GRAVITATIONAL LENSING BY A COLLECTION OF OBJECTS WITH RADIAL DENSITIES

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Abstract. In a recent paper [1] (2006), the authors considered a certain class of gravitational lenses consisting of n non-overlapping objects with radial densities. They concluded that there can be at most $6(n-1)+1$ lensed images of a single light source. We show that, without additional assumptions, this claim is false.

1. Introduction

Beginning with the first observation of a multiple-image lens in 1979 [8], the field of gravitational lensing has grown rapidly and is now perhaps the most important means of indirect measurement in astrophysics. A problem of interest on the theoretical side is to determine the number of images that can be produced by a given class of gravitational lenses. The survey [4] gives an overview of the progress on this problem. Some recent efforts have used complex analysis to study models for which the so-called "thin-lens approximation" is valid. Briefly, if the mass does not deviate too far from residing in a common plane (the *lens plane*) orthogonal to our line of sight, then lensed images of a background source are modeled by complex-valued solutions z of the following *lens equation*, where μ is the mass density projected to the lens plane and w is the position of the source projected onto the *lens plane* (see [2] for a more detailed description).

$$z = \int_\Omega \frac{\mu(\zeta) dA(\zeta)}{\bar{\zeta} - \bar{z}} + w. \tag{1.1}$$

(Here, $dA(\zeta)$ denotes the area element.) Consider for instance, the case when $\mu = \phi(r)$ is radial and, say, supported on a disk of radius R centered at the origin. Then it was shown in [2] that the integral appearing in Eq. (1.1) equals $M/\bar{z}$ for $r > R$, where M is the total mass, and for $r < R$ it equals

$$\frac{2\pi}{\bar{z}} \int_0^{|z|} \phi(r) r dr. \tag{1.2}$$

In this note, we consider a class of examples consisting of n non-overlapping objects so that each has radial symmetry (about its own center). Outside the region occupied by the n objects, the lensing effect is the same as that of n point masses (cf. [3, Cor. 3]).

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In [1], the authors suggested to study the case of n non-overlapping radially symmetric objects by starting with a configuration of point masses and replacing consecutively each point mass with an object of equal mass having radial projected mass density. If each object does not overlap with other objects and also does not overlap pre-existing images, then they argued that *"...the maximum number of images for a system of n distributed and non-overlapping lenses is $(5n-5)+n = 6(n-1)+1$."* The only assumption they made on the projected mass density of each object is that it was radial and did not diverge faster than $1/r$, where r is the distance to the center of the object.

In the next section we give an example that violates this claim. In Section 3 we consider a more restricted class of examples.

2. A Counterexample

Example 1: Using complex numbers to denote position in the lens plane, we first consider a lens consisting of point masses. Place point masses at $\pm i$ and ± 1 each with total mass 1, and place a point mass at the origin with mass .05. For this lens, the integral appearing in Eq. (1.1) becomes a finite sum. Thus, the lensing equation becomes

$$z = \frac{1}{\bar{z}-1} + \frac{1}{\bar{z}+1} + \frac{1}{\bar{z}+i} + \frac{1}{\bar{z}-i} + \frac{.05}{\bar{z}} + w, \tag{2.1}$$

or, simplifying,

$$z = \frac{4\bar{z}^3}{\bar{z}^4-1} + \frac{.05}{\bar{z}} + w. \tag{2.2}$$

This is based on an example of S. Rhie. With a source positioned at the origin $w = 0$, this lens equation has 20 solutions (see [5]).

Now replace the four point masses at $\pm i$ and ± 1 with radial objects of radius .2 with uniform projected density and the same total mass 1. Replace the central point mass with an object of the same mass of radius .2 with density that projects to the lens plane as proportional to r^2 (other quadratic densities lead to the same number of images) so that the total mass is the same. According to [1] this configuration produces at most $6(5-1)+1 = 25$ images.

Outside the mass support, each of these objects acts as a point mass, together producing the same 20 images. Some additional images appear inside the mass support. In [1] it was claimed that only one additional image can appear inside the support of each object. This is indeed the case for each of the objects centered at ± 1 and $\pm i$, but several images appear inside the support of the central object, where the lensing equation becomes

$$z = \frac{4\bar{z}^3}{\bar{z}^4-1} + \frac{(.05)z^2\bar{z}}{(.2)^4}. \tag{2.3}$$

Considering just the real solutions of Eq. (2.3), $x = 0$ is a solution, and it is easy to check analytically that there is another solution between $x = 0$ and $x = 0.2$. Real solutions of Eq. (2.3) are zeros of $g(x) = x - \left(\frac{4x^3}{x^4-1} + .05\frac{x^3}{.2^4}\right)$. Notice that $g(.2)$ is negative and $g'(0) = 1$ is positive. So $g(x)$ must have a zero between $x = 0$ and $x = 0.2$. Obviously, by symmetry, there is yet another real solution between -0.2 and 0. This shows that there are at least 3 solutions inside the support of the central object. Checking for solutions numerically, one finds that there are actually 9 (see Fig. 1), so that the total number of images lensed by this configuration is $20 + 4 + 9 = 33$.

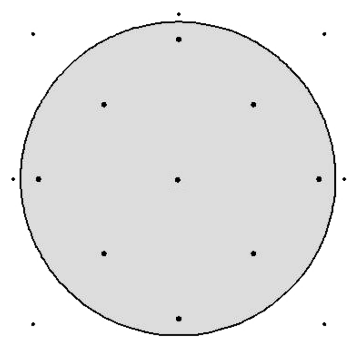

FIGURE 1. 9 images appear inside the support of the central object. The approximate positions of these 9 images are $(\pm 0.12, \pm 0.12)$, $(0, \pm 0.19)$, $(\pm 0.19, 0)$, and $(0, 0)$.

We conclude that "radial and not diverging faster than $1/r$" is too general a class of densities to consider. Therefore, we restrict our attention to the case when the projected mass density of each object is uniform, and we show that the bound $6(n-1)+1$ holds but only if the radii are sufficiently small.

3. Gravitational lensing by n uniform disks

Let us define the *lensing map*, $z \to z - \int_\Omega \frac{\mu(\zeta)dA(\zeta)}{\zeta - \bar{z}}$, so that solutions of Eq. 1.1 are preimages of w under the lensing map. For the density considered in the following proposition, the lensing map is harmonic inside and outside the support, and the solutions are isolated, so that the generalized *argument principle* applies (see [7]).

Proposition 3.1. *Suppose μ is supported on n non-overlapping disks and is constant on each. If the radii are decreased while adjusting μ so that on each disk the total mass stays the same, then for sufficiently small radii, the total number of solutions to Eq. (1.1) is bounded by* $6(n-1)+1$ (cf. [3, Cor. 3]).

Proof. Outside the mass support, the number of images is bounded by $5(n-1)$. Inside the mass support of one of the objects, the lensing equation has the form $z = \sum_{i=1}^{n-1} \frac{m_i}{\bar{z}-\bar{z}_i} + \frac{m_n}{R^2}(z - z_n)$. The Jacobian of the lensing map is $|\frac{m_n}{R^2} - 1|^2 - |\sum_{i=1}^{n-1} \frac{m_i}{\bar{z}-\bar{z}_i}|^2$, which for small enough R is strictly positive over the disk of radius R centered at z_n. Thus, the lensing map is orientation-preserving in this region and the number of zeros is given by the degree (winding number) of the map restricted to the boundary. But the lensing map on the boundary coincides with the lensing map of the exterior of the total mass support. If R is small enough that the exterior map does not have any zeros inside the disk of radius R, then we claim that the winding number on the boundary is 1. Indeed, the exterior map $z - \sum_{i=1}^{n-1} \frac{m_i}{\bar{z}-\bar{z}_i}$ extends inside the disk of radius R centered at z_n where it has one orientation-reversing pole and no zeros. Hence, the increment of the argument of $z - \sum_{i=1}^{n-1} \frac{m_i}{\bar{z}-\bar{z}_i} - \frac{m_n}{R^2}(z - z_n)$ along the boundary of the disk centered at z_n is 2π. Therefore, it has precisely one zero there. □

By starting with one of S. Rhie's extremal examples and replacing each point mass with a uniform disk of small radius, one can obtain examples showing that the bound $6(n-1)+1$ is sharp.

Referring to Proposition 3.1, one might ask if the condition that the disks do not overlap each other already ensures that the radii are sufficiently small. In the next example we show that this is not the case by describing a configuration of $n = 5$ non-overlapping uniform disks lensing 27 images ($6(n-1)+1 = 25$ for $n = 5$).

Example 2: Place objects at the points $\pm i$ each with total mass 1 supported in a circle of radius .5, and place objects at ± 1 each with total mass 1.13 and radius .5. Then place an object with total mass .05 and radius .245 at the origin. For each of these objects we choose a projected mass density that is uniform. The ensemble is depicted in Fig.s 2 and 3 along with the resulting 27 lensed images of a background source located at the origin.

Again, it is necessary to separate the domain for the lensing equation into 6 regions: one inside the support of each object along with the region exterior to the total mass support. In the exterior of the support, the objects act as point masses and the lensing equation is

$$z_s = z - \left(\frac{1.13}{\bar{z}-1} + \frac{1.13}{\bar{z}+1} + \frac{1}{\bar{z}+i} + \frac{1}{\bar{z}-i} + \frac{.05}{\bar{z}} \right),$$

where z_s is the position of the source projected onto the lensing plane, and solutions z give lensed images. We choose $z_s = 0$. Then in the region governed by this lensing equation, there are 16 solutions. The approximate positions of these 16 images are

$$(\pm 2.115, 0), (0, \pm 2.141), (\pm 1.329, \pm 1.517), (\pm .144, \pm .226), (\pm .264, \pm .318)$$

The region inside the support of an object is governed by a different lensing equation. Since we assumed uniform projected mass density, the lensing equation can be obtained from the one above by replacing the relevant term $\frac{m}{\bar{z}-\bar{z}_0}$ by $\frac{m}{R^2}(z - z_0)$, where m, R, and z_0 are respectively the mass, radius, and central position of the object (see [1] and [2, Sec 7]). Thus, inside the support of the object centered at the origin, the lensing equation is

$$0 = z - \left(\frac{1.13}{\bar{z}-1} + \frac{1.13}{\bar{z}+1} + \frac{1}{\bar{z}+i} + \frac{1}{\bar{z}-i} + \frac{.05}{.245^2} z \right).$$

This equation has 7 solutions all located within a radius of .245 (see figure 3). The approximate positions of these 7 images are

$$(\pm .066, \pm .187), (0, \pm .148), (0, 0)$$

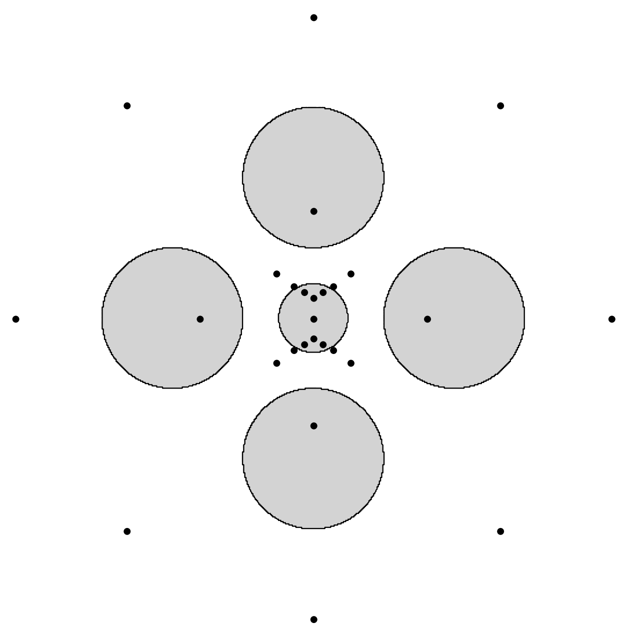

FIGURE 2. 16 images appear outside the support of the mass, 7 appear inside the support of the central object, and one image appears inside the support of each of the other four objects. The approximate positions of the images are listed in the text.

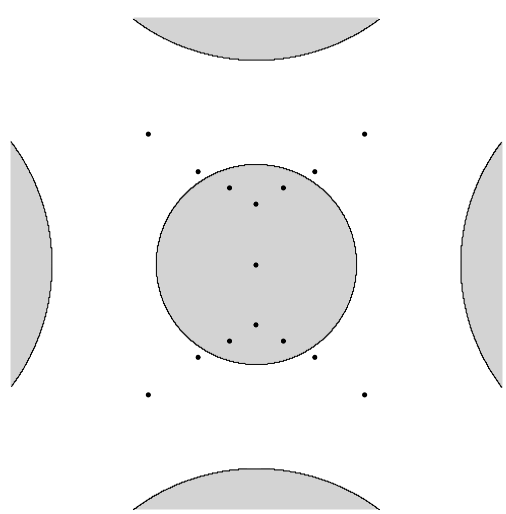

FIGURE 3. Detail for Fig. 2: 7 images appear inside the support of the central object.

The lensing equations for the interiors of the other four objects each give one additional image, for a total of $16 + 7 + 4 = 27$.

This example leaves open the following question.

Question: Referring to the statement of Proposition 3.1, suppose the disks do not overlap each other or any of the images produced by the configuration of point masses positioned at the center of each disk and having the same mass. Does this condition ensure that the radii are sufficiently small so that the bound $6(n-1)+1$ holds?

For the moment, we do not see any reason why the answer to this question should be "yes".

Concluding Remarks: The argument in [1] consists of two steps: (i) outside the support of mass the known result for point masses [3] bounds the number of images appearing by $5(n-1)$ and (ii) inside the mass support of each object the authors of [1] approximated the lensing equation by its first order expansion about the object's center. Even restricting to uniform densities as we did in Section 3, there is no reason to expect, as claimed in [1], that the number of images throughout the entire support of the object's mass will coincide with the number of solutions to the linear approximation. The proof of Proposition 3.1 instead focuses on the orientation of the lensing map. This seems to be the true crux of executing a correct version of the argument, and the Example in Section 3 indicates that the lensing map can change orientation within the support of a single object. In summary, the linear approximation does not yield a rigorous bound on the number of images, and the general problem of counting all images (both "bright" and "dim") produced by multiple radial objects is richer than previously suggested.

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