A REMARK ON "GRAVITATIONAL LENSING AND THE MAXIMUM NUMBER OF IMAGES": 5 RADIAL OBJECTS CAN LENS 27 IMAGES

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Abstract. In a recent paper [1] (2006), authors considered a class of gravitational lenses consisting of n non-overlapping objects. If each of the objects has radial symmetry (about its own center), and the projected mass density does not diverge at the center of any object faster than $1/r$, where r is the distance to the center of the object, then they claimed there can be at most $6(n-1)+1$ images. We demonstrate a simple example of 5 non-overlapping, radially-symmetric objects that can create $27 = 6(5-1)+1+2$ images of a background source. Our example draws on a construction due to S. Rhie [4].

In this note, we describe a mathematical model of a gravitational lens which violates the claim made in [1]. Throughout, we assume the thin-lens approximation, and we follow [1] (cf. [5]) in using the complex-variable formulation of the lensing theory.

Using complex numbers to denote position in the Lensing Plane, we place objects at the points $\pm i$ each with total mass 1 supported in a circle of radius .5, and we place objects at ± 1 each with total mass 1.13 and radius .5. Then we place an object with total mass .05 and radius .245 at the origin. For each of these objects we choose a projected mass density that is uniform. The ensemble is depicted in figures 1 and 2 along with the resulting 27 lensed images of a background source located at the origin.

It is necessary to separate the domain for the lensing equation into 6 regions: one inside the support of each object along with the region exterior to the total mass support. In the exterior of the support, the objects act as point masses and the lensing equation is

$$z_s = z - \left(\frac{1.13}{\bar{z}-1} + \frac{1.13}{\bar{z}+1} + \frac{1}{\bar{z}+i} + \frac{1}{\bar{z}-i} + \frac{.05}{\bar{z}} \right),$$

where z_s is the position of the source projected onto the lensing plane, and solutions z give lensed images. We choose $z_s = 0$. Then in the region governed by this lensing equation, there are 16 solutions. The approximate positions of these 16 images are

$$(\pm 2.115, 0), (0, \pm 2.141), (\pm 1.329, \pm 1.517), (\pm .144, \pm .226), (\pm .264, \pm .318)$$

The region inside the support of an object is governed by a different lensing equation. Since we assumed uniform projected mass density, the lensing equation can be obtained from the one above by replacing the relevant term from $\frac{m}{\bar{z}-\bar{z_0}}$ to $\frac{m}{R^2}(z-z_0)$ where m, R, and z_0 are respectively the mass, radius, and central position of the object (see [1] and

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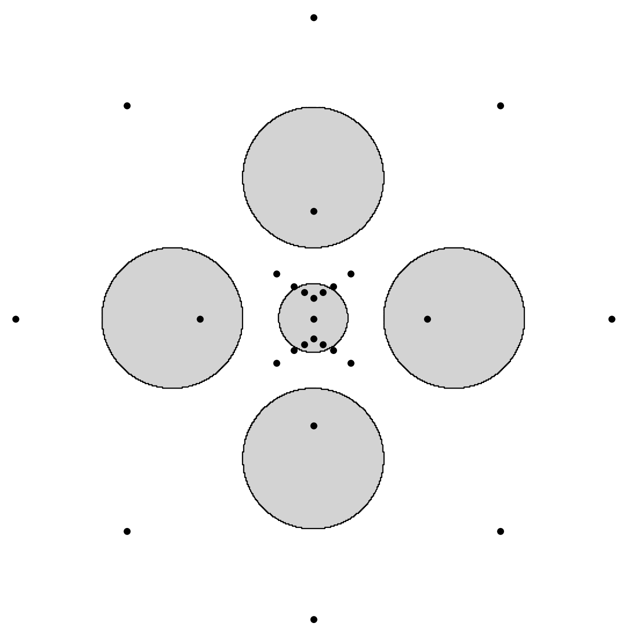

FIGURE 1. 16 images appear outside the support of the mass, 7 appear inside the support of the central object, and one image appears inside the support of each of the other four objects.

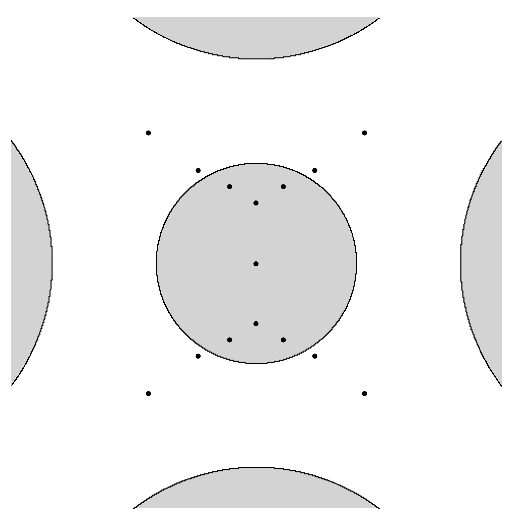

FIGURE 2. Detail for figure 1: 7 images appear inside the support of the central object.

[2, Sec 7]). Thus, inside the support of the object centered at the origin, the lensing equation is

$$0 = z - \left(\frac{1.13}{\bar{z}-1} + \frac{1.13}{\bar{z}+1} + \frac{1}{\bar{z}+i} + \frac{1}{\bar{z}-i} + \frac{.05}{.245^2} z \right),$$

This equation has 7 solutions all located within a radius of .245 (see figure 2). The approximate positions of these 7 images are

$$(\pm.066, \pm.187), (0, \pm.148), (0, 0)$$

The lensing equations for the interiors of the other four objects each give one additional image, for a total of $16 + 7 + 4 = 27$.

Concluding Remarks: The question arises if some small assumption can be added to salvage the result under consideration. The argument in [1] consists of two steps: (i) outside the support of mass the known result for point masses [3] bounds the number of images appearing by $5(n-1)$ and (ii) inside the mass support of each object the authors of [1] approximated the lensing equation by the terms up to first order in its Taylor series expanded about the object's center. There is no fault with Step (i), but there is no reason to expect, as claimed in Step (ii), that the number of images throughout the entire support of the object's mass will coincide with the number of solutions to the linear approximation. There needs to be an assumption that the radius of each of the objects be sufficiently small, where "sufficiently small" will depend on the position and mass density of each object. Before one can begin to quantify such dependence, we would suggest to fix attention on a more specific class of densities than "radial and not diverging faster than $1/r$". It seems necessary to choose a class whose instances are controlled by finitely many parameters. Also, it is natural and expedient to require the density to be monotone decreasing with respect to r. We do not go into detail since the examples in this case are not of much physical interest, but if the mass is allowed to increase with respect to r, one can obtain even simpler counterexamples than the one we have given, and if the mass density is even allowed to oscillate one can produce examples with arbitrarily many images. In conclusion, it appears that the linear approximation does not yield robust bounds on the number of images, and that the general problem of counting images produced by multiple radial objects is richer than previously suggested.

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