Nonlinear extremal problems for analytic functions

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Historic Development

Solving extremal problems has been one of the major stimuli for progress in complex analysis.

Linear problems in Hardy type spaces

- H. A. Schwarz' lemma

- C. Carathéodory-L. Fejér problem (coefficients of bounded analytic functions)

- E. Landau, S. Kakeya, F. Riesz, J. Doob, H. Milloux, A.Denjoy..

- Use of duality in linear extremal problems (S. Ya. Khavinson, also W. Rogosinski and H. S. Shapiro, late 40s early 50s)

Common Wisdom

Theorem 1. *(Metatheorem) Solutions to "good" extremal problems are MUCH "better" than a generic representative of the class for which the problem is posed.*

TRUE even in Hardy-Orlicz spaces. (V. Terpigoreva, '1960s).

Linear Problems in Bergman Spaces

- V. Ryabych, early 60s

- K. Osipenko and M. Stessin, 1991

- Contractive divisors in Bergman spaces (H. Hedenmalm, then P. Duren, D. Khavinson, H. S. Shapiro, C. Sundberg), 90s

- D. Khavinson and M. Stessin, D. Vukotic, General linear extremal problems in Bergman spaces, 1995-96.

In this talk, I will focus on nonlinear problems, specifically, on problems for non-vanishing functions. This talk is based on a recent survey paper with C. Beneteau.

Hardy Spaces

I. General Theory

Definition 1. *For* $p > 0$,

$$H_0^p := \{f : f \text{ is analytic and non-vanishing in } \mathbb{D}, \quad \|f\|_p^p := \sup_{0<r\leq 1} \frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^p dt \leq 1\}.$$

General ("good") extremal problem: Given $\tau_0, \tau_1, \ldots, \tau_m \in \mathbb{C}$, find

$$\sup_{f\in H_0^p} \{Re \sum_{k=0}^{m} \tau_k \frac{f^{(k)}(0)}{k!}\} \qquad (1)$$

and identify the corresponding extremal function(s).

Convex analysis approach (S. Ya. Khavinson - V. M. Terpigoreva, 1960s).
(Some of the results (not the methods) were re-discovered in '70s-'80s by J. Hummel, S. Scheinberg, T. Suffridge, L. Zalcman).

Define the set

$$A_m(H_p^0) \;:=\; \{\vec{c} = < c_0, c_1, \ldots, c_m > \in \mathbb{C}^{m+1} : \\ f(z) = \sum_{k=0}^{m} c_k z^k + \ldots, f \in H_p^0\}.$$

Notice that if

$$f^*(z) = \sum_{k=0}^{m} c_k^* z^k + \ldots$$

is a solution to a problem of type (1) then

$$\vec{c^*} = < c_0^*, c_1^*, \ldots, c_m^* >$$

is a boundary point of $A_m(H_p^0)$.

Obstacle: $\overline{A_m}(H_p^0) = A_m(H_p^0) \cup \{0\}$ is *not* a convex set!

Natural step: Write $f(z) = \exp(q(z))$ and let Q_p^* be the class of logarithms $q(z)$ of functions in H_p^0.

$A_m(Q_p^*)$, the set of the first $m+1$ coefficients of all elements of Q_p^*, **IS** a convex set.

$$\overline{A_m}(H_p^0) \simeq A_m(Q_p^*)$$

Finite boundary points of $A_m(Q_p^*)$ correspond to non-zero boundary points of $A_m(H_p^0)$.

Goal: Study the boundary points of $A_m(Q_p^*)$.

Let

$$\vec{a}^* = < a_0^*, a_1^*, \ldots, a_m^* >$$

be a boundary point of $A_m(Q_p^*)$.

There exists a supporting hyperplane passing through that point: that is, there exist constants $d \in \mathbb{R}$ and $\gamma_0, \gamma_1, \ldots, \gamma_m \in \mathbb{C}$ such that

$$Re(\sum_{k=0}^{m} \gamma_k a_k) \leq d$$

for every $\vec{a} \in A_m(Q_p^*)$ and

$$Re(\sum_{k=0}^{m} \gamma_k a_k^*) = d.$$

Problem restated: Find, given $\gamma_0, \gamma_1, \ldots, \gamma_m \in \mathbb{C}$ fixed,

$$\lambda_p^* = \sup\{Re(\sum_{k=0}^{m} \gamma_k a_k) : q(z) = \sum_{k=0}^{\infty} a_k z^k \in Q_p^*\}. \quad (2)$$

Theorem 2. *(S. Ya. Khavinson, 1960). Given a non-zero boundary point*

$$\vec{c}^* =< c_0^*, c_1^* \ldots, c_m^* >\in \overline{A_m}(H_p^0),$$

there exists a unique function $f \in H_0^p$ *such that* $f(z) = \sum_{k=0}^m c_k^* z^k + \ldots$. *This function has the form*

$$f(z) = C \prod_{k=1}^{m} (1 - \bar{\alpha}_k z)^{\frac{2}{p}} \exp(- \sum_{|\alpha_k|=1} \lambda_k \frac{\alpha_k + z}{\alpha_k - z}), \tag{3}$$

where $C > 0$ *is a constant such that*

$$\frac{1}{2\pi} \int_0^{2\pi} |f(e^{it})|^p dt = 1,$$

$|\alpha_k| \leq 1$, *and* $\lambda_k \geq 0$.

Discussion

- The extremal function $f(z)$ has representation

$$exp\left(\frac{1}{2p\pi}\int_{\mathbf{T}}\frac{e^{i\theta}+z}{e^{i\theta}-z}d\mu(\theta)\right),$$

where measure $d\mu = log|f(e^{i\theta})|d\theta+d\nu,\ d\nu\perp d\theta$ and $d\nu \leq 0$.

- μ is extremal in finding for a nonegative trigonometric polynomial P of degree m the maximum of

$$\int_{\mathbb{T}} P\, d\mu,$$

over a convex subset of measures μ with non-positive singular part and à priori bound on the Hardy norm $\|exp\left(\frac{1}{2\pi}\int_{\mathbf{T}}\frac{e^{i\theta}+z}{e^{i\theta}-z}d\mu(\theta)\right)\|_{H^2}$.

Conclusions

- It is not difficult to see that the maximum is determined by the absolutely continuous part of the measure μ and is attained for $\log|f| = C \log|P|$.

- Finally, not to violate the maximum of

$$\int_{\mathbb{T}} P \, d\mu,$$

the singular part ν of the measure μ must be supported exclusively by the zeros of P on $\mathbf{T}$.

II. Some Examples

Problem. *Find*

$$\sup_{f \in H^\infty} \{Re \frac{f^{(m)}(0)}{m!} : \|f\|_\infty \leq 1,\ f \text{ non-vanishing in } \mathbb{D}\}.$$

What's known from the previous theorem: $f^*(z)$ is a singular function with at most m point masses on the circle.

Conjecture 1. *(Krzyż)*

$$\sup_{f \in H^\infty} \{Re \frac{f^{(m)}(0)}{m!} : \|f\|_\infty \leq 1,\ f \neq 0 \text{ in } \mathbb{D}\} = \frac{2}{e}.$$

(Open for $m > 5$!) C. Horowitz (1978) showed that $|\frac{f^{(m)}(0)}{m!}| < 1$ for all m!

Problem. *Find, for a positive integer* m,

$$\sup_{f \in H_0^p} \{Re \frac{f^{(m)}(0)}{m!}\}. \tag{4}$$

What's known from the previous theorem:

$$f^*(z) = (p_m(z))^{2/p} S(z),$$

where p_m is a polynomial of degree m and $S(z)$ is a singular function with atomic masses at at most m points on the circle (which can only occur where the polynomial has roots on the circle).

Hummel-Scheinberg-Zalcman conjecture.

Conjecture 2. *(Hummel-Scheinberg-Zalcman)*

$$\sup_{f\in H_0^p}\{Re\frac{f^{(m)}(0)}{m!}\} = \left(\frac{2}{e}\right)^{1/q},$$

where q is conjugate to p.

(Open for $p > 1$ and $m \geq 3$!)

(Various results obtained by J. Brown (1985), T. Suffridge (1989), C. Beneteau and B. Korenblum (2001), ...).

In 1990 K. Samotij showed à la Horowitz that for each $p > 1$ the supremum in H-S-Z conjecture is strictly < 1.

Bergman Spaces

I. Known Results

Definition 2. *For* $0 < p < \infty$, *let*

$$A^p = \{f \text{ analytic in } \mathbb{D} : \left(\int_{\mathbb{D}} |f(z)|^p dA(z)\right)^{\frac{1}{p}} =: \|f\|_{A^p} < \infty\},$$

where dA *denotes normalized area measure in the unit disk* $\mathbb{D}$.

$A_0^p :=$ set of non-vanishing A^p functions.

Model Problem:

Find

$$\inf\{\int_{\mathbb{D}} |f|^p dA : f \in A_0^p : f^{(j)}(0) = c_j, 0 \leq j \leq m\}, \tag{5}$$

where the c_j are given non-zero complex numbers.

Remarks:

- Without loss of generality, $p = 2$.
- Solution always exists and is unique.
- Rewrite $f = e^q$, and solve

$$\inf\{\|e^q\|_{A^2} : q \text{ is analytic in } \mathbb{D}, \\ q^{(j)}(0) = a_j, 0 \leq j \leq m\}.$$

The Solution is...

Theorem 3. *(D. Aharonov, C. Beneteau, D. K., H. Shapiro, 2005) The extremal function f^* is in H^∞. If the singular measure in the representation of f^* is supported on a Carleson set, then the outer factor of f is a polynomial of degree m.*

Conjectured form of the extremal:

$$f^*(z) = C \prod_{j=1}^{m} (1 - \bar{\alpha}_j z) \exp\Big(\sum_{j=1}^{k} -\lambda_j \frac{e^{i\theta_j} + z}{e^{i\theta_j} - z} \Big), \tag{6}$$

$|\alpha_j| \leq 1$, $k \leq m$ and $\lambda_j \geq 0$.

A Special Case: Find, for $c > 1$ fixed,

$$\inf\{\|f\|_{A^2} : f \neq 0 \text{ in } \mathbb{D}, f(0) = 1, f'(0) = c\}. \tag{7}$$

Theorem 4. *(ABKS, '05) If the singular part of the extremal function f^* has a single point mass, then*

$$f^*(z) = C(z - 1 - \mu_0)e^{-\mu_0 \frac{1+z}{1-z}}, \tag{8}$$

where the constants C and μ_0 are uniquely determined by c.

Remark: Recall Hardy space extremal

$$f^*(z) = C(z - 1)e^{-\mu_0 \frac{1+z}{1-z}}. \tag{9}$$

It differs in an essential way (which is not at all the case in the linear problems!)

II. Further Conjectures

Where should we expect the atoms of the singular measure to be?

We know that the singular part μ^* of the extremal measure ν^* is non-trivial for many values of the data. The example above shows that it may be possible that the extremal functions in Bergman spaces need not be continuous in the closed disk, so the atoms are not expected to be at the zeros of the polynomial $R \geq 0, R = |\prod_{j=1}^{m}(1 - \bar{\alpha}_j z)|^2$ corresponding to the modulus of the outer part of the extremal function. Where are they?

Discussion

There is another non-negative trigonometric polynomial P that appears if one follows the S. Ya. Khavinson's ideas to construct the logarithmically convex coefficient body for Bergman functions. Essentially, P appears from the equation of the supporting hyperplane at a boundary point of that coefficient body corresponding to the extremal function.

Conjecture 3. *(ABKS) If $P > 0$ on the unit circle $\mathbb{T}$, then the singular part μ^* of the extremal measure ν^* is supported on the set of local minimum points of P on $\mathbb{T}$.*

The characterization of a boundary point for the convex body leads to maximizing the integral

$$\int_{\mathbb{T}} P \, d\mu,$$

over a convex subset of measures μ with non-positive singular part and à priori bound on the Bergman (!) norm

$$\|exp\left(\frac{1}{2\pi}\int_{\mathbf{T}} \frac{e^{i\theta}+z}{e^{i\theta}-z} d\mu(\theta)\right)\|_{A^2}.$$

Intuitively, in order to maximize the integral, we are best off if we concentrate all the negative contributions from the singular factor at the points where P is smallest. Yet, the main difficulty is to sort out the dependence of the Bergman norms of singular inner function on supports of the corresponding singular measures.

A Krzyż type conjecture for the Bergman space.

Consider the problem

$$\max\{Re(f^{(m)}(0)/m!) : f \in A_p^0, \|f\|_{A^p} \le 1\}. \tag{10}$$

Suppose f^* has the conjectured form

$$f^*(z) = C(z-1-\mu)\exp(\mu\frac{1+z}{z-1}), \tag{11}$$

where $\mu > 0$ and $C = C(\mu)$ is a constant such that $\|f^*\|_{A^p} = 1$.

If we denote by F^* the antiderivative of f^* such that $F^*(1) = 0$, then

$$F^*(z) = \frac{C}{2}(z-1)^2 \exp(\mu \frac{1+z}{z-1}).$$

Using the complex form of Green's theorem together with the fact that

$$|\exp(\mu \frac{1+z}{z-1})| = 1 \text{ a.e. on } \mathbb{T},$$

we can calculate that

$$\int_{\mathbb{D}} |f^*(z)|^2 dA = \frac{i}{2\pi} \int_{\mathbb{T}} F^*(z) \overline{f^*(z)} d\overline{z} = \frac{C^2}{2}(3+2\mu).$$

Since $\int_{\mathbb{D}} |f^*(z)|^2 dA = 1$, we get that $C = \sqrt{\frac{2}{3+2\mu}}$, where $\mu > 0$. Substituting C, we obtain

$$(f^*)'(0) = \sqrt{\frac{2}{3+2\mu}}\, e^{-\mu}(1 + 2\mu + 2\mu^2). \quad (12)$$

It is not hard to see that this function of μ, when $\mu > 0$, is maximized when $\mu = 1$. We thus obtain

Conjecture 4.

$$\max\{Re(f'(0)) : \|f\|_{A^2} \leq 1, f \neq 0\} = \sqrt{2}\frac{\sqrt{5}}{e}. \quad (13)$$

A wild conjecture. It is also natural then to expect that an extremal function for any m in the problem

$$\max\{Re(f^{(m)}(0)) : \|f\|_{A^2} \leq 1, f \neq 0\} \qquad (14)$$

would be $c_m f^*(z^m)$, where f^* is the extremal for the first derivative, and c_m is the normalizing constant.

Conjecture 5.

$$\max\{Re(f^{(m)}(0)/m!) : \|f\|_{A^2} \leq 1, f \neq 0\} \asymp$$
$$\asymp \sqrt{m+1}\frac{\sqrt{5}}{e}.$$

Let $\Lambda_m = \Lambda_{m,p}$ for $1 < p < \infty$ be defined by

$$\Lambda_m = \max\{Re(f^{(m)}(0)/m!) : \|f\|_{A^p} \leq 1, f \neq 0 \text{ in } \mathbb{D}\}. \tag{15}$$

Conjecture 6.

$$\limsup_{m\to\infty} \frac{\Lambda_m}{(\frac{mp+2}{2})^{\frac{1}{p}}} < 1.$$

Denote by λ_m the analog of Λ_m in the H_0^p-context. A priori, of course, $\Lambda_m \geq \lambda_m$.

Question. What are the asymptotics of Λ_m? Is $\frac{\Lambda_m}{\lambda_m} \asymp m^{1/p}$?

We think that perhaps with the advances in the theory of Bergman spaces in the last decade, the time has come for a thorough study of these fundamental extremal problems.

THANK YOU!