

Boundary behaviour of universal Taylor series

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Abstract

A power series that converges on the unit disc \mathbb{D} is called *universal* if its partial sums approximate arbitrary polynomials on arbitrary compacta in $\mathbb{C} \setminus \mathbb{D}$ that have connected complement. This paper shows that such power series grow strongly and possess a Picard-type property near every boundary point.

1 Introduction

Let $D(z, r)$ denote the open disc of centre z and radius r in the complex plane \mathbb{C} , let $\mathbb{D} = D(0, 1)$ and $\mathbb{T} = \partial\mathbb{D}$. A power series $\sum a_n z^n$ with radius of convergence 1 is said to belong to the class \mathcal{U} if, for every compact set $K \subset \mathbb{C} \setminus \mathbb{D}$ with connected complement and every continuous function $g : K \rightarrow \mathbb{C}$ that is holomorphic on K° , there is a subsequence (N_k) of \mathbb{N} such that $\sum_0^{N_k} a_n z^n \rightarrow g$ uniformly on K . Members of \mathcal{U} are called *universal Taylor series*. Nestoridis [18] has shown that such universal behaviour is a generic property of holomorphic functions on the unit disc; that is, \mathcal{U} is a dense G_δ subset of the space of all holomorphic functions on \mathbb{D} endowed with the topology of local uniform convergence.

A significant avenue of investigation concerns the boundary behaviour of functions in \mathcal{U} : see [19], [16], [8], [14], [15], [7], [17], [2], [4], [6], [10], [11]. In this paper we will improve several known results by showing that universal Taylor series have very strong growth properties at *every* boundary point.

Theorem 1 *Let $\psi : [0, 1) \rightarrow (0, \infty)$ be an increasing function such that*

$$\int_0^1 \log^+ \log^+ \psi(t) dt < \infty. \quad (1)$$

If $f(z) = \sum a_n z^n$ and $|f(z)| \leq \psi(|z|)$ on $D(w, r) \cap \mathbb{D}$ for some $w \in \mathbb{T}$ and $r > 0$, then $f \notin \mathcal{U}$.

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The special case of Theorem 1 where the inequality $|f(z)| \leq \psi(|z|)$ is required to hold on all of \mathbb{D} is due to Melas (Theorem 3 of [14]), who also showed that condition (1) is close to being sharp (see Theorem 8 of [14]).

We can deduce the following analogue of Picard's theorem for the boundary behaviour of universal Taylor series.

Corollary 2 *Let $f \in \mathcal{U}$. Then, for every $w \in \mathbb{T}$ and $r > 0$, the function f assumes every complex value, with at most one exception, infinitely often on $D(w, r) \cap \mathbb{D}$.*

This answers a question of Costakis (private communication). Costakis and Melas [8] had previously proved that f assumes every complex value, with at most one exception, infinitely often on \mathbb{D} ; their argument shows that there is at least one point $w \in \mathbb{T}$ with the stated Picard-type property. The exceptional value can actually arise: it was shown in Theorem 1.3 of [8] that there exist zero-free members of \mathcal{U} . Corollary 2 also improves a particular case of Theorem 1 in [10], where it was established that $\mathbb{C} \setminus f(D(w, r) \cap \mathbb{D})$ must be a polar set. Following its proof in Section 2 we will indicate two respects in which it can be generalized.

We can further show that any function f in \mathcal{U} must assume all but one complex value in any angle at "most" boundary points. We denote angular approach regions at a point $w \in \mathbb{T}$ by

$$\Gamma_\alpha^t(w) = \{z : |z - w| < \alpha(1 - |z|) < \alpha t\} \quad (\alpha > 1, 0 < t \leq 1),$$

and recall that a set $E \subset \mathbb{T}$ is called residual if $\mathbb{T} \setminus E$ is a countable union of nowhere dense subsets of \mathbb{T} (that is, $\mathbb{T} \setminus E$ is of first Baire category with respect to \mathbb{T}). The following result complements Theorem 2 of [11], which says that $\overline{f(\Gamma_\alpha^t(e^{i\theta}))} = \mathbb{C}$ almost everywhere ($d\theta$).

Corollary 3 *Let $f \in \mathcal{U}$. Then there is a residual set $E \subset \mathbb{T}$ such that*

$$\mathbb{C} \setminus f(\Gamma_\alpha^t(w)) \text{ contains at most one point } (w \in E, \alpha > 1, 0 < t \leq 1).$$

Finally, we observe that membership of \mathcal{U} is incompatible with any local Bergman-type integrability condition.

Corollary 4 *Let $f \in \mathcal{U}$. Then, for every $w \in \mathbb{T}$ and $r > 0$, and every $\alpha > -1$,*

$$\int_{D(w, r) \cap \mathbb{D}} \log^+ |f(z)| (1 - |z|^2)^\alpha dA(z) = \infty.$$

In particular, f does not belong to any Bergman or Bergman-Nevanlinna class on \mathbb{D} .

Proofs of the above results may be found in the next section. Subsequently we indicate an analogue of Theorem 1 for universal polynomial expansions of harmonic functions in terms of homogeneous polynomials.

2 Proofs of results

Proof of Theorem 1. Let ψ, f, w, r be as in the statement of Theorem 1. We may assume that $r < 1$. Let $r_0 = r/3$ and

$$A_t = \{tz : z \in D(w, r_0) \cap \mathbb{T}\} \quad (t > 0). \quad (2)$$

Clearly

$$D(\zeta, r_0) \subset D(w, r) \cap \mathbb{D} \quad (\zeta \in A_{1-r_0}). \quad (3)$$

The growth hypothesis on f shows that, when $0 < \rho < r_0$ and $\zeta \in A_{1-r_0}$, we have

$$\int_0^{2\pi} \left| f(\zeta + \rho e^{i\theta}) \right|^2 \frac{d\theta}{2\pi} \leq \{\psi(1 - r_0 + \rho)\}^2,$$

and so

$$\sum_{n=0}^{\infty} \left| \frac{f^{(n)}(\zeta)}{n!} \right|^2 \rho^{2n} \leq \{\psi(1 - r_0 + \rho)\}^2.$$

Writing

$$S_N(f, \zeta)(z) = \sum_{n=0}^N \frac{f^{(n)}(\zeta)}{n!} (z - \zeta)^n \quad (N \in \mathbb{N}, \zeta \in \mathbb{D}, z \in \mathbb{C}),$$

it follows that

$$\begin{aligned} \int_0^{2\pi} \left| S_N(f, \zeta)(\zeta + \rho e^{i\theta}) \right|^2 \frac{d\theta}{2\pi} &= \sum_{n=0}^N \left| \frac{f^{(n)}(\zeta)}{n!} \right|^2 \rho^{2n} \\ &\leq \{\psi(1 - r_0 + \rho)\}^2 \quad (\rho \in (0, r_0), \zeta \in A_{1-r_0}) \end{aligned} \quad (4)$$

Let $z \in A_{1-\varepsilon}$, where $\varepsilon \in (0, r_0)$. Putting $\zeta = (1 - r_0)z/|z| \in A_{1-r_0}$ and $\rho = r_0 - \varepsilon/2$, we can now use the subharmonicity of $|S_N(f, \zeta)|^2$ and the Poisson integral formula for the disc $D(\zeta, \rho)$ to see that

$$\begin{aligned} |S_N(f, \zeta)(z)|^2 &\leq \int_0^{2\pi} \frac{\rho^2 - (r_0 - \varepsilon)^2}{|z - (\zeta + \rho e^{i\theta})|^2} \left| S_N(f, \zeta)(\zeta + \rho e^{i\theta}) \right|^2 \frac{d\theta}{2\pi} \\ &\leq \frac{4r_0}{\varepsilon} \{\psi(1 - r_0 + \rho)\}^2 \quad (N \in \mathbb{N}), \end{aligned}$$

by (4). Hence

$$|S_N(f, \zeta)(z)| \leq 2\sqrt{\frac{r_0}{\varepsilon}} \psi(1 - \varepsilon/2) \quad (z \in A_{1-\varepsilon}, \varepsilon \in (0, r_0), N \in \mathbb{N}). \quad (5)$$

We now suppose, for the sake of contradiction, that $f \in \mathcal{U}$. Then we can choose a subsequence (N_k) of \mathbb{N} such that

$$S_{N_k}(f, 0) \rightarrow 0 \quad \text{uniformly on } \{tz : 1 \leq t \leq 2, z \in \overline{A_1}\}. \quad (6)$$

In fact, according to Section 9 of Melas and Nestoridis [15], we can simultaneously arrange that

$$S_{N_k}(f, \zeta)(z) - S_{N_k}(f, 0)(z) \rightarrow 0 \text{ locally uniformly with respect to } (\zeta, z) \in \mathbb{D} \times \mathbb{C}.$$

Let $\phi : (1 - r_0, 2) \rightarrow (0, \infty)$ be given by

$$\phi(t) = \begin{cases} 2(1-t)^{-1/2}\psi((1+t)/2) & (1-r_0 < t < 1) \\ 1 & (1 \leq t < 2) \end{cases}.$$

Then (5) and (6) together show that, for all sufficiently large k ,

$$|S_{N_k}(f, 0)(z)| \leq \phi(|z|) \quad \text{on } \{tz : z \in A_1, 1 - r_0 < t < 2\}.$$

Since

$$\int_{\{tz:1-r_0<t<1,z \in I\}} \log^+ \log^+ \phi(|z|) dA(z) < \infty,$$

by (1), we can now apply a well known “log log” result (see, for example, Section 3 of Domar [9]) to see that $(S_{N_k}(f, 0))$ is a normal family on $\{tz : z \in A_1, 1 - r_0 < t < 2\}$. In view of (6) and the local uniform convergence of $(S_{N_k}(f, 0))$ to f on \mathbb{D} , this leads to the absurd conclusion that $f \equiv 0$. Hence $f \notin \mathcal{U}$. ■

Proof of Corollary 2. Let $w \in \mathbb{T}$ and $r \in (0, 1)$, and suppose that f is a holomorphic function on \mathbb{D} that omits two values, say 0 and 1, on $D(w, r) \cap \mathbb{D}$. Let $r_0 = r/3$ and A_t be as defined in (2), let $z \in A_{1-\varepsilon}$, where $0 < \varepsilon < r_0$, and let $\zeta = (1 - r_0)z/|z| \in A_{1-r_0}$. Then Schottky’s theorem, as refined by Ahlfors (see Theorem B in [1]), can be applied to f on the disc $D(\zeta, r_0)$ to see that

$$\log |f(z)| \leq \frac{2r_0}{\varepsilon} (7 + \log^+ |f(\zeta)|),$$

in view of (3). Hence

$$\log |f(z)| \leq \frac{2r_0}{1-|z|} (7 + \sup \{\log^+ |f(\zeta)| : \zeta \in A_{1-r_0}\}) \quad (z \in D(w, r_0) \cap \mathbb{D}).$$

It now follows from Theorem 1 that $f \notin \mathcal{U}$. ■

Remark 5 We can give a quantitative version of Corollary 2, which improves Theorem 2 of [14], as follows. Let $f \in \mathcal{U}$. Then, for any $w \in \mathbb{T}$ any $r, \kappa > 0$, and all but at most one complex number a , the distinct zeros $(z_j(a))$ of $f - a$ in $D(w, r) \cap \mathbb{D}$ satisfy

$$\sum (1 - |z_j(a)|)^\kappa = \infty. \tag{7}$$

To prove this, suppose that the above series converges for two distinct choices of a . Then $\log |f(z)| \leq C(1 - |z|)^{-\kappa-1}$ on $D(w, r/2) \cap \mathbb{D}$, by Proposition 2 of [14] (which relies on Nevanlinna value distribution theory), combined with a suitable conformal mapping from $D(w, r) \cap \mathbb{D}$ to \mathbb{D} . Theorem 1 can now be invoked to obtain a contradiction. In fact, (7) can be even further strengthened, along the lines of Theorem 4 of [14]. We are grateful to Vassili Nestoridis for raising the question of whether such a result might hold.

Remark 6 There is a natural extension, $\mathcal{U}(\Omega, \zeta)$, of the collection \mathcal{U} to general simply connected domains $\Omega \subset \mathbb{C}$, where $\zeta \in \Omega$ denotes the centre of expansion of the Taylor series (see [19]). It was shown recently [10] that, for any $f \in \mathcal{U}(\Omega, \zeta)$, $w \in \partial\Omega$ and $r > 0$, the set $\mathbb{C} \setminus f(D(w, r) \cap \Omega)$ is polar. The stronger conclusion we obtained in Corollary 2 readily extends, by a similar argument, to simply connected domains Ω which satisfy the following geometric condition: for any $w \in \partial\Omega$ and $r > 0$, there exists $w_0 \in \partial\Omega$ and $r_0 > 0$, and a family of discs $\{D(z_\alpha, \rho_\alpha)\}$ in $D(w, r) \cap \Omega$ such that $\inf_\alpha \rho_\alpha > 0$ and $D(w_0, r_0) \cap \Omega \subset \cup_\alpha D(z_\alpha, \rho_\alpha)$.

Proof of Corollary 3. Let f be a holomorphic function on \mathbb{D} . For each $j, k, l \in \mathbb{N}$ we define the set

$$E_{j,k,l} = \left\{ w \in \mathbb{T} : \text{there exist } a_w, b_w \in D(0, l) \setminus f\left(\Gamma_{j+1}^{1/k}(w)\right) \text{ satisfying } |a_w - b_w| \geq l^{-1} \right\}.$$

Clearly $E_{j,k,l}$ increases with each of j , k and l . Suppose now that the conclusion of the corollary fails to hold. Then there exist $j_0, k_0, l_0 \in \mathbb{N}$ such that the set \overline{F} , where $F = E_{j_0, k_0, l_0}$, has non-empty interior I with respect to \mathbb{T} . For each w in F we choose distinct points

$$a_w, b_w \in D(0, l_0) \setminus f\left(\Gamma_{j_0+1}^{1/k_0}(w)\right) \text{ such that } |a_w - b_w| \geq l_0^{-1}$$

and define

$$f_w(z) = \frac{f(z) - a_w}{b_w - a_w} \quad (z \in \mathbb{D}),$$

so that f_w omits the values 0 and 1 on $\Gamma_{j_0+1}^{1/k_0}(w)$. Next, there exist $\rho, \tau \in (0, 1/2)$, depending only on j_0, k_0 , such that

$$D((1-t)w, 2\rho t) \subset \Gamma_{j_0+1}^{1/k_0}(w) \quad (w \in F, 0 < t \leq \tau).$$

If $w \in F$, then Schottky's theorem can be applied on $D((1-\tau)w, 2\rho\tau)$ to see that

$$\log |f_w((1-t)w)| \leq 3(7 + \log^+ |f_w((1-\tau)w)|) \quad ((1-\rho)\tau \leq t \leq \tau).$$

Repeated application of this inequality shows that

$$7 + \log^+ |f_w((1-t)w)| \leq 4^n (7 + \log^+ |f_w((1-\tau)w)|)$$

whenever $(1 - \rho)^n \tau \leq t \leq (1 - \rho)^{n-1} \tau$ and $n \in \mathbb{N}$. Hence there is a constant $\alpha \geq 1$, depending only on ρ and τ , such that

$$\log |f_w((1 - t)w)| \leq t^{-\alpha} (7 + \log^+ |f_w((1 - \tau)w)|) \quad (0 < t \leq \tau).$$

Since f is bounded on $\{(1 - \tau)w : w \in F\}$, and $a_w, b_w \in D(0, l_0)$, where $|a_w - b_w| \geq l_0^{-1}$, there exists $C > 0$ such that

$$\log |f((1 - t)w)| \leq Ct^{-\alpha} \quad (w \in F, 0 < t \leq \tau).$$

Continuity now yields

$$\log |f(z)| \leq C(1 - |z|)^{-\alpha} \quad (z \in \{tw : w \in I, 0 < t \leq \tau\}),$$

and Theorem 1 shows that $f \notin \mathcal{U}$. ■

Proof of Corollary 4. Let $w \in \mathbb{T}$, $r \in (0, 1)$ and $\alpha > -1$, and suppose that f is a holomorphic function on \mathbb{D} such that $J < \infty$, where

$$J = \int_{D(w, r) \cap \mathbb{D}} \log^+ |f(z)| (1 - |z|^2)^\alpha dA(z).$$

By the subharmonicity of $\log^+ |f|$ we deduce that

$$\begin{aligned} \log^+ |f(\zeta)| &\leq \frac{4}{\pi(1 - |\zeta|)^2} \int_{D(\zeta, (1 - |\zeta|)/2)} \log^+ |f(z)| dA(z) \\ &\leq \frac{C(\alpha)}{(1 - |\zeta|)^{\alpha+2}} J \quad (\zeta \in D(w, r/2) \cap \mathbb{D}), \end{aligned}$$

where $C(\alpha)$ is a positive constant depending only on α . It now follows again from Theorem 1 that $f \notin \mathcal{U}$. ■

3 Universal polynomial expansions of harmonic functions

Let $B(x, r)$ denote the open ball of Euclidean space \mathbb{R}^d ($d \geq 2$), let $\mathbb{B} = B(0, 1)$ and $\mathbb{S} = \partial\mathbb{B}$. Any harmonic function h on \mathbb{B} has a unique expansion of the form

$$h(x) = \sum_0^\infty h_n(x) \quad (x \in \mathbb{B}), \quad (8)$$

where h_n belongs to the space of homogeneous harmonic polynomials of degree n in \mathbb{R}^d . We say that h belongs to the collection \mathcal{U}_H , of harmonic functions on \mathbb{B} with universal homogeneous polynomial expansions, if, for any compact set $K \subset \mathbb{R}^d \setminus \mathbb{B}$ with connected complement and any harmonic function u on a neighbourhood of K , there is a subsequence (N_k) of \mathbb{N} such that $\sum_0^{N_k} h_n \rightarrow u$ uniformly on K . Such universal harmonic functions have been studied in [5], [12] and [13]. We can now give the following analogue of Theorem 1.

Theorem 7 Let $\psi : [0, 1) \rightarrow (0, \infty)$ be an increasing function such that

$$\int_0^1 \log^+ \psi(t) dt < \infty. \quad (9)$$

If (8) holds and $|h(x)| \leq \psi(|x|)$ on $B(w, r) \cap \mathbb{B}$ for some $w \in \mathbb{S}$ and $r > 0$, then $f \notin \mathcal{U}_H$.

The proof is directly analogous to the argument given in Section 2. The reason why only one “log” appears in (9), in contrast to (1), is that we apply Domar’s result to subharmonic functions of the form $|h|$ rather than $\log |f|$. The required versions for harmonic functions of the results we used from [15] were recently established by Manolaki [13] (we can use a combination of Theorems 3 and 4 of that paper).

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