int_0^{\pi/2} \frac{1}{\sqrt{1-k^2\sin^2(\theta)}} d\theta$$

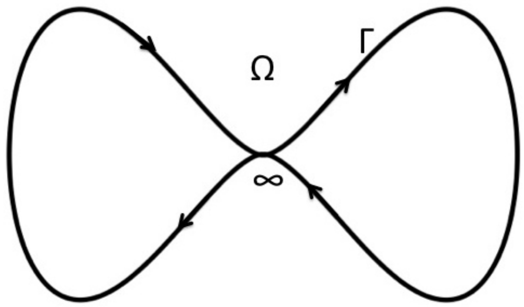


FIGURE 1. Local geometry of the boundary $\Gamma = \partial\Omega$ near infinity, for the case when supp $(\mu)$ in equation (9.1) consists of two points.

It diverges for $k = 1$ and equals $\pi/2$ for $k = 0$.

Then it is straightforward to check that (6.1) is satisfied by $f(z)$, due to the identity [1, p. 573, §16.9]:

$$1 = [-\mathrm{sn}(z)\cos(\theta) + \mathrm{cn}(z)\sin(\theta)]^2 + [\mathrm{cn}(z)\cos(\theta) + \mathrm{sn}(z)\sin(\theta)]^2.$$

Let $\gamma$ be the pre-image of $\partial\Omega$ under $z(\zeta)$: it consists of two pieces $\gamma_\pm, \gamma_- = -\gamma_+$, dividing the fundamental domain $\mathbb{F}$ into three sub-domains. Denote the component which contains the origin by $D_0$, then since $f(0) = 1$, we conclude that $\Re f(z) > 0$ for $z \in D_0 \setminus \gamma_\pm$, and we have proven the following lemma.

**Lemma 6.2.** *The exceptional domain* $\Omega$ *is the image of the domain* $D_0(k)$ *under the map* $z(\zeta) = \phi(\zeta) + \cos(\theta)sn(\zeta, k) - \sin(\theta)cn(\zeta, k)$.

(ii) The case discussed in the proof of the theorem corresponds to the degenerate elliptic modulus $k = 0$. Then the domain $\mathbb{F}$ becomes the infinite strip

$$T_1(0) = 4K(1) \to \infty, \quad T_2(0) = 4K(0) = 2\pi,$$

while the functions $f, g$ become (using the fact that $\mathrm{dn}(z, 1) \equiv 1$)

$$z(\zeta) = \zeta + \sinh(\zeta), \quad f(z(\zeta)) = \cosh(\zeta).$$

As noted before, the conditions $\Re f(\zeta)|_{\gamma_\pm} = 0$ give the pre-image $\gamma_\pm := z^{-1}(\partial\Omega) = \{\Im\zeta = \pm\frac{\pi}{2}\}$, and the pre-image of the domain, $D_0$, becomes the strip $|\Im\zeta| \leq \frac{\pi}{2}$.

(iii) The reparametrization invariance of the solution $f(z)$ of (6.1) under rescaling of the elliptic modulus $k$ is indicative of a deeper invariance of the solution: as we briefly discuss in the concluding remarks, all the specific solutions in $\mathbb{C}$ discussed here are associated with fixed points in the moduli space of Riemann surfaces. Returning to the case at hand, the domain $D_0(k)$ is the pre-image of the unit disk under the map $\zeta(w) : \mathbb{F} \to \mathbb{D}$,

$$\zeta(w) = \frac{\mathrm{sn}(w, k) - i}{\mathrm{sn}(w, k) + i}, \quad k \in [0, 1],$$

with the support of $\mu$ at points $\zeta_\pm = \pm\frac{1-ik}{1+ik}$. The case $k \to 0$ corresponds to the strip domain and to $\zeta_\pm = \pm 1$.

## 7. An Extension of Theorem 3.1 to Higher Dimensions

In this section, we give a proof of Theorem 3.1 in higher dimensions. However, since complex analytic methods are not available, we must impose higher regularity assumptions on the boundary of $\Omega$.

**Theorem 7.1.** *Suppose $\Omega$ is an exceptional domain in $\mathbb{R}^n$ whose exterior is bounded and connected. If $\partial\Omega$ is $C^{2,\alpha}$-smooth, $\alpha > 0$, then $\partial\Omega$ is a sphere.*

*Proof.* Let $u$ be a roof function for $\Omega$, and let $v(s) = \frac{1}{|s|^{n-2}}$ denote the Newtonian kernel. Fix $y \in \Omega$ and take a small ball $B_\varepsilon$ centered at $y$. Take also a large ball $B_R$ of radius $R$ that containes both $B_\varepsilon$ and the complement of $\Omega$.

Since $u(x)$ and $v(x-y)$ are harmonic in $\Omega \setminus B_\varepsilon$, Green's second identity gives

$$\int_{\partial B_R + \partial\Omega - \partial B_\varepsilon} (v(x-y)\partial_n u(x) - u(x)\partial_n v(x-y))\, d\sigma_x = 0. \tag{7.1}$$

Letting $R \to \infty$, we can drop the integration over $\partial B_R$, since again by Bôcher's Theorem [3, Ch. 3], near infinity $u(x) \approx |x|^{2-n}$.

Since, $u(x) = 0$ on $\partial\Omega$ and $\partial_n u(x) = 1$ on $\partial\Omega$,

$$\int_{\partial\Omega} v(x-y) d\sigma_x = \int_{\partial B_\varepsilon} (v(x-y)\partial_n u(x) - u(x)\partial_n v(x-y))\, d\sigma_x. \tag{7.2}$$

Let $U$ be the bounded domain such that $\mathbb{R}^n \setminus \overline{U} = \Omega$. The outward normal for $\partial U$ is opposite to that of $\partial\Omega$, and since $v(x-y) = \frac{1}{\varepsilon^{n-2}}$ on $\partial B_\varepsilon$,

$$\int_{\partial U} v(x-y) d\sigma_x = \int_{\partial B_\varepsilon} \left(-\tfrac{1}{\varepsilon^{n-2}}\partial_n u(x) + u(x)\partial_n v(x-y)\right) d\sigma_x. \tag{7.3}$$

For the first term on the right-hand-side, we have

$$\int_{\partial B_\varepsilon} \tfrac{1}{\varepsilon^{n-2}}\partial_n u(x) d\sigma_x = \int_{B_\varepsilon} \Delta u(x) dV = 0.$$

So,

$$\int_{\partial U} v(x-y) d\sigma_x = \int_{\partial B_\varepsilon} u(x)\partial_n v(x-y) d\sigma_x = u(y).$$

So, $u(y)$ is the single layer potential with charge density one on the surface $\partial U$. The normal derivative of a single layer potential has a jump across the surface equal to the charge density (e.g., see Lemma 1.3 in [8], or for a detailed exposition, see Theorem VI, Sec. 5, Ch. VI in Kellogg's classic [14]). In this case the charge density is constant. Thus, the normal derivative into $U$ of the single-layer potential vanishes, so that the single-layer potential inside is constant.

If the single layer potential of a charge density is constant inside the domain, then the charge density is proportional to the *equilibrium measure* (see [22, pp. 28, 48]). Thus, the equilibrium measure of $U$ is proportional to surface measure. Under the assumption that the boundary is $C^{2,\alpha}$, this implies that $U$ is a ball by W. Reichel's proof of P. Gruber's conjecture [18] (see also the discussion near the end of the introduction in [8]).

□

## 8. Nonexistence of a Higher-dimensional Analog of the $\cosh(z)$ Example

The authors in [10] expressed a suspicion (see Remark 2.1 in [10]) that there exist n-dimensional, rotationally-suspicion examples similar to the two-dimensional example $\{(x,y) \in \mathbb{R}^2 : |y| < \frac{\pi}{2} + \cosh(x)\}$ that appeared in Section 6. We show, perhaps surprisingly, that such an example does not exist in $\mathbb{R}^4$. Specifically, we show that there does not exist an exceptional domain in $\mathbb{R}^4$ whose boundary is generated by rotation about the $x$-axis of the (two-dimensional) graph of an even function.

**Theorem 8.1.** *There does not exist a rotationally-symmetric exceptional domain $\Omega$ in $\mathbb{R}^4$ that contains its own axis of symmetry and whose boundary is obtained by rotating the (two-dimensional) graph of an even real-analytic function about the $x$-axis.*

*Remark.* (i) Our proof will rely heavily on two tricks, one exploiting the assumption that $n = 4$, and the other using the assumption that the generating curve is symmetric. However, we strongly suspect a more general non-existence of such examples in $\mathbb{R}^n$ for any $n > 2$. Therefore, we conjecture the following.

**Conjecture 8.2.** *For $n > 2$, there does not exist an axially symmetric, exceptional domain in $\mathbb{R}^n$ that contains its own axis of symmetry.*

(ii) The assumption that the domain contains its axis of symmetry rules out the exteriors of balls and circular (or spherical) cylinders, respectively (which are clearly exceptional domains as was noted in [10]).

*Proof of Theorem 8.1.* Suppose that $\Omega$ is such a domain in $\mathbb{R}^4$. Namely, the boundary $\partial\Omega$ is obtained from rotation of $\gamma := \{(x,y) \in \mathbb{R}^2 : y = g(x)\}$, with $g(-x) = g(x)$. i.e., the boundary of $\Omega$ is given by

$$\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : \sqrt{x_2^2 + x_3^2 + x_4^2} = g(x_1)\}.$$

Considering the boundary data, the rotational symmetry of the domain will be passed to the roof function, so that, abusing notation, we can write

$$u(x_1, x_2, x_3, x_4) = u(x, y).$$

For clarity, we emphasize that the $x$-axis corresponds to the axis of symmetry and the $y$-coordinate gives the distance from the axis of symmetry.

For axially symmetric potentials $v$ in $\mathbb{R}^n$ the cylindrical reduction of Laplace's equation is:

$$\Delta_{(x,y)} v + \frac{(n-2)v_y}{y} = 0,$$

where $x = x_1$ and $y = \sqrt{x_2^2 + ... + x_n^2}$. Moreover, in the case we are considering, when $n = 4$, $u$ satisfies the equation $\Delta u + \frac{2u_y}{y} = 0$, if and only if $yu(x,y)$ is a harmonic function of two variables $x$ and $y$. Indeed,

$$\Delta(yu) = y\Delta u + 2\nabla u \cdot \nabla y + u\Delta y = y\Delta u + 2u_y.$$

(The trick that reduces axially symmetric potentials in $\mathbb{R}^4$ to harmonic functions in the meridian plane is well known, cf. [15] and [12].)

Since $yu(x,y)$ is then harmonic in the unbounded two-dimensional domain $D$ bounded by $\gamma$ and its reflection (which we denote by $\bar{\gamma}$) with respect to the $x$-axis, this implies $\frac{\partial}{\partial z}(yu(x,y))$ is analytic in the domain $D$, where as usual $z = x+iy$. The Cauchy data (originally posed in $\mathbb{R}^4$) imply that $u_z = \frac{1}{2}(u_x - iu_y)$ coincides with $\sqrt{-S'(z)}$ on $\gamma$ and $\bar{\gamma}$. This implies that the analytic function

$$W(z) := (yu)_z = \frac{-i}{2}u + yu_z \tag{8.1}$$

coincides with $\frac{z-S(z)}{2i}\sqrt{-S'(z)}$ on $\gamma$ and $\bar{\gamma}$. Since $\frac{z-S(z)}{2i}\sqrt{-S'(z)}$ is analytic, this actually gives a formula for $W(z)$ valid throughout $D$:

$$W(z) = \frac{z-S(z)}{2i}\sqrt{-S'(z)}. \tag{8.2}$$

Let $f(\zeta)$ be the conformal map from the strip $\Sigma := \{|\Im\zeta| < \frac{1}{2}\}$ to $D$ such that $f(0) = 0$ and $\arg\{f'(0)\} = 0$. The two-fold symmetry of $D$ implies that $f(\zeta)$ is an odd function. Indeed, otherwise $h(\zeta) = -f(-\zeta)$ gives another conformal map from the strip $\Sigma$ to $D$. But, $h(0) = -f(0) = 0$ and $h'(0) = f'(0)$ implies $h = f$, by the uniqueness of the conformal map (up to choice of $f(0)$ and argument of $f'(0)$).

The Schwarz functions $S_t$, $S_b$ of the top and bottom edges of the strip $\Sigma$ are $S_t(\zeta) = \zeta - i$, and $S_b(\zeta) = \zeta + i$. In terms of the conformal map $f(\zeta)$, the pull-back to the $\zeta$-plane of the Schwarz functions $S_+$ and $S_-$ of $\gamma$ and $\bar{\gamma}$ (resp.) satisfy (see [4, Ch. 8, Eq. 8.7])

$$S_\pm(f(\zeta)) = f(\zeta \mp i), \text{ and} \tag{8.3}$$

$$S'_\pm(f(\zeta)) = \frac{f'(\zeta \mp i)}{f'(\zeta)}. \tag{8.4}$$

Substituting these into (8.2), we obtain two expressions for the pullback of $W(z)$ to the strip $\Sigma$:

$$\frac{f(\zeta) - f(\zeta \mp i)}{2i}\sqrt{-\frac{f'(\zeta \mp i)}{f'(\zeta)}} \tag{8.5}$$

Even though $W(f(\zeta))$ is analytic throughout $\Sigma$, we caution that these two expressions (one expression for "+" and one for "−") may only be valid near the bottom and top sides (respectively) of the strip $\Sigma$.

**Claim:** The function $W(f(\zeta))$ is odd.

Before proving the Claim, let us see how it is used to finish the proof of the Theorem. The fact that $W(f(\zeta))$ is odd implies $W(0) = W(f(0)) = 0$. By (8.1) we then have that $\frac{-i}{2}u + yu_z$ vanishes at $z = 0$, which implies that $u(0,0) = 0$. This contradicts the positivity of $u$.

*Proof of Claim.* We wish to show that $V(\zeta) = W(f(\zeta)) + W(f(-\zeta))$ vanishes identically. Use each of the expressions in (8.5) above to represent $W(f(\zeta))$ and $W(f(-\zeta))$, respectively.

$$V(\zeta) = \frac{f(\zeta) - f(\zeta - i)}{2i}\sqrt{-\frac{f'(\zeta - i)}{f'(\zeta)}} + \frac{f(-\zeta) - f(-\zeta + i)}{2i}\sqrt{-\frac{f'(-\zeta + i)}{f'(-\zeta)}} \tag{8.6}$$

We show that this formula vanishes where it is valid, which then implies that $V(\zeta)$ vanishes identically throughout $\Sigma$. For this, we use the fact that $f$ is odd and consequently $f'$ is even.

$$V(\zeta) = \frac{f(\zeta) - f(\zeta - i)}{2i}\sqrt{-\frac{f'(\zeta - i)}{f'(\zeta)}} + \frac{-f(\zeta) + f(\zeta - i)}{2i}\sqrt{-\frac{f'(\zeta - i)}{f'(\zeta)}} = 0. \tag{8.7}$$

This establishes the Claim. □

□

## 9. Concluding Remarks and Some Generalizing Conjectures

The problem investigated in this work seems to have a remarkably large number of nontrivial connections with a variety of distinct open problems from different areas of mathematics. Moreover, the exact solutions discussed above appear as elementary realizations of a much more general result, as we describe in these remarks. In order to elaborate on these generalizations, let us begin by noting that (leaving aside the obvious generic solution for the exterior of the unit ball), all the solutions found in the two-dimensional case (sections 5 and 6) can be expressed by first writing down the general solution to the Dirichlet problem, and then imposing the Neumann condition to functions in that class. Let $h(\zeta)$ be a conformal map from the disk to $\Omega$ so that $h^{-1}(\infty) = S_\mu \subset \mathbb{T}$ is the pre-image of the point at infinity in $\Omega$. (The reason for this notation will become clear when we realize $S_\mu$ as the support of a measure $\mu$.) Adding a complex constant if necessary, we may assume that $f(h(0)) > 0$ is real. By the Herglotz Theorem (see [11, Ch. 3], [6, Ch. 1]), we can represent $f(h(\zeta))$, which (being analytic in $\mathbb{D}$ with positive real part) is in Carathéodory class, as

$$f(h(\zeta)) = \int_{\mathbb{T}} \frac{e^{i\theta} + \zeta}{e^{i\theta} - \zeta} d\mu(\theta), \tag{9.1}$$

for some finite positive measure $\mu$ on $\partial\mathbb{D}, \mu \perp d\ell$, so that $\mu > 0$ on $S_\mu$ and zero elsewhere on $\mathbb{T}$. The solutions presented in sections 5 and 6 correspond, in turn, to the measure $\mu$ having either a single atom of mass 1, or two atoms of equal masses $1/2$, placed at diametrically opposite points on $\mathbb{T}$.

Towards the end of the article [10], the authors seek to construct more general solutions of this type, i.e. corresponding to discrete measures $\mu$, supported at $n$ points on the unit circle, for $n > 2$. Not finding any example of this type, they consequently pose the question whether such solutions can exist at all.

As we prove in the following, it is indeed impossible to have a solution of this type with $\#(S_\mu) = n > 2$. Moreover we conjecture that it is impossible to find such solutions for any singular, positive measure $\mu$ on $\mathbb{T}$, if the cardinality of its support (defined here as the set where $\mu$ is strictly non-zero) is larger than 2 (including the case when $\mu$ is supported on a continuum). We then discuss how this result relates to representations of Kleinian groups in the hyperbolic halfplane and to functional determinants for the Laplace-Beltrami operator and trace-class Toeplitz operators.

9.1. **The case of discrete measures supported at finitely many points.** In view of the argument of the preceding section, we consider only the case of solutions given by the Poisson kernel with measures supported at finitely many points. (This generalizes the case described by Corollary 4.2 by removing the assumption that the tangent vector has a limit when approaching the point at infinity, and retaining only the condition that the number of boundary components is finite.)

**Theorem 9.1.** *Suppose infinity is a non-isolated boundary point of a simply connected exceptional domain* $\Omega$*, then the analytic completion* $f$ *of the function* $u$ *is a Schwarz integral,*

$$f(h(z)) = \int_{\mathbb{T}} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta),$$

*where* $h(z)$ *is the conformal map from the unit disk* $\mathbb{D}$ *to* $\Omega$*, and* $\mu$ *is a finite positive measure on* $\mathbb{T}$*, singular with respect to the arc-length measure. Suppose card*$(\{z \in \mathbb{T} : \mu(z) > 0\}) = n$ *is finite. Then* $n = 1$ *or* $2$*.*

*Proof.* As before, let $h(z)$ be a conformal map from the disk to $\Omega$ so that $h^{-1}(\infty) = S_\mu$, where $S_\mu \equiv \{z \in \mathbb{T}, \mu(z) > 0\}$ has zero Lebesgue measure. By the Herglotz Theorem (see [11, Ch. 3], [6, Ch. 1]), we can represent $f(h(z))$, which is in Carathéodory class, as

$$f(h(z)) = \int_{\mathbb{T}} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta),$$

and $\mu$ is a finite positive measure with support on $\partial\mathbb{D}, \mu \perp d\ell, \mu(\mathbb{T}) = 1$.

Let now $n = \text{card}(S_\mu) \in \mathbb{N}$, so that $S_\mu$ consists of a finite number of isolated points, $S_\mu = \{z_1, z_2, \ldots, z_n\}$. The arcs defined by consecutive endpoints $\{z_k, z_{k+1}\}$, mod $(n)$, have as images under $h(z)$ the corresponding boundary arcs $\gamma_k$, $\cup_{k=1}^n \gamma_k = \partial\Omega$. Noting that on each arc $\gamma_k \in \partial\Omega$, $f(\zeta) = i\ell(\zeta)$, where $\ell(\zeta)$ denotes arc-length, it follows that we can define an isometry taking every arc $\gamma_k$ into $\gamma_{k+1}$, mod $(n)$, since each $\gamma_k$ is isometric to $\mathbb{R}$. The pull-back of this transformation is an orientation-preserving diffeomorphism of $\mathbb{T}$, which we denote by $\psi(z)$, with the obvious property $\psi^n(z) = z$, for all $z \in \mathbb{T}$. Thus, it is simply a global rotation by $2\pi/n$.

Therefore, $\arg(z_k) = \frac{2\pi}{n} + \arg(z_{k-1}), k = 1, 2, \ldots, n$. The diffeomorphism $\psi(z) = e^{\frac{2\pi i}{n}} z$ takes $S_\mu$ into itself, which means that $f(h(z))$ is invariant under composition with $\psi$, so $f(h(z))$ has a regular expansion in powers of $z^n$, $f(h(z)) = F(z^n)$, for some function $F$ analytic in the unit disk. We also have that $[f'(\zeta)]^2 = -S'(\zeta)$ on $\partial\Omega$, and by the invariance under isometry of the Schwarz function, $[f'(h(z))]^2$ must also be invariant under composition with $\psi$. But then $[f'(h(z))]^2 = [nz^{n-1}F'(z^n)]^2$ has a regular expansion as a function of $z^n$, so that there is some integer $k \geq 0$ such that

$$2(n-1) = kn \Rightarrow n = 1, k = 0 \quad \text{or} \quad n = 2, k = 1.$$

It follows that $S_\mu = \{z \in \mathbb{T} | \mu(z) \neq 0\}$ can consist of either one or two atoms. In the case $n = 2$, invariance under rotation by $\pi$ implies that they have an argument difference equal to $\pi$, and the theorem is proved. □

*Remark.* In fact, a stronger result seems to hold, namely the conclusion of the previous theorem remains valid even in the case of the measure $\mu$ having support of arbitrary cardinality.

While our investigations convincingly point towards this conclusion, the methods employed in a tentative proof are so different from the rest of this paper, and the areas where the result is relevant are so diverse, that we must content ourselves with merely formulating the result as a conjecture here; a detailed analysis will be provided in a future publication. Only some general comments regarding these aspects are presented in the next section.

Theorems 3.1, 5.1, and 6.1 make certain a priori assumptions on the boundary of an exceptional domain. However, we believe that these hypotheses are redundant and conjecture (generalizing a similar claim first made in [10]) the following.

**Conjecture 9.2.** *The only exceptional domains in $\mathbb{R}^2$ are the exterior of the unit disk, the halfplane, and the domain described in Theorem 6.1.*

9.2. **Further connections and possible generalizations.** As discussed in the Introduction, the problem considered here is nontrivially different from all the "classical" variations: one must find the pair $(\Omega, u)$, not just one of elements, so it cannot be formulated as a Caldèron problem; the domain $\Omega$ is unbounded, so there is no variational "Dirichlet principle" in the sense of Riemann and Weierstrass; and the dynamics formulation through the Ricci flow leads to a nonlinear heat equation reducible to the Liouville equation, for which no general solution is known.

We offer here two approaches currently used to prove Conjecture 9.2 rigorously:

- Given an exceptional domain $\Omega$, we can search among the functions in the Carathéodory class the one with minimum energy on the unit circle. But since $f$ is analytic in $\mathbb{D}$ and its real part vanishes a.e. on $\mathbb{T}$, the only non-zero contribution to the energy comes from the double-layer potential (*P.V.* denotes principal value)
$$W[\mu] = P.V. \oint\oint \frac{1}{\sin^2(\frac{\theta-\phi}{2})} d\mu(\theta) d\mu(\phi),$$
subject to the linear constraint $\oint d\mu(\theta) = 1$. In other words, we are searching for the extremum of a quadratic form whose kernel is the positive-definite Toeplitz kernel $K(\theta, \phi) = \sin^{-2}(\frac{\theta-\phi}{2})$, with a uniform linear constraint (thus, extremum can only be a minimum). An elementary argument shows that the minimizing measure is equidistributed on the circle. Thus, the minimizing measure is either discrete uniform with $n$ atoms of equal masses $1/n$, or the absolutely continuous uniform distribution on $\partial\mathbb{D}$. The latter contradicts the assumption $\mu \perp d\ell$, so $\text{card}(S_\mu) \in \mathbb{N}$ and Theorem 9.1 applies.
- The connection to the Schwarz function in Section 4 reveals that exceptional domains are *arclength null-quadrature domains*. That is, for any function $f$, say analytic in $\Omega$, continous in $\overline{\Omega}$, integrable over the boundary, and vanishing at infinity, we have $\int_{\partial\Omega} f ds = 0$. Indeed, $\int_{\partial\Omega} f ds = \int_{\partial\Omega} f(z) \frac{1}{T(z)} T(z) ds = \int_{\partial\Omega} f \sqrt{S'(z)} dz$, where $T(z)$ is the complex unit tangent vector (see Section 4), and now the contour in this last integral can be deformed to infinity in order to see that the integral vanishes. Null-quadrature domains have only been previously studied in the case of area measure. They were characterized in the plane by M. Sakai [19]. Our current study can be seen as a counterpart in the setting of null-quadrature domains for arclength (but we emphasize that in Sakai's study, he made no assumptions on the regularity or topology of the domain).
- Finally, a comment regarding the natural generalization of the problem to the case where the constant Neumann data can be different on different boundary components, in the context of symmetry considerations. Let

again $f(h(z))$ be the analytic completion of a solution written in the form of Thm. 9.1, and denote by $\mathcal{G}$ the group of transformations which leaves $\text{supp}(\mu)$ invariant up to a global rotation. It follows that $f$ is an automorphism of the quotient of the group of linear fractional transformations by $\mathcal{G}$, which can be in general a Kleinian group [16]. The limit set (accumulation points of the orbits of the group) can be finite (in which case it can consist of only 0, 1, or 2 points), or infinite. It is known ([2], Thm. 10.3.4.) that the set of homeomorphic solutions for a quasilinear elliptic equation of Laplace-Beltrami type forms a group only in the case of finite limit set [2]. The Kleinian groups are called degenerate in this case, and they correspond to either finite groups (with empty limit set), or the cyclic groups (generated by one element, with limit set consisting of 1 or 2 points). These correspond to the solutions described in the present paper (isolated point at infinity, respectively simple and double boundary point at infinity).

Adopting this approach, our preliminary study of this general case (to be presented in a forthcoming publication) seems to indicate that the exceptional domains and their corresponding harmonic functions (for generalized constant Neumann data conditions) correspond to Kleinian groups generated from fundamental polygons with edges identified via homeomorphisms reflecting the different values of Neumann data.

## References


1. M. Abramowitz and I. A. Stegun, eds., *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, New York: Dover, 1965.
2. K. Astala, T. Iwaniec, and G. Martin, *Elliptic Partial Differential Equations and Quasiconformal Mappings in the Plane*, Princeton Univ. Press, New Jersey, 2009.
3. S. Axler, P. Bourdon, and W. Ramey, *Harmonic Function Theory*, Springer, New York 1992.
4. P. J. Davis, *The Schwarz Function and its Applications*, The Mathematical Association of America, Buffalo, N. Y., 1974. The Carus Mathematical Monographs, No. 17.
5. L. De Castro and D. Khavinson, *Analytic functions in Smirnov classes with real boundary values*, Complex Analysis and Operator Theory, to appear.
6. P. Duren, *Theory of $H^p$ spaces*, Dover Publications, 2000.
7. P. Duren, H. S. Shapiro, and A. L. Shields, *Singular measures and domains not of Smirnov type*, Duke Math. J., 33 (1996), 247-254.
8. P. Ebenfelt, D. Khavinson, and H.S. Shapiro, *A free boundary problem related to single layer potentials*, Ann. Acad. Sci. Fenn., Vol 27 (2002), 22-46.
9. S. D. Fisher, *Function Theory on Planar Domains: A Second Course in Complex Analysis*, Dover Publications, 2007.
10. L. Hauswirth, F. Hélein, and F. Pacard, *A note on some overdetermined elliptic problem*, Pacific J. Math, 250 (2011), 319-334.
11. K. Hoffman, *Banach Spaces of Analytic Functions*, Prentice-Hall, New Jersey, 1968.
12. L. Karp, *Construction of quadrature domains in $\mathbb{R}^n$ from quadrature domains in $\mathbb{R}^2$*, Complex Var. Elliptic Eq., 17 (1992), 179-188.
13. M. Keldysh and M. Lavrentiev, *Sur la représentation conformeé des domaines limités par des courbes rectifiables*, Ann. Sci. École Norm. Sup. 54 (1937), 1-38.
14. O. D. Kellogg, *Foundations of Potential Theory*, Ungar, New York, 1929.
15. D. Khavinson, *On reflection of harmonic functions in surfaces of revolution*, Complex Variables 17 (1991), 7-14.
16. H. Poincaré, *Papers on Fuchsian Functions*, Springer-Verlag, New York, 1985.
17. Ch. Pommerenke, *Boundary Behavior of Conformal Maps*, Springer-Verlag, Berlin, 1992.
18. W. Reichel, *Radial symmetry for elliptic boundary-value problems on exterior domains*, Arch. Rational Mech. Anal. 137 (1997), 381-394.
19. M. Sakai, *Null quadrature domains*, J. Analyse Math. 40 (1981), 144-154.

20. H. Shahgholian, *A characterization of the sphere in terms of single-layer potentials*, Proc. AMS, Vol 115, No. 4 (1992), 1167 - 1168.
21. P. Topping, *Lectures on the Ricci Flow*, C.U.P.. ISBN 0-521-68947-3, 2006.
22. J. Wermer, *Potential Theory*, Lecture Notes in Math 408, Springer-Verlag, 1974.