# Harold Seymour Shapiro 1928-2021; Life in Mathematics, in memoriam. 

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#### Abstract

This is a (very) personal outlook on the life and mathematical achievements of Harold S. Shapiro, who has passed away in March of 2021, few days short of his 92d birthday. It is based on the author's long (almost 40 years) collaboration and friendship with Harold; so it might be interpreted as "kaddish".


## Prologue

I heard the name Harold Shapiro spoken many times by my father, S. Ya. Khavinson, since the time I can remember anything at all. My father told me on our walks, that he and that mysterious Harold Shapiro, had discovered the same thing in mathematics simultaneously without knowing about each other's research. The story, not the meaning of what they had discovered together while apart, stuck with me. Very possibly, during Harold's two-year stay in Moscow, we actually met at some mathematical parties at our apartment, or at B. Shabat's, or A. Markushevich, my dad's advisor, or G. Ts. Tumarkin, my dad's lifelong friend and collaborator, possible, but I don't remember meeting him then. And even if we did, I was only interested in soccer and hockey in those days, so naturally neither of us would have remembered the meeting. While preparing to leave the former Soviet Union for good, I had written to several mathematicians in the West asking for advice on how to pursue a career as a mathematician. Harold's letter, I still remember the blue airmail envelope in which it arrived, was friendly and invited me to come to Stockholm and
study with him for a Ph.D. For various reasons, the path of an immigrant from USSR at that time couldn't go through Sweden, so our actual meeting happened only five years later at a conference at Purdue University dedicated to celebrating the proof of the Bieberbach conjecture by L. de Branges. I remember clearly being at the cocktail party, not daring to talk to any of the grand apostles of complex analysis, when B. Korenblum, who had met me as a graduate student at Brown, came over and said that Harold Shapiro would like to meet me. And there came Harold, and we started talking, and my mathematical life turned much better in an instant! Harold was so easy to talk to, forgetting everything, we started a project right then and there. It took 18 years and substantial input from P. Ebenfelt to finally become a paper [1], but from that point on there would not be a week that we would not exchange letters, discussing everything - problems, courses we were teaching, gossip - everything. In few years, thanks to the e-mail, the weekly blue envelopes from Sweden were replaced by daily e-mails, back and forth between Arkansas and Stockholm. Often, very often actually, we exchanged several e-mails a day. That continued for almost four decades until a debilitating illness prevented Harold from typing. I am saying this before addressing Harold's mathematical life, to stress that (by all means) I cannot pretend to be a neutral observer and evaluator of Harold's role and mathematical input. I am biased - even, dare I say - extremely biased and, therefore, what follows is a very personal outlook on Harold Shapiro's mathematical life which, for forty years, was very tightly connected to my own.

Let me finish this introduction by saying that already during that first meeting in Indiana, when listening to a young lad's complaints that no one seemed to like the problems I was interested in while I am not particularly keen on problems that everyone else seems to be "gaga" about, Harold smiled (I still remember that smile) and said: "If mathematics is your life, you might as well try and enjoy it." That sentence stayed with me from that time on. So, with that preamble, here it goes.

## 1 Harold's Life and Mathematics, an Overview

Harold Seymour Shapiro, Professor Emeritus of the Royal Institute of Technology in Stockholm, passed away on March 5, 2021.

He was born in 1928 in Brooklyn, NY. His father was a dentist by profession and an inventor at heart. Harold's mother was a homemaker. Harold was the middle child, flanked by two sisters. He obtained his undergraduate degree from City College, NY, where he was a prominent member of the "mathematical table", that included L. Flatto, D. J. Newman, M. Davis, L. Rubel, and J. Schwartz. He received his Ph.D. from MIT in 1952 under the direction of Norman Levinson. Having spent a couple of years at Bell Labs, then eight years at New York University, Shapiro became a professor at the University of Michigan in 1962. In 1970, he moved to Sweden and became a professor at the Royal Institute of Technology in 1972.

His main area of interest was Classical Analysis and its interplay with Functional Analysis. He has left significant imprints in Approximation Theory, Complex and Harmonic Analysis, Potential Theory, and the Theory of Functions. During his long and productive mathematical life, Harold authored and co-authored over 150 research papers and four books, many of which are (nowadays) standard references for mathematicians working in a wide spectrum of fields - cf., e.g., [2-5]. Shapiro's 1951 M.S. thesis introduced special trigonometric polynomials with coefficients $\pm 1$ as solutions of a certain extremal problem. The main feature of these polynomials is that their $L^{\infty}$-norm on the unit circle happens to be controlled by their $L^{2}$-norm. The polynomials, nowadays called Golay-Rudin-Shapiro polynomials, are systematically used in engineering applications: communications theory, antenna designs, data compression, etc.

Harold's Ph.D. thesis, in 1953, introduced a completely novel approach to a wide class of extremal problems in complex analysis based on Hahn-Banach duality. (S. Ya. Khavinson, in the USSR, made the same discovery independently in his M.S. Thesis in 1949 [6]. Harold's results appeared as a joint paper [7] with W. Rogosinski in Acta. Math. in 1953. The idea of studying together the extremal problem in a given function space and the dual approximation problem in the dual space, based on the Hahn-Banach theorem, is nowadays the main tool in Approximation Theory. There was an instructive story that Harold told me regarding that Acta paper with Rogosinski. The two never met. Rogosinski had written a paper on extremal problems in Hardy spaces, continuing his previous Acta paper with Macyntire. Rogosinski's paper was already accepted to appear in Acta when Walter Rudin, who was in Europe for the summer, told Rogosinski about Harold's Ph.D. thesis. Due to the use of duality and F. and M. Riesz' theorem, Harold obtained, in a much more straightforward way, the same results as Rogosinski, yet including the case of $H^{\infty}$ that Rogosinski didn't have. And Rogosinski, a well-known, well-established mathematician, put the name of the unknown youngster as an equal co-author on the paper. Just pause and think of the likelihood of something similar happening today. Sadly, by the time Harold traveled to Europe for the first time several years later, Rogosinski had passed away.

In a groundbreaking paper with D. Aharonov in 1976, the fruitful subject of quadrature domains in the complex plane was introduced. The topic remains a very active field today. It turned out to have far-reaching application in the dynamics of oil spills, tumor growth, etc., unified by the name of the HeleShaw processes. The beginning of this field of research can be found in Harold's monograph [2].

From the early days of his research, Harold was fascinated by the idea of forced analytic continuation. In the, by now absolutely classical paper [8], joint with R. Douglas and A. Shields, a life-long friend of Harold's, the authors drew out connection lines between [an] analytic continuation theme and [the] central problem of functional analysis, the problem of invariant subspaces for linear
operators. Further results and references are contained in the monograph [9] joint with W. Ross.

The topic of analytic continuation can be traced as a red thread through Harold's mathematical life. Starting in the mid-1980s, he turned to a more general problem of analytic continuation, not exclusively for holomorphic functions, but also for more general solutions of analytic linear PDEs. At this point, our mathematical paths intersected and our collaboration that has lasted for nearly four decades started. Some results, references and, most importantly, many attractive problems that remain, can be found in an unpublished report [10] and, much later, in the monograph [11], dedicated to Harold's 90th birthday.

As mentioned above, Harold Shapiro has published over 150 research articles and 4 books, making it impossible to revisit all of his works in-depth. In Section 3, we shall provide a more detailed glance into some of the topics mentioned in this section.

## 2 Pedagogy

Harold's legacy as a mathematician is organically connected with his passion for teaching and supervising graduate students. His encouragement to students, young colleagues and collaborators, often supported by a quote from Divine Comedy: "Follow your path and let people talk ...", served as a guiding light for many.

In his first years in Stockholm, he had launched a special Problem Solving seminar that attracted numerous students. That was a novelty in a rather cool atmosphere of Swedish mathematics at the time. Many of the first attendees graduated with a Ph.D. under Harold's supervision: M. Benediks, B. Gustafsson, L. Svensson, G. Johnsson, and K. Ullemar. Also, later on, Harold supervised the Ph.D. thesis of P. Ebenfelt, H. Shahgholian, and J. Annianssson. He later co-advised L. Karp with D. Aharonov at Technion. Harold was unbelievably generous with his time for his students, or even simply visitors streaming through his office. Being a professor at an engineering school, he often had visitors from other departments or even off the street (so to speak), knocking on his door seeking a response to a query. All of his students remember that there was never any need to make an appointment with him to share their latest idea. One would simply knock on his door. There he was, sitting behind his desk, drinking tea, waving for a visitor to come in and go to the blackboard to show him what the idea was. The tea in Harold's office was a sacred ritual, enriched by a tasty pastry from a nearby Konditori shop if the theorem was taking shape or a Lemma was proved. He was also always honest - gentle, but direct. If your idea had little to no chance of working out, he would let you know right away and would explain why that was the case. He was usually right. He would also be very excited if he saw some potential in the idea. He was a fantastic Ph.D. advisor, to which many of his students can attest.

Shapiro was a model speaker and an outstanding expositor. He has given over 70 plenary talks at various national and international meetings. His guiding principle, that he repeated again and again to me was: "Less is more". Another important point that he loved to iterate was that the one and only purpose of giving a talk is to communicate something to the audience; not to show off, but actually try and explain the ideas, concepts, results, and conjectures. Make the audience your collaborators and friends, instead of instilling fear and awe in them regarding how smart the speaker is and how inadequate the audience is in comparison. In my 40+ years of being a professional mathematician, I can only think of a very few lectures matching Harold's ability to explain and engage the audience. Another pet peeve of his was clarity. I think he almost felt that by allowing some fog and unclear statements in, most likely as pretence to make the subject matter look much deeper than it was, is a betrayal of the very essence of mathematics. I strongly recommend his paper [3] on the subject, especially to young mathematicians just starting their pedagogical careers. It is a true gem.

Besides his mathematical and pedagogical input, Harold Shapiro will always be remembered by his many students, friends and collaborators for his human and scientific generosity, his love and enthusiasm for mathematics, his curiosity, his humbleness, gentleness, his readiness to discuss mathematics at all times with anyone (visitors, generally from the KTH Engineering Department, often came to his office with questions). Harold was also known for his erudition in many areas outside of mathematics (he spoke five languages: English, French, German, Swedish and Russian), he loved literature and poetry, he sang and played guitar, and always displayed kind Brooklynish/Eastern European humor. On many occasions, my father repeatedly said to me that Harold is "like Vitya Khavin" (Professor V. P. Khavin, a founder of the celebrated school at the St. Petersburg University in 1960s-1970s Russia). In my dad's set of values, no one could climb higher than that.

## 3 Harold's Mathematics: a Bird's-Eye View

(In this section I've tried to stay away from our joint work as much as possible for obvious reasons.)

### 3.1 Shapiro's M.S. Thesis, MIT - 1951 (Golay-Rudin-Shapiro Polynomials)

These polynomials, with $\pm 1$ coefficients, were independently discovered by M. J. E. Golay (1949), H. S. Shapiro (1951) and W. Rudin (1959) as complex polynomials of degree $N$ with $\pm 1$ coefficients, whose $L^{\infty}$-norm on the unit circle is dominated by their $L^{2}$-norm. More precisely, denoting by $\mathbf{T}$ the unit
circle $\{z:|z|=1\}$ in the complex plane

$$
\begin{equation*}
\sup _{z \in \mathbf{T}}\left|\sum_{0 \leq n \leq N} a_{n} e^{i n \theta}\right| \geq\left\|\sum_{0 \leq n \leq N} a_{n} e^{i n \theta}\right\|_{2}=\sqrt{n+1} \tag{1}
\end{equation*}
$$

where $a_{n} \in\{-1,1\}$, and $\|f\|_{2}=\left(\frac{1}{2 \pi} \int_{0}^{2 \pi}\left|f\left(e^{i \theta}\right)\right|^{2} d \theta\right)^{1 / 2}$ denotes the usual $L^{2}$-norm on T. Moreover, for almost every sequence $\left(a_{n}\right)_{n \geq 0}$ with each $a_{n} \in$ $\{-1,1\}$,

$$
\begin{equation*}
\sup _{z \in \mathbf{T}}\left|\sum_{0 \leq n \leq N} a_{n} e^{i n \theta}\right|=O(\sqrt{N \log N}) \tag{2}
\end{equation*}
$$

- cf. [12]. However, Golay and Shapiro discovered special polynomials $P_{N}$ of this kind for which, roughly speaking, an estimate opposite to (1) holds, i.e.,

$$
\begin{equation*}
\sup _{z \in \mathbf{T}}\left|P_{N}(z)\right| \leq C \sqrt{N+1} \tag{3}
\end{equation*}
$$

(It is conjectured that one can take $C=\sqrt{6}$, still an open question.) M. J. E. Golay was motivated in his discovery by possible applications to spectrometry and published it in [13].

Harold Shapiro arrived at what is now called the "Shapiro sequence" via intricate and technically ingenious study of the extremal problem

$$
\begin{equation*}
\max \left\{\frac{1}{2 \pi} \int_{0}^{2 \pi}\left|P\left(e^{i \theta}\right)\right| d \theta: P(z)=\sum_{0}^{N} a_{n} z^{n},\left|a_{n}\right| \leq 1\right\} \tag{4}
\end{equation*}
$$

Specifically, for $N=2^{k}-1$, the extremal polynomials satisfy the recurrence relation $\left(P_{k}:=P_{N}, N=2^{k}-1\right)$ :

$$
\begin{gather*}
P_{k+1}(z)=P_{k}\left(z^{2}\right)+z P_{k}\left(z^{2}\right) \\
P_{0}=1, P_{1}=1+z, P_{2}=1+z+z^{2}-z_{1}^{3}  \tag{5}\\
P_{3}=1+z+z^{2}-z^{3}+z^{4}+z^{5}-z^{6}+z^{7}, \ldots
\end{gather*}
$$

and enjoy the estimate (cf. to (2)!)

$$
\begin{gathered}
\left\|P_{k}\right\|_{L^{\infty}(\mathbf{T})} \leq 2^{k+1}=2\left(\operatorname{deg} P_{k}+1\right) \\
\left\|P_{k}\right\|_{L^{2}(\mathbf{T})}=\operatorname{deg} P_{k}+1, \text { i.e. } \\
\left\|P_{k}\right\|_{L^{\infty}(\mathbf{T})} \leq \sqrt{2}\left\|P_{k}\right\|_{L^{2}(\mathbf{T})}
\end{gathered}
$$

The literature nowadays dedicated to applications of these remarkable polynomials, in particular, the engineering literature, is simply enormous.

### 3.2 Harold's Ph.D. Dissertation (1953). Linear Extremal Problems for Analytic Functions - cf. [6, 7]

Recall the Hahn-Banach distance formula. Let $X$ be a Banach space, $E \subset X$ - a subspace. As always, $X^{*}=\left\{\varphi^{*}: X \rightarrow \mathbb{C}, \varphi^{*}\right.$ bounded and linear $\}$ denotes the dual space of $X, E^{\perp} \subset X$ stands for $\operatorname{Ann} E=\left\{\varphi^{*} \in X^{*}:\left.\varphi^{*}\right|_{E}=0\right\}$. The Hahn-Banach distance formula then yields for $\varphi^{*} \in X^{*}$ :

$$
\begin{equation*}
\sup _{x \in E,\|x\|_{X} \leq 1}\left|\varphi^{*}(x)\right|=\min _{\psi^{*} \in E^{\perp}}\left\|\varphi^{*}-\psi^{*}\right\|_{X^{*}} \tag{6}
\end{equation*}
$$

( $\|\cdot\|_{X^{*}}$ denotes the usual norm of linear functionals.) The second version serving at present as the point of departure for Approximation Theory is as follows. Let $w \in X$. Then,

$$
\begin{equation*}
\inf _{x \in E}\|w-x\|=\max _{\substack{\psi^{*} \in E^{\perp} \\\left\|\psi^{*}\right\|_{X^{*}} \leq 1}}\left|\psi^{*}(w)\right| \tag{7}
\end{equation*}
$$

Now apply this "functional analysis wisdom" to the following situation (for the definition, properties, etc., of Hardy spaces we refer the reader to, e.g., [14, 15]

$$
\begin{gathered}
E=\left.H^{p}\right|_{\mathbf{T}} \subset X=L^{p}(\mathbf{T}, d \theta), \quad 1 \leq p \leq \infty \\
w \in L^{q}(\mathbf{T})=X^{*}, \quad \frac{1}{p}+\frac{1}{q}=1
\end{gathered}
$$

For $f \in E, w(f)=\langle w, f\rangle:=\frac{1}{2 \pi} \int_{\Pi} w(\xi) f(\xi) d \xi$. This is the context that allowed Shapiro [7, 16] and S. Ya. Khavinson [6] unify most linear extremal problems for analytic functions in, say, Hardy spaces, into one general framework. For example, consider

$$
\begin{gathered}
w:=\sum_{j=1}^{N} \sum_{k=0}^{n_{j}} \frac{a_{k j}}{\left(\xi-z_{j}\right)^{k+1}},\left|z_{j}\right|<1, \quad j=1, \ldots, N, \\
n_{j} \text { are some positive integers. }
\end{gathered}
$$

The extremal problem (6) corresponding to (8) is then finding

$$
\begin{equation*}
\sup _{\|f\|_{H^{p}} \leq 1}\left|\sum_{j=1}^{N} \sum_{k=1}^{n_{j}} a_{k j} f^{(k)}\left(z_{j}\right)\right| . \tag{9}
\end{equation*}
$$

It is rather general, embracing many particular problems studied by ad hoc methods for specific values of $z_{j}, a_{k j}$ by L. Fajer, O. Sasz, F. Riesz, ..., to name just a few. (For more detailed history, we refer the reader to the Notes in $[14,15,17]$.) Why do (6)-(7) turn out to be so efficient?

The celebrated F. and M. Riesz Theorem (cf. [17-19]) provides the description of the annihilator of $E$ :

$$
\left(H^{p}\right)^{\perp}=H_{0}^{q}=\left\{f \in H^{q}: f(0)=0\right\} \text { on } \mathbf{T}, \frac{1}{p}+\frac{1}{q}=1
$$

A routine proof of the existence of the extremals $f^{*}, \psi^{*}$ plus the case of equality in Hölder's inequality ([7, 14, 15, 17]) leads to the key identity holding on $\mathbf{T}$ for the extremals:

$$
\begin{equation*}
f^{*}(\xi)\left(w(\xi)-\psi^{*}(\xi)\right) d \xi=\left|w(\xi)-\psi^{*}(\xi)\right| d \theta \tag{10}
\end{equation*}
$$

The key point of (10) is that a certain "analytic differential" in the LHS turns out to be non-negative on $\mathbf{T}$. The Schwarz reflection principle then does the trick and the detailed structural formulas for the extremal functions $f^{*}$ and $\psi^{*}$ easily follow even in the most general problem (9). I will finish on a funny (curious?) note. I've already noted that Harold never met Rogosinski. Even more surprising, perhaps, is that neither Harold (while writing his Ph.D. thesis in 1953), nor my father in his M.S. thesis (equivalent) [6], actually knew the Hahn-Banach theorem! Both of them proved equalities ((6), (7)) in the context of (9) by sheer tour-de-force function theory arguments (pretty awful and technical at that)! A good illustration to a well-known remark by Allen Shields: "Functional Analysis moves us horizontally, pointing out where to dig, while Classical Analysis allows us to dig vertically."

### 3.3 Quadrature Domains (1976) D. Aharonov and H. S. Shapiro Work

In the early 1970s D. Aharonov and H. Shapiro tried to solve the "minimal area" extremal problem in the celebrated class $S:=\left\{f(z)=z+a_{2} z+\cdots\right\}$ of univalent in the unit disk $\mathbb{D}:=\{|z|<1\}$ functions. The solution at the time escaped them and was only obtained, almost 25 years later, with the major contribution from Alex Solynin - cf. [20, 21]. Yet, as an early by-product (a reward for very prolonged and hard work?) came the notion of a quadrature domain and the Schwarz function (christened by Ph. Davis [22], also cf. [2, 23]). A quadrature domain (QD) in the complex plane is a domain $\Omega \in \mathbb{C}$, such that there exist points $\left\{z_{j}\right\}_{1}^{N}$ in $\Omega$ and coefficients $\left\{c_{k j}\right\}_{k=1}^{n_{j}}$ so that the quadrature identity

$$
\begin{equation*}
\int_{\Omega} f d A=\sum_{j=1}^{N} \sum_{k=1}^{n_{j}} c_{k j} f^{(k)}\left(z_{j}\right) \tag{11}
\end{equation*}
$$

holds for all analytic functions $f$ in $\Omega$, integrable with respect to the area measure. As was shown by Dov and Harold in [24], such domains are smoothly bounded. In fact, their boundaries are algebraic (!) - [24]. Hence, the boundary $\partial \Omega$ can be written in the form $\{\bar{z}=S(z)\}$, where $S(z)$, called the Schwarz function (SF), is analytic in a tubular neighborhood of $\partial \Omega$. As was shown in
[24], (11) is equivalent to $S(z)$ having meromorphic continuation inside $\Omega$ with poles at $z_{j}$ of order $n_{j}$.

A much more subtle and difficult work of M. Sakai [25] established the existence of the Schwarz function for a much larger class of QD "in a wide sense" - cf. [23] for detailed references.

Of course, as almost always happens in mathematics, both concepts were noted before. Let's just note in passing (for a curious reader) that G. Herglotz (1914), R.-L. Wavre ( $\sim 1930$ ), E. Schmidt ( $\sim 1912$ ) and even H. Bruns ( $\sim$ 1871) all noted the idea of QD in some form. (Although not a household name, Bruns, a student of K. Weierstrass, had 27 Ph.D. students among which were F. Hausdorff and H. König.)

The problem these authors set out to solve, some (G. Herglotz, E. Schmidt, R. Warre) in general, some (H. Bruns) in particular (for a torus) was this:

Question. How far into $\Omega$ does the gravitational (logarithmic in $2 D$ ) potential of the uniform mass distribution extend harmonically?

The answer is: $\Omega \backslash\{$ singularities of the Schwarz function (in 2D) . An answer in higher dimensions is much more subtle - cf. [10, 11].

I must add that it was Harold (fluent in German) who had discovered (almost entirely forgotten) Herglotz' memoir that, in spite of having won a prize, appeared in 1914, at the brink of WWI.

These concepts and results turned out to be closely related to many fundamental processes studied by mathematical physicists and known under the name of Hele-Shaw processes (including evolution of oil spills, growth of tumors, diffusion limited aggregation growths (DLA processes) and plenty of other applications). A Google search shows several hundreds citations for the Aharanov-Shapiro paper; 83 citations are on MathSciNet.

Working an extension of the concepts of QD and SF to dimensions greater than 2 prompted Harold's interest in studying global behavior of solutions of linear holomorphic PDEs.

### 3.4 Harold's Wide-Scope Program of Studying Global Behavior of Solutions of Holomorphic PDEs

It started naïvely in [10], essentially in ignorance of [the] monumental work of Hörmander, Leray, Tsumo, Hamada and many, many others, motivated by trying to extend Herglotz' question on analytic continuation of potentials to higher dimensions. Revisit Herglotz' question as to how far inside a bounded domain $\Omega \subset \mathbb{R}^{n}, n \geq 2$, the Newtonian potential

$$
\begin{equation*}
u_{\Omega}=c_{n} \int_{\Omega} \frac{d y}{|x-y|^{n-2}} \tag{12}
\end{equation*}
$$

( $c_{n}$ is the dimensional constant) extends harmonically, if we assume, e.g., that the boundary $\Gamma$ of $\partial \Omega$ is an algebraic surface. The answer is $\Omega \backslash\left\{\operatorname{sing} u_{\Gamma}\right\}$,
where $u_{\Gamma}$ ("the modified Schwarz potential" of $\Omega$ ) is defined as a solution of the Cauchy problem

$$
\left\{\begin{array}{l}
\Delta u_{\Gamma}=1 \text { near } \Gamma ; \quad\left(\Delta:=\sum_{1}^{n} \frac{\partial^{2}}{\partial x_{j}^{2}}\right)  \tag{13}\\
\left.u_{\Gamma}\right|_{\Gamma}=\left.\operatorname{grad} u_{\Gamma}\right|_{\Gamma}=0
\end{array}\right.
$$

- cf. [2, 10, 11].

Now, Harold's program stems from several simple-minded questions:
Q. 1 What if we replace the uniform density in (12) by an arbitrary, say, polynomial density $p(x)$ and consider

$$
\begin{equation*}
u_{p, \Omega}(x)=c_{n} \int_{\Omega} \frac{p(y) d y}{|x-y|^{n-2}}, \quad n>2 . \tag{14}
\end{equation*}
$$

The conjecture (cf. [10, 11, 23] and references therein), still unproven in general, is that the answer is the same: $\Omega \backslash \operatorname{sing} u_{\Gamma}$ (!). It has been proven true in many specific cases, e.g., ellipsoids. The latter case under closer consideration, led to a rather subtle question.
Q. 2 Why do the singularities differ (algebraic vs. transcendental) for oblate and prolate spheriods. From that point on, Shapiro's "high ground master strokes" program takes off. For example,
Q. 3 If we solve the Dirichlet problem (vs. Cauchy problem)

$$
\begin{gather*}
\Delta u=0 \text { in } \Omega \\
\left.u\right|_{\Gamma}=\left.p\right|_{\Gamma}, \text { a polynomial, } \Gamma:=\partial \Omega, \tag{15}
\end{gather*}
$$

where would singularities of $u$ be outside $\Omega$ and how are they related to the geometry of $\Gamma$ ?
Q. 4 Similarly, where are the singularities of the eigenfunctions of the Laplacian lie outside $\Gamma$ ?
Q. 5 Is the Schwarz reflection principle bound to $2 D$ ?

Most of these and many other related questions (cf. [11]) are still open, in a large part. Only Q. 5 has been answered more or less completely - see the discussion and references in [11]. But the program is attractive, incorporates a beautiful interplay between analysis, algebra and geometry, and surely will be catching the eyes of young mathematicians in the years to come.

## 4 Some Final Thoughts

Harold possessed a magic ability to create an atmosphere of intellectual excitement while working on a problem and, at the same time, never letting his co-authors feel any inferiority. To the contrary, knowing very well his powers as an analyst, his abilities, his technical skills, he was never arrogant, rather
humble, one might say, always encouraging input from whoever he was working with. Personally, I "lucked out" having the opportunity to work closely with him for most of my mathematical life.

In spite of his deteriorating health - not being able to write and then, later, type - Harold never stopped being a Mathematician. Two years ago, during our last meeting, he showed me his proof of the prime factorization theorem without using the Euclidean algorithm. An excellent exercise, incidentally, for all teaching Elementary Number Theory courses.

Stealing from E. Schmidt, one can affirm that with his departure "Mathematics lost one of it's true knights, and I lost my best friend."

And still (most likely, for the rest of my life), when I first open my e-mail in the morning, for a split second I am hoping to find this in the message from the server: "You have $x$ new messages, the most recent one is from Harold Shapiro..."

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