Centre de Recherches Mathématiques
CRM Proceedings and Lecture Notes

# Questions Related to the Spectral Properties of Several Classical Integral Operators and Geometry

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Abstract. These informal notes represent a lecture given as part of the CRM Summer School on Spectral Theory and Applications at Laval University in July 2016.

## 1. Classical Integral Operators

Let's start with a very simple operator, the Cauchy integral operator.
Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, $\Gamma = \partial\Omega$ is assumed to be sufficiently smooth.

$$L^2(\Omega) = \left\{ f : \Omega \to \mathbb{C}, \|f\|_2^2 = \int_\Omega |f|^2 dA(\xi) < \infty, \, dA = \text{area measure} \right\}.$$

$$(Cf)(z) = \left[ -\frac{1}{\pi} \int_\Omega \frac{f(\xi)}{\xi - z} \, dA(\xi) \right].$$

An easy exercise shows that $C$ is bounded, even compact, yet not Hilbert–Schmidt.

**Remark 1.1.** Note that the Cauchy integral operator on curves is well-known but it is not compact. The above Cauchy operator is compact and reproduces all $C_0^\infty$ functions

$$\varphi(z) = -\frac{1}{\pi} \int_\Omega \frac{\partial \varphi}{\partial \bar{\xi}} \, \frac{1}{\xi - z} \, dA(\xi).$$

**Question.** What is the norm of $C$, for say domains that are simple, say a disk? What can be said about spectral asymptotics of $C$?

The inverse questions one plentiful: from the information about the spectrum, can one derive some information regarding geometry of $\Omega$? Can one characterize disks among the domains of the same area based on the spectral information of the operator? Can we "hear the shape" of a drum, i.e., are there two different domains with the same spectrum, and so on?

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2010 *Mathematics Subject Classification.* Primary .

The author gratefully acknowledges the NSF support of the Summer School under the grant DMS 1653027.

Two other operators $L$, $N$ naturally come to mind:

$$(Lf)(z) = \frac{1}{2\pi} \int_{\Omega} \log \frac{1}{|z-\xi|}\, dA(\xi);$$
$$(Nf)(x) = \frac{1}{(n-2)\omega_n} \int_{\Omega} \frac{1}{|x-y|^{n-2}} f(y)\, dV(y).$$

(Here, $\omega_n$ denotes the area of the unit sphere. $dV$ is the Lebesgue measure in $\mathbb{R}^n$.) All of the above questions can legitimately be asked for $L$, $N$. Also, one can certainly consider comparison (in, say, norm) of $C$, $L$, $N$. In 1989 Anderson and Hinkkanen [1] showed that

$$\|C\|_{L^2(\mathbb{D})} = \frac{2}{j_0},$$

where $j_0 \equiv 2.408\ldots$, is the smallest positive zero of the zero Bessel function

$$J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{x}{2}\right)^{2k};$$
$$(\mathbb{D}) = \{z : |z| < 1\} \text{ is the unit disk in } \mathbb{C}.$$

The proof was a rather ingenious computation and application of an inequality of Hardy type and an ODE result due to P. Boyd (1969).

However, if one notices ([3]) that $j_0^2 = \lambda_1$ is the smallest eigenvalue for the Dirichlet Laplacian in $\mathbb{D}$, i.e., the smallest positive $\lambda > 0$ such that

$$\Delta\varphi + \lambda\varphi = 0 \text{ in } \Omega,\ \varphi = 0 \text{ on } \partial\Omega,$$

the problem immediately connects to mathematical physics. Furthermore, let

$$g(x,y) = \begin{cases} \dfrac{1}{(n-2)w_{n-1}} \dfrac{1}{|x-y|^{n-2}} + u_x(y), & x, y \in \Omega \quad n \geq 3 \\ \dfrac{1}{2\pi} \log \dfrac{1}{|x-y|} + u_x(y), & x, y \in \Omega \\ 0, \text{ elsewhere} & \end{cases}$$

be Green's function [13] of $\Omega$, $u_x(y)$ is harmonic in $\Omega$ and, $g(x,y) \mid_{y \in \partial\Omega} = 0$, and consider the operator

$$(Gf)(x) = \int_{\Omega} g(x,y) f(y)\, dV(y).$$

Then, $G^{-1} = -\Delta$ on the Sobolev space $W_0^{1,2}$. The eigenvalues of $G$ are reciprocals of those of $\Delta$. Hence, the Anderson–Hinkkanen results can be read as

$$\left\| \frac{1}{4} C^* C \right\|_{\mathbb{D}} = \frac{1}{j_0^2} = \frac{1}{\lambda_1} = \mu_1 := \|G\|_{\mathbb{D}}$$

It turns out that not all of these are coincidences.

**Theorem 1.2** ([2, 6])**.**

$$\left\| \frac{1}{4} C^* C \right\|_{\Omega} = \left( \frac{1}{2} \|C\| \right)^2_{\Omega} = \|G\|_{\Omega} = \frac{1}{\lambda_1} = \mu_1.$$

Moreover, in view of the Raleyigh–Faber–Krahn theorem [4], we have

**Corollary 1.3.** *Among the domains of equal area, the norm for $C$ (and $G$) is maximized over a disk, i.e.*

$$\|C\| \le \frac{2}{j_0}\sqrt{\frac{\text{Area}\,\Omega}{\pi}}$$

Moreover, combining the results from [2], [3] and [5], we have the following results on spectral asymptotics:

(i) The operator $C$, or, rather, $\frac{C^*C}{4}$: The singular numbers $s_n$, i.e., the eigenvalues of the self-adjoint operator $\left(\frac{C^*C}{4}\right)^{1/2}$ behave asymptotically as $O\left(\frac{1}{\sqrt{n}}\right)$.

All the eigenvalues are of double multiplicities so, $C \in S_{2,\infty}$, in particular, $C \in S_p$, $p > 2$ but $C$ is not Hilbert–Schmidt ($S_p$, $S_2$ stand for the corresponding Schatten classes — cf. [3]).

Remarkably, if we restrict $C$ to the Bergman space of square-integrable analytic functions, the singular numbers of $CP$, where $P$ denotes the orthogonal projection onto the Bergman space, become $O\left(\frac{1}{n}\right)$, hence $CP \in S_{1,\infty}$, i.e., "2 times better than $C$".

(ii) The operator $L$. The spectra of $L$ and $\frac{1}{4}C^*C$ are quite similar (and coincide for the disk) although in the disk $L$ has the whole series of triple eigenvalues. The singular numbers are $O\left(\frac{1}{n}\right)$ and $L \in S_{1,\infty}$. Again the operator $PLP$ is "twice as nice" ($\in S_{1/2,\infty}$), the eigenfunctions in the disk are simply monomials.

In 2007, M. Dostanić [7] established further refinement on asymptotics of $C$ and $L$.

$$s_n(C) = \sqrt{\frac{\text{Area}(\Omega)}{\pi}\frac{1}{n}} + O\left(\frac{1}{n}\right);$$
$$\lambda_n(L) = \frac{\text{Area}(\Omega)}{4\pi}\cdot\frac{1}{n} + O\left(\frac{1}{n^{3/2}}\right).$$

Thus, as in the case of the Dirichlet Laplacian, the spectral behavior of $C$, $L$ allow us to determine the area of $\Omega$. Note that $L$ is "two times better" than $C$. As noted above, the norms of $C$, $L$ allow us to determine when the domain is a disk (or, not — more often) in view of the Faher–Krahn theorem. The following questions certainly pose themselves:

(i) Are there domains with identical spectra for both operators $C_\Omega$ and $L_\Omega$ ? (Most likely "yes", but virtually nothing has been done.)

(ii) The eigenfunctions for disks, balls for the operators $C$, $L$, $N$ are products of Bessel functions and spherical harmonics. Thus entire functions of exponential type. Does this property characterize balls? Probably not, since some of these statements hold for ellipsoids as well. Hence, it is true that the growth of the eigenfunctions characterize ellipsoids. For the operator $G$, or equivalently, for the Dirichlet Laplacian the eigenfunctions for ellipsoids are also entire functions of exponential type (Bessel functions combined with spherical harmonics for balls). It is tempting to suggest that that property does characterize ellipsoids, in other words for all other domains some eigenfunctions develop singularities somewhere outside the domains. At present, there is virtually no progress on the problem. A closely-related problem is associated with the so-called Khavinson–Shapiro conjecture — cf. [10, 11]. The conjecture asserts that ellipsoids are the only domains for which all solutions of the Dirichlet problem $\Delta u = 0$ in $\Omega$, $u\,|_{\Gamma:=\partial\Omega}= f$, with entire data $f$ are themselves entire functions.

The problem where do the singularities of solutions of the Dirichlet problem with a "nice", entire or polynomial, data appear from is deep, difficult and, mostly, widely open. Some recent progress has been achieved mostly by H. Render, and , also, by S. Bell–P. Ebenfelt–D. Khavinson–H. S. Shapiro and D. Khavinson–E. Lundberg – cf. [10, 11] and references therein. In two dimensions the problem leads very quickly to some intriguing and non-standard algebraic geometry: complex "lightning bolts" on Riemann surfaces, etc. – cf. [10].

(iii) Do the spectra of $C$, $L$, $N$ transmit the information about the geometry of the base domain $\Omega$? For example, can we detect corners, or cusps on $\Gamma = \partial\Omega$ from the spectral properties of those operators. Since the crude asymptotics ($O\left(\frac{1}{\sqrt{n}}\right)$ for $C$ and $O\left(\frac{1}{n}\right)$ for $L$) are the same for all bounded domains, the problem appears quite delicate. It appears as a worthy task to investigate in detail the situation for $C$ in cardioids since the conformal map from the disk onto cardioids is just a quadratic polynomial. Hence, via a simple change of variables, the problem could be moved back to the disk. Similarly, the example of a square should be possible to handle directly for the operator $L$.

## 2. Single-Layer Potentials

Set $\Gamma = \partial\Omega$, $\Omega$ is as above. Let

$$u(x) = \int_\Gamma E_n(x-y)f(y)\,dS(y),$$

where

$$E_n(x) = \begin{cases} -\dfrac{1}{2\pi}\log|x|, & n = 2; \\ \dfrac{1}{(n-2)\omega_n}|x|^{2-n}, & n > 2, \end{cases}$$

where as above $\omega_n$ is the area of the unit sphere $S^{n-1}$ in $\mathbb{R}^n$. $dS$ is Lebesgue measure on $\Gamma$. Obviously, $u$ is harmonic in $\mathbb{R}^n \setminus \Gamma$ and is continuous in $\mathbb{R}^n$. The operator $S : f \mapsto u|_\Gamma$ is self-adjoint on $L^2(\Gamma)$ and it is not difficult to see that $S$ is compact, even Hilbert–Schmidt. $S$ is injective in $\mathbb{R}^n$, $n \geq 3$, and has at most 1 dimensional kernel in $\mathbb{R}^2$. It is not difficult to calculate the spectrum of $S$ in $\mathbb{D}$, $n = 2$ or $B := \{|x| < 1\}$ in $\mathbb{R}^n$, $n \geq 3$. The eigenfunctions are spherical harmonics, the multiplicity of every eigenvalue $\lambda_n$ in $\mathbb{R}^2$ is 2 while for $n \geq 3$ it is $A(n,m) - A(n,m-2)$, $A(n,m) = \frac{n(n+1)\cdots(n+m-1)}{m!}$, the dimension of the space of spherical homogeneous harmonics of degree $m$. The eigenvalues are $\frac{1}{2m}$ for $n = 2$ and $\frac{1}{2m+n-2}$ for $n > 2$. This observation provides us with crude asymptotics of the spectra for all bounded domains in $\mathbb{R}^n$.

Note that the constant functions are eigenfunctions for $S$ in all dimensions on the ball.

**Question.** Does this properly characterize balls?

The answer is "Yes" ([9]), provided that $\Gamma$ satisfies certain mild smoothness assumptions. (For example, for $n = 2$, $\Omega^e = \mathbb{C} \setminus \overline{\Omega}$ is assumed to be a Smirnov domain.) The result is proved in two dimensions by function theory via the Riemann mapping.

**2.1. Quadrature identity.** One can show that $S$ has 1 as an eigenfunction if and only if the domain $\Omega'$ obtained from $\mathbb{R}^n \setminus \Omega$ by inversion has the following quadrature property

$$u(0) = c \int_{\Gamma' := \partial\Omega'} |x|^{-n} u(x)\, dS, \text{ for all harmonic } u \text{ in } \Omega.$$

Then it is indeed true that such domain must be a ball (W. Reichel, S. Shahgholian $\sim$ '97 – cf. references in [9]) but it is only known for $C^{2\alpha}$ boundaries in $\mathbb{R}^n$, $n \geq 3$.

Perhaps, one wonders, even a more ambitious conjecture is true: If for $\alpha \in \mathbb{R}$

$$(*) \qquad u(0) = c \int_{\Gamma} |x|^{\alpha} u(x)\, dS(x)$$

for all $u$ harmonic in $\overline{\Omega}$, then $\Omega$ is a ball. $(*)$ of course, means that the harmonic measure $\omega$ at 0 is represented as $\omega(0, \Omega, \Gamma) = c|x|^{\alpha}\, dS(x)$.

**Theorem 2.1** ([9])**.**

(i) *For $n = 2$ and $\alpha = -2$, $(*)$ holds for ALL disks containing the origin.*
(ii) *For $n = 2$, $\alpha = -3, -4, -5, \ldots \exists$ solutions (domains) $\Omega$ which are NOT disks (!)*
(iii) *For all other $\alpha$, $\Omega$ is a disk centered at the origin.*

(*For (ii), $\Omega$ is obtained from $\mathbb{D}$ via conformal maps* $\varphi(w) = \frac{w}{(A+Bw^k)^{1/k}}$, $k = 2, 3, \ldots$ *with appropriate $A$, $B$.*)

**Questions.**

(i) How can the above theorem be extended to $n \geq 3$?
(ii) Can one extend the results of Reichel, Shahgholian, etc., to less-smooth boundaries?
(iii) All the questions from the previous section I, e.g., isospectrality, information about the boundary, e.g., cusps, corners, etc., make sense for the single-layer operator $S$ as well.
(iv) We have shown that under some mild restrictions only balls have constants as eigenfunctions for $S$. What about higher-degree harmonic polynomials?

S. Zoalroshd [14] showed that if in $\mathbb{R}^2$ a harmonic polynomial $h = Re\, p(z)$ is an eigenfunction for $S$ and all but one zeros of $p$ are inside $\Omega$, then $\Gamma = \partial\Omega$ is a circle. It is natural to conjecture that the latter hypothesis is not necessary. Is it true? Is his result true in $\mathbb{R}^n$, $n \geq 3$?

## 3. Double-Layer Potentials

Let as above $E(x, y) := E_n(x, y) = c_n \begin{cases} |x|^{2-n}, & n \geq 3 \\ \log|x| & n = 2 \end{cases}$ be the usual kernel in potential theory.

Let $K : L^2(\Gamma) \to L^2(\Gamma)$ be defined

$$(Kf)(x) = 2 \int_{\Gamma} \frac{\partial}{\partial n_y} E(x - y) f(y)\, d\sigma(y),$$

$K$ is called the Neumann–Poincaré operator. If we let $x \in \Omega$, we have a harmonic function whose boundary values equal $f - Kf = (I - K)f$ on $\Gamma$. Thus, proving that the operator $(I \pm K)$ is surjective allowed Fredholm to solve the Dirichlet problem (Fredholm's alternative) in terms of double layer potentials in rather general domains provided that $I \pm K$ is injective.

Unfortunately, $K$ is never self-adjoint unless $\Omega$ is a ball, cf. [12]. However, using the so-called symmetrization procedure one can show that $K$ indeed has a real spectrum and its eigenfunctions do span the $L^2-$ space on the boundary. Poincaré has conjectured (by analogy with the sphere) that the spectrum of $K$ is always nonnegative in $\mathbb{R}^n$, $n \geq 3$. This is now proven to be false ([12]). There is enormous literature on eigenvalues and eigenfunctions for $K$ — cf. the references in [12]. Let me just touch on the most fundamental property of $K$:

**Question.** Is $K$ injective, i.e., does $\ker K = \{0\}$?

Obviously, it is true for $\Gamma = S^{n-1}$, $n \geq 3$, where the operator $K$, as is easily verified, is a scalar multiple of $S$, which is obviously injective. So, is it true for other smooth surfaces $\Gamma$ in $\mathbb{R}^n$, $n \geq 3$? The answer is probably "no", but it is still unknown. In 2 dimensions the situation is even more intriguing. On $\mathbb{T} := \partial\mathbb{D}$,

$$Kf = \frac{1}{\pi} \int_0^{2\pi} f\left(e^{i\theta}\right) ds$$

has rank 1. So $\ker K = L^2(\mathbb{T}) \ominus \mathbb{C}$ is virtually almost everything. In the 1980's, D. Khavinson and H. S. Shapiro showed (unpublished) that $\ker K$ cannot be finite dimensional.

In [8] it was proven that not only if $\ker K \neq \{0\}$, then $\dim K = \infty$, but moreover it is an algebra! Namely, $F \in \ker K \Leftrightarrow \exists f, g$ analytic in $\Omega, \Omega^e$ respectively, $g(\infty) = 0$ and such that

$$(**) \qquad f = \bar{g} \text{ on } \Gamma.$$

**Example 3.1.**

(i) $\Omega = \mathbb{D}$, $f = z^n$, $n \geq 1$.

(ii) Yet, the "matching problem" $(**)$ has other solutions!

Rational lemniscates ([8]) are defined as Jordan curves $\Gamma$ such that

$$\Gamma = \{|R(z)| = c,\ c > 0\},$$

where $R$ is a rational function, $R$ has no zeros in $\Omega^e$, and no poles in $\Omega$. The pair $R$, $\frac{C}{R}$ obviously solves $(**)$.

**Conjecture 3.2.** *Rational lemniscates are the only curves on which the Neumann-Poincaré is not injective .*

**Remark 3.3.** It is truly important to identify the $\ker K$ since these are precisely the densities whose double layer potentials give directly explicit solutions of the Dirichlet problem.

Again, all the questions in Section 1 apply to $K$ as well.

Another open venue for research is to study in depth the relationships between the spectra of $C$, $L$, $N$ and well-known problems in mathematical physics. Even in $\mathbb{D}$ (or, $B = \{|x| < 1\}$, $n \geq 3$) those spectra are close, but not identical. So understanding precisely, in geometric terms, the "corrections" between the eigenvalues is already a worthy project.

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