

Analytic Functions in Smirnov Classes E^p with Real Boundary Values II

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Abstract Multiply connected Smirnov domains with non-smooth boundaries may admit non-trivial functions of Smirnov class E^p with real boundary values for certain $p \geq 1$. This paper describes the particular geometric boundary characteristics of multiply connected Smirnov domains that make the existence of such functions possible. This extends the similar results in [2] obtained for simply connected domains.

Keywords Smirnov classes · Hardy classes · boundary values

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1 Introduction.

Let G be a finitely connected domain in the complex plane bounded by Jordan rectifiable curves $\gamma_1, \dots, \gamma_n$ and let $\Gamma = \bigcup_{i=1}^n \gamma_i$. Recall the following definition (cf. [1], Ch. 10).

Definition 1.1. An analytic function $f(z)$ in G is said to belong to the class $E^p(G)$ for $p : 0 < p \leq \infty$ if there exists a sequence of rectifiable curves $\{\Gamma_i\}$ in G converging

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to Γ such that

$$\limsup_{i \rightarrow \infty} \int_{\Gamma_i} |f(z)|^p |dz| < \infty.$$

Define $\varphi : K \rightarrow G$ to be the conformal mapping of an n -connected circular domain K onto G . Let $l_1, \dots, l_n$ be the circular boundary curves of K . Then $\varphi_i : K_i \rightarrow G_i$, $i = 1, \dots, n$, will denote the conformal mapping of the domain bounded by l_i containing K onto the domain bounded by γ_i and containing G . We shall call G a Smirnov domain, $G \in S$, if $\varphi(w)$ is a purely outer function (cf. [1] Ch. 10, [7]). S. Ya. Khavinson and Tumarkin have proven that this definition is also equivalent to $\varphi_i(w)$ being purely outer functions for all $i = 1, \dots, n$ ([8]).

It is well known that for $p \geq 1$ analytic functions in Smirnov classes E^p can be represented as Cauchy integrals of their non-tangential boundary values. If all of the boundary curves of G are smooth, then $E^p(G) = H^p(G)$, that is the Smirnov class $E^p(G)$ coincides with the Hardy class $H^p(G)$ as sets. Since functions in H^p can be represented as Poisson integrals of their boundary values (cf. [3] Ch. 4, [5]), this implies that in smoothly bounded domains any $f(z) \in E^p(G)$ with real boundary values is a constant.

In [7] D. Khavinson began the study of the existence of non-constant analytic functions of Smirnov class E^p with real boundary values. These results rested upon the smoothness of the boundary of G . It is proved in [7] that if G is completely non-Smirnov, i.e., $G_i \notin S$, $\forall i = 1, \dots, n$, then there does exist a non-constant analytic function $f(z) \in E^p(G)$ for each p with $0 \leq f(z) \leq 1$ a.e. on Γ . Conversely, if $G \in S$ and $f(z) \in E^1(G)$ with $f(\zeta)$ real valued and bounded on Γ then $f(z) \equiv \text{const}$. Furthermore, there is an example due to S. Ya. Khavinson in [7] showing that when the condition of boundedness is removed and G is a simply connected domain with a cusp, then there does exist a non-constant $f(z) \in E^1(G)$ with real boundary values.

In our previous paper [2] we classified those simply connected Smirnov domains that admit non-constant analytic functions of class E^p with real boundary values. In this paper we will explore the case when G is a finitely connected domain and discuss the boundary characteristics of G that are sufficient for non-constant analytic functions with real boundary values in class E^p for $p \geq 1$ to exist.

This paper is organized as follows. The second and fourth sections will contain preliminary results that are necessary for the fifth section in which we will discuss the main results of the paper. In the second section we show that if an analytic function has real boundary values everywhere except at finitely many points, where it has polar singularities, then it can be represented by a linear combination of the derivatives of Schwarz type kernels. This result (Cor. 2.4) will prove instrumental in the following sections. The third section explores a physical interpretation associated with the results of the second section. The fourth section focuses on harmonic functions in finitely connected domains that can be represented by the Green-Stieltjes integral with a measure consisting of a finite sum of point masses. In this section we will discuss the necessary and sufficient conditions under which such a function will have a single-valued conjugate. The fifth section describes the geometric boundary characteristics for which non-constant analytic functions of Smirnov class E^p exist. The last section of this paper presents an open question regarding an extension of Neuwirth-

Newman's theorem to functions in Smirnov classes in finitely connected Smirnov domains.

2 Functions representable as a linear combination of the derivatives of Schwarz kernels.

Let $\mathbb{D}$ be the unit disk and $\mathbb{T} = \partial\mathbb{D}$. Let Ω be a simply connected domain in the complex plane bounded by a real analytic curve. We shall define $R_\Omega(z, \zeta)$ to be the Schwarz kernel of Ω . This means that

$$R_\Omega(z, \zeta) = \frac{\partial P(z, \zeta)}{\partial n_\zeta}, \text{ where, } P(z, \zeta) = g(z, \zeta) + i\tilde{g}(z, \zeta),$$

$z \in \Omega$, $\zeta \in \partial\Omega$, $\frac{\partial}{\partial n_\zeta}$ is the inner normal derivative, and g and $\tilde{g}$ are respectively the Green function of Ω with pole at z and its harmonic conjugate. For example, when $\Omega = \mathbb{D}$ then $R_\mathbb{D}(z, \zeta) = \frac{1}{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z}$. Let $\frac{\partial}{\partial \tau}$ denote the tangential derivative.

Lemma 2.1. *Suppose $f = u + iv$ is analytic in $\mathbb{D}$, continuous in $\overline{\mathbb{D}} \setminus \{1\}$, and $v \equiv 0$ on $\mathbb{T} \setminus \{1\}$. If $|f| = O\left(\frac{1}{|1-z|^k}\right)$ near 1, then*

$$f(z) = \sum_{n=0}^{k-1} \lambda_n \frac{\partial^n}{\partial \tau^n} iR_\mathbb{D}(z, 1) + c,$$

where $\lambda_{k-1}, \dots, \lambda_0, c \in \mathbb{R}$.

Proof. We proceed by induction. Suppose $k = 1$. Then by the generalized Schwarz reflection principle f extends to be analytic in $\mathbb{C} \setminus [1, \infty)$. So $|f(z)| = O\left(\frac{1}{|1-z|}\right)$ in $\overline{\mathbb{D}}$. Consider $F = f \circ \tau$, where $\tau^{-1}(z) = \frac{1+z}{1-z}$ is the conformal mapping of $\mathbb{D}$ onto the right half plane. Then $|F(z)| = O(|z|)$ and $\text{Im } F(z) = 0$ on $\{it : t \in \mathbb{R}\}$. By the Schwarz reflection principle F extends to be an entire function. Therefore $F(z) = Az + B$, where $A, B \in \mathbb{C}$. This implies that

$$f(z) = A \frac{1+z}{1-z} + B.$$

Since $\text{Im } f(z) = 0$ on $\mathbb{T} \setminus \{1\}$, then $A = \lambda i$ and $\lambda, B \in \mathbb{R}$. Suppose the statement holds for some k . Again, looking at $f \circ \tau$, we conclude that f is a rational function with a pole at 1 of order k . Since

$$\left| \frac{\partial^k}{\partial \theta^k} R_\mathbb{D}(z, 1) \right| = O\left(\frac{1}{|1-z|^{k+1}}\right),$$

then $\lambda_k \in \mathbb{R} \setminus \{0\}$ can be chosen so that

$$\left| f(z) - \lambda_k \frac{\partial^k}{\partial \theta^k} iR_\mathbb{D}(z, 1) \right| = O\left(\frac{1}{|1-z|^j}\right),$$

where $j \leq k$. Since

$$\operatorname{Im} \left(f(z) - \lambda_k \frac{\partial^k}{\partial \theta^k} iR_{\mathbb{D}}(z, 1) \right) = 0$$

on $\mathbb{T} \setminus \{1\}$, then

$$f(z) - \lambda_k \frac{\partial^k}{\partial \theta^k} iR_{\mathbb{D}}(z, 1) = \sum_{n=0}^{k-1} \lambda_n \frac{\partial^n}{\partial \tau^n} iR_{\mathbb{D}}(z, 1) + c,$$

where $\lambda_{k-1}, \dots, \lambda_0, c \in \mathbb{R}$. Therefore $f(z) = \sum_{n=0}^k \lambda_n \frac{\partial^n}{\partial \tau^n} iR_{\mathbb{D}}(1, z) + c$. $\square$

Remark 2.2. If an analytic function $f(z)$ in $\mathbb{D}$ has real boundary values on $\mathbb{T} \setminus \{1\}$ and $|f(z)| = O\left(\frac{1}{|1-z|^\beta}\right)$ where $\beta \in \mathbb{R}_+$ then $|f(z)| = O\left(\frac{1}{|1-z|^{\lfloor \beta \rfloor}\right)$. To see that this is true note that by the generalized Schwarz reflection principle f extends to be analytic in $\mathbb{C} \setminus [0, \infty)$ so $|f(z)| = O\left(\frac{1}{|1-z|^\beta}\right)$ in $\overline{\mathbb{D}}$. Letting $F = f \circ \tau$ where $\tau^{-1}(z) = \frac{1+z}{1-z}$ is the conformal mapping of $\mathbb{D}$ onto the right half plane, then $|F(z)| = O(|z|^\beta)$ and $\operatorname{Im} F(z) = 0$ on $\{it : t \in \mathbb{R}\}$. By the Schwarz reflection principle F extends to be an entire function. Therefore, by an easy corollary of Liouville's Theorem, $F(z) = a_k z^k + \dots + a_1 z + a_0$, where $k \leq \lfloor \beta \rfloor$. Thus

$$f(z) = a_k \left(\frac{1+z}{1-z} \right)^k + \dots + a_1 \left(\frac{1+z}{1-z} \right)^{k-1} + a_0.$$

Therefore $|f(z)| = O\left(\frac{1}{|1-z|^{\lfloor \beta \rfloor}\right)$.

Corollary 2.3. If $f = u + iv$ is analytic in Ω , where Ω is a simply connected domain with analytic boundary, $v \equiv 0$ on $\partial\Omega \setminus \{\zeta\}$, continuous in $\bar{\Omega} \setminus \{\zeta\}$, and $|f(z)| = O\left(\frac{1}{|\zeta-z|^k}\right)$ near ζ then

$$f(z) = \sum_{n=0}^{k-1} \lambda_n \frac{\partial^n}{\partial \tau^n} iR_{\Omega}(z, \zeta) + c,$$

where $\lambda_{k-1}, \dots, \lambda_0, c \in \mathbb{R}$.

Proof. We proceed by induction. Let $\varphi : \mathbb{D} \rightarrow \Omega$ be the conformal mapping of $\mathbb{D}$ onto Ω , where $\varphi(1) = \zeta$. Suppose $k = 1$. Then $f \circ \varphi$ satisfies the conditions of Lemma 2.1 and so

$$(f \circ \varphi)(w) = \lambda_0 iR_{\mathbb{D}}(w, 1) + c.$$

Thus

$$f(z) = \lambda_0 iR_{\Omega}(z, \zeta) + c.$$

Assume that for some k the statement holds. Then, for $k+1$, $|f(z)| = O\left(\frac{1}{|\zeta-z|^{k+1}}\right)$, $f \circ \varphi$ is as in the proof of Lemma 2.1, a rational function of $\frac{1+w}{1-w}$ and degree k , and $\left| \frac{\partial^k}{\partial \tau^k} R_{\Omega}(z, \zeta) \right| = O\left(\frac{1}{|\zeta-z|^{k+1}}\right)$ near ζ . Choose $\lambda_k \in \mathbb{R} \setminus \{0\}$ so that

$$\left| f(z) - \lambda_k \frac{\partial^k}{\partial \tau^k} iR_{\Omega}(z, \zeta) \right| = O\left(\frac{1}{|\zeta-z|^j}\right),$$

where $j \leq k$. Since $\operatorname{Im} \left(f(z) - \lambda_k \frac{\partial^k}{\partial \tau^k} iR_\Omega(z, \zeta) \right) = 0$ on $\partial\Omega \setminus \{\zeta\}$ and $f(z) - \lambda_k \frac{\partial^k}{\partial \tau^k} iR_\Omega(z, \zeta)$ is analytic in Ω , then

$$f(z) - \lambda_k \frac{\partial^k}{\partial \tau^k} iR_\Omega(z, \zeta) = \sum_{n=0}^{k-1} \lambda_n \frac{\partial^n}{\partial \tau^n} iR_\Omega(z, \zeta) + c,$$

where $\lambda_{k-1}, \dots, \lambda_0, c \in \mathbb{R}$. Therefore, $f(z) = \sum_{n=0}^k \lambda_n \frac{\partial^n}{\partial \tau^n} iR_\Omega(z, \zeta) + c$. $\square$

The next corollary shows that Lemma 2.1 and Corollary 2.3 are, in fact, local properties.

Corollary 2.4. *Let Ω be as above. Let u be harmonic in Ω and smooth in $\overline{\Omega} \setminus \{\zeta_0\}$. If on the arc $\gamma_\varepsilon \subset \partial\Omega$ where $\zeta_0 \in \gamma_\varepsilon$, $u(\zeta) \equiv 0$ on $\gamma_\varepsilon \setminus \{\zeta_0\}$ and $|u(z)| = O\left(\frac{1}{|\zeta_0 - z|^k}\right)$ near ζ_0 then*

$$u(z) = \int_{\partial\Omega \setminus \gamma_\varepsilon} \frac{\partial}{\partial n_\zeta} g(z, \zeta) u(\zeta) ds + \sum_{j=0}^{k-1} \lambda_j \frac{\partial^j}{\partial \tau^j} \left(\frac{\partial}{\partial n_{\zeta_0}} g(z, \zeta_0) \right),$$

where $\lambda_{k-1}, \dots, \lambda_0 \in \mathbb{R}$.

Proof. Let

$$u_1(z) = \int_{\partial\Omega \setminus \gamma_\varepsilon} \frac{\partial}{\partial n_\zeta} g(z, \zeta) u(\zeta) ds$$

and let v be the harmonic conjugate of $u - u_1$. Let $f = v + i(u - u_1)$. Then $|f| = O\left(\frac{1}{|\zeta_0 - z|^k}\right)$ and satisfies the hypothesis of Corollary 2.3. Therefore,

$$f(z) = \sum_{j=0}^{k-1} \lambda_j \frac{\partial^j}{\partial \tau^j} iR_\Omega(z, \zeta_0) + c,$$

where $c, \lambda_0, \dots, \lambda_{k-1} \in \mathbb{R}$. Since $\operatorname{Re} R_\Omega(z, \zeta_0) = \frac{\partial}{\partial n_{\zeta_0}} g(z, \zeta_0)$,

$$(u - u_1)(z) = \sum_{j=0}^{k-1} \lambda_j \frac{\partial^j}{\partial \tau^j} \left(\frac{\partial}{\partial n_{\zeta_0}} g(z, \zeta_0) \right).$$

Therefore,

$$u(z) = \int_{\partial\Omega \setminus \gamma_\varepsilon} \frac{\partial}{\partial n} g(z, \zeta) u(\zeta) ds + \sum_{j=0}^{k-1} \lambda_j \frac{\partial^j}{\partial \tau^j} \left(\frac{\partial}{\partial n_{\zeta_0}} g(z, \zeta_0) \right),$$

where $\lambda_{k-1}, \dots, \lambda_0 \in \mathbb{R}$. $\square$

Remark 2.5. Note that if a harmonic function u satisfies the hypothesis of Corollary 2.4 with the exception that $|u(z)| = O\left(\frac{1}{|\zeta_0 - z|^\beta}\right)$ near ζ_0 where $\beta \in \mathbb{R}_+$, then $|u(z)| = O\left(\frac{1}{|\zeta_0 - z|^{\lfloor \beta \rfloor}}\right)$. This statement follows now from Remark 2.2 and the proof of Corollary 2.4.

3 A digression: the potential function of a fluid flow.

Consider a two-dimensional incompressible fluid in a bounded simply connected domain Ω in the complex plane. Let $\Phi = \phi + i\psi$, be the fluid potential function of this system. This function describes the nature of the fluid flow at a fixed time. More precisely,

ϕ is the velocity potential of the fluid flow.

$\nabla\phi = \mathbf{v} = (u(x, y), v(x, y))$ is the velocity of the fluid.

$\nabla^2\phi = \nabla \cdot \mathbf{v}$ is the divergence of the fluid flow.

ψ is known as the stream function. The level curves of ψ are the stream lines of the fluid where the value of the level curve denotes the flux of the fluid across the curve.

$\nabla\psi = *\mathbf{v} = (-v(x, y), u(x, y))$ is the force on the rotational components (vortices) of the fluid (cf. [4]).

$\nabla^2\psi = \nabla \times \mathbf{v} = -\omega$ is the vorticity of the fluid.

For a general reference regarding the fluid potential function we direct the reader to [10] Ch. 3, Sec. 2. A function of the type described in Lemma 2.1 and Corollary 2.3 corresponds to a fluid flow in which 2^{k-1} sources and 2^{k-1} sinks of increasing strength approach each other from opposite sides of the boundary of Ω to form “multipoles”. Let us take up the case when $k = 1$, Φ is analytic in the right half plane, and the source and sink meet at $\zeta = 0$ along the real axis. Here we will let $\psi_+(z) = \frac{-1}{2\varepsilon} \log|z - \varepsilon|$ represent the source located at the point $z = \varepsilon$ and $\psi_-(z) = \frac{1}{2\varepsilon} \log|z + \varepsilon|$ represent the sink located at the point $z = -\varepsilon$ (cf. [10] Ch. 3, Sec. 2). Define

$$\psi_\varepsilon(z) := \psi_+(z) + \psi_-(z) = \frac{1}{2\varepsilon} (\log|z + \varepsilon| - \log|z - \varepsilon|).$$

Then we have that

$$\begin{aligned} \psi_\varepsilon(z) &= \frac{1}{4\varepsilon} [\log(z + \varepsilon) - \log(z - \varepsilon) + \log(\bar{z} + \varepsilon) - \log(\bar{z} - \varepsilon)] \\ &= \frac{1}{4\varepsilon} \left[\log\left(1 + \frac{\varepsilon}{z}\right) - \log\left(1 - \frac{\varepsilon}{z}\right) + \log\left(1 + \frac{\varepsilon}{\bar{z}}\right) - \log\left(1 - \frac{\varepsilon}{\bar{z}}\right) \right]. \end{aligned}$$

Since

$$\log(1 - z) = - \sum_{n=1}^{\infty} \frac{z^n}{n},$$

then

$$\psi_\varepsilon(z) = \frac{1}{4\varepsilon} \left[\frac{2\varepsilon}{z} + O(\varepsilon^3) + \frac{2\varepsilon}{\bar{z}} + O(\varepsilon^3) \right] = \frac{1}{2} \left[\frac{1}{z} + \frac{1}{\bar{z}} + O(\varepsilon^2) \right].$$

Letting $\varepsilon \rightarrow 0$ we have that

$$\psi(z) = \frac{1}{2} \left[\frac{1}{z} + \frac{1}{\bar{z}} \right] = \frac{\operatorname{Re}(z)}{|z|^2}.$$

This implies that

$$\phi(z) = \frac{\operatorname{Im}(z)}{|z|^2},$$

and thus

$$\Phi(z) = \frac{i}{z}.$$

Thus, $\operatorname{Im} \Phi(z) = 0$ on $\{it : t \in \mathbb{R} \setminus \{0\}\}$ and $|\Phi(z)| = \frac{1}{|z|}$. When $k = 1$, Ω is the unit disk, and $\zeta = 1$ then, using the conformal map $w = \tau(z) = \frac{1-z}{1+z}$, we see that $\Phi \circ \tau^{-1} = i \frac{1+w}{1-w}$ which is as to be expected by Lemma 2.1.

Essentially, when $k = 1$, the fluid potential function models a dipole which lies on the boundary of Ω . This situation is very similar to the case in electrodynamics in which an electric dipole is formed on a boundary by the meeting of two point charges of opposite and increasing strength. The difference lies in the assignment of the real and imaginary parts of the respective functions. Yet, similar to the electrodynamic case, if $k > 1$, then Φ models 2^k multipoles that are situated at the point ζ on the boundary of Ω .

4 Green-Stieltjes integrals with a single-valued conjugate.

Let G be an n -connected domain in $\mathbb{C}$, $n \geq 2$, with analytic boundary $\Gamma = \bigcup_{i=1}^n \gamma_i$. Let $u(z)$ be a harmonic function in G that can be represented by the Green-Stieltjes integral with Borel measure μ . That is,

$$u(z) = \frac{1}{2\pi} \int_{\Gamma} \frac{\partial g(z, \zeta)}{\partial n_{\zeta}} d\mu(\zeta),$$

where $g(z, \zeta)$ is the Green's function of G with pole at z and $\frac{\partial}{\partial n_{\zeta}}$ is the derivative at ζ in the direction of the inner normal. If $v(z)$ is the harmonic conjugate of $u(z)$, then $v(z)$ may be multivalued. The period of $v(z)$ along each γ_i is (cf. [3] Ch. 4, [6])

$$\Delta_{\gamma_i} v = - \int_{\Gamma} \frac{\partial \omega_i(\zeta)}{\partial n_{\zeta}} d\mu(\zeta),$$

where $\omega_i(\zeta) = 1$ for $\zeta \in \gamma_i$ and $\omega_i(\zeta) = 0$ for $\zeta \in \Gamma \setminus \gamma_i$ is the harmonic measure of γ_i . So if the system of equations

$$\sum_{j=1}^{n-1} \lambda_j \int_{\gamma_j} \frac{\partial \omega_i(\zeta)}{\partial n_{\zeta}} d\mu(\zeta) = 0, \quad i = 1, \dots, n-1, \quad \lambda_j \in \mathbb{R}, \quad (4.1)$$

has a non-trivial solution then the function

$$\bar{u}(z) = \frac{1}{2\pi} \int_{\Gamma} \frac{\partial g(z, \zeta)}{\partial n_{\zeta}} d\bar{\mu}(\zeta) \quad \text{where} \quad d\bar{\mu} \Big|_{\gamma_j} = \lambda_j d\mu \Big|_{\gamma_j}$$

has a single-valued conjugate.

Proposition 4.1. *Let $u(z)$ be a harmonic function in G such that $u(z)$ is represented by the Green-Stieltjes integral with $d\mu = \sum_{j=1}^m \delta_{\zeta_j}$, where δ_{ζ_j} is the unit point mass at ζ_j . If $m \geq n$ then there exist real numbers $\lambda_1, \dots, \lambda_m$ that are not all equal to zero such that the function*

$$\bar{u}(z) = \lambda_1 \frac{\partial g(z, \zeta_1)}{\partial n_{\zeta_1}} + \dots + \lambda_m \frac{\partial g(z, \zeta_m)}{\partial n_{\zeta_m}}$$

has a single valued conjugate.

Proof. Consider the matrix A associated with the system of equations (4.1). Then

$$A = \|a_{ij}\| \text{ where } a_{ij} = \frac{\partial \omega_i(\zeta_j)}{\partial n_{\zeta_j}}. \quad (4.2)$$

Let $\lambda = (\lambda_1, \dots, \lambda_m) \neq 0$. Then $\bar{u}$ will have a single-valued conjugate if $A\lambda = 0$. Since $A : \mathbb{R}^m \rightarrow \mathbb{R}^{n-1}$ is a linear map given by the matrix A , it suffices to show that $\ker(A) \neq \{0\}$. Since $m = \dim(\ker(A)) + \dim(\text{rank}(A))$ and $\dim(\text{rank}(A)) \leq n-1$ then $\dim(\ker(A)) \geq 1$. Therefore $\ker(A) \neq \{0\}$. $\square$

The following example illustrates a case when $\text{rank}(A) = n-1$.

Example 4.2. If $m \geq n$ and there exists j where $1 \leq j \leq m$ such that $\zeta_j \in \gamma_i$ for each $i = 1, \dots, n$ then there exist $\lambda_1, \dots, \lambda_m$ not all equal to zero such that $\bar{u}$ has a single-valued conjugate. Without loss of generality we may assume that $\zeta_1 \in \gamma_1, \dots, \zeta_{n-1} \in \gamma_{n-1}$. Then by Lemma 1 in [6] the minor of the matrix A in (4.2) formed by the first $(n-1)$ columns and first $(n-1)$ rows has rank $(n-1)$ and therefore, $A\lambda = 0$ has a non-zero solution.

The converse of Proposition 3.1 is not true. It can happen that $m < n$ and yet $\bar{u}$ has a single-valued conjugate.

Example 4.3. In [9] S. Ya. Khavinson provides the following example: Let G be the domain bounded by the circles $\gamma_1 = \{z : |z| = 1\}$, $\gamma_2 = \{z : |z + 1/2| = 1/4\}$, and $\gamma_3 = \{z : |z - 1/2| = 1/4\}$. Let $\zeta_1 = -1/2 + 1/4i$ and $\zeta_2 = -1/2 - 1/4i$. Then $\{\zeta_1, \zeta_2\} \subset \gamma_2$ and ζ_1 is symmetric to ζ_2 with respect to the horizontal diameters of γ_i , $i = 1, 2, 3$. Thus by symmetry, $\frac{\partial \omega_i(\zeta_1)}{\partial n_{\zeta_1}} = \frac{\partial \omega_i(\zeta_2)}{\partial n_{\zeta_2}}$ for $i = 1, 2$. So the two rows of A are linearly dependent, and the equations

$$\lambda_1 \frac{\partial \omega_i(\zeta_1)}{\partial n_{\zeta_1}} + \lambda_2 \frac{\partial \omega_i(\zeta_2)}{\partial n_{\zeta_2}} = 0, \quad i = 1, 2$$

have non-zero solutions $\lambda_1 = -\lambda_2$.

5 Analytic functions in class E^p with real boundary values.

Let G , K , $\varphi(w)$, and $\varphi_i(w)$ be defined as in the introduction. Recall the Keldysh-Lavrentiev theorem for multiply connected domains (cf. [7], [8]).

Theorem 5.1.

$$f(z) \in E^p(G) \text{ if and only if } f(\varphi(w))[\varphi'(w)]^{1/p} \in H^p(K)$$

Let us also recall the following Lemma (cf. [7],[8]).

Lemma 5.2. *There exist constants c_1 and c_2 such that the inequality*

$$0 < c_1 \leq \left| \frac{\varphi'(w)}{\varphi_i'(w)} \right| \leq c_2 < \infty$$

holds near l_i for all $i = 1, \dots, n$.

The proof follows at once after one observes that $\varphi \circ \varphi_i^{-1}$ preserves l_i , is one to one near it, and hence extends analytically across l_i .

We emphasize here that in Lemma 5.2 the boundary of G is assumed to be merely Jordan and rectifiable.

Theorem 5.3. *Let G be an n -connected domain in $\mathbb{C}$ bounded by the curves $\gamma_1, \dots, \gamma_n$ which are real analytic except at the points $z_1, \dots, z_m \in \Gamma$, where there are corners with interior angles $\alpha_1, \dots, \alpha_m$, $\pi < \alpha_j \leq 2\pi$ for all $j = 1, \dots, m$. Then every $f(z) \in E^p(G)$ with real boundary values is a constant whenever $p \geq \frac{\alpha}{\pi}$, where $\alpha = \min\{\alpha_q : f \text{ is unbounded near } z_q, q \in \{1, \dots, m\}\}$.*

Note that an interior angle equal to 2π means that there is a cusp on the boundary pointing into the domain. It is readily seen (as noted in [2] for simply connected domains) that the case $\alpha_j \leq \pi$ for all j , yields no non-constant $E^1(G)$ functions with real boundary values. Hence we shall omit it.

Proof. Suppose that there exists such an $f(z) \in E^{\alpha/\pi}(G)$. Then by the Keldysh-Lavrentiev theorem for multiply connected domains

$$f(\varphi(w))[\varphi'(w)]^{\pi/\alpha} \in H^{\alpha/\pi}(K).$$

Let $w_j \in \partial K$ such that $\varphi(w_j) = z_j$ for each $j = 1, \dots, m$. Assume that $\varphi_i(w_j) = z_j$ whenever $z_j \in \gamma_i$. Then, cf. [12], Ch. 3, Sec. 4, near each circular arc of l_i containing only w_j , we have

$$\varphi_i'(w) = (w - w_j)^{\alpha_j/\pi - 1} g_j(w),$$

where $g_j(w)$ is bounded away from 0 and ∞ . Then

$$m_j |w - w_j|^{\alpha_j/\alpha - \pi/\alpha} \leq |\varphi_i'(w)|^{\pi/\alpha} \leq M_j |w - w_j|^{\alpha_j/\alpha - \pi/\alpha} \quad (5.1)$$

near each arc of l_i containing only w_j where $m_j = \min |g_j(w)|$ and $M_j = \max |g_j(w)|$. By Lemma 5.2 and (5.1) there exist constants C_{1j} and C_{2j} depending on j such that

$$C_{1j} |w - w_j|^{\alpha_j/\alpha - \pi/\alpha} \leq |\varphi'(w)|^{\pi/\alpha} \leq C_{2j} |w - w_j|^{\alpha_j/\alpha - \pi/\alpha}, \quad (5.2)$$

near each arc of l_i containing w_j . Since $0 < \alpha_j - \pi \leq \pi$, then $0 < \frac{\alpha_j - \pi}{\alpha} \leq \frac{\pi}{\alpha} < 1$. This implies that $f \circ \varphi$ is to be unbounded near some w_j (otherwise $f \equiv \text{const.}$, cf. [1], [3], [5]). Since $(f \circ \varphi)[\varphi']^{\pi/\alpha} \in H^{\alpha/\pi}(K)$ then repeating the argument in the proof of Theorem 2 in [2] (in particular, formula (4) and ff.) we obtain

$$|f \circ \varphi(w)| = O\left(\frac{1}{|w - w_j|^k}\right)$$

near each arc of l_i containing only w_j , where $0 < k < 2$. However, Remark 2.5 implies that k must equal 1. For each $q \in \{q_1, \dots, q_s\} \subseteq \{1, \dots, m\}$ such that f is unbounded near z_q , let $U_q \subset K$ be a simply connected domain such that the boundary of U_q contains the arc of l_i containing w_q and $w_r \notin \partial U_q \cap \partial K$ when $q \neq r$. Since $|f \circ \varphi(w)| = O\left(\frac{1}{|w - w_q|}\right)$ near w_q and $\text{Im}(f \circ \varphi(w)) \equiv 0$ on $\partial K \setminus \{w_{q_1}, \dots, w_{q_s}\}$ then, by Corollary 2.4 we have in U_q ,

$$\text{Im}(f \circ \varphi(w)) = v(w) = h_q(w) + \lambda_q \frac{\partial}{\partial n_{w_q}} g(w, w_q),$$

where h_q is a bounded harmonic function, $\lambda_q \in \mathbb{R}$ and $g_q(w, w_q)$ is the Green function of U_q . This implies that $|v|$ is integrable near each w_q and hence has a harmonic majorant in K . Since $v(w) \neq 0$ only at $\{w_{q_1}, \dots, w_{q_s}\}$, where f is unbounded, then

$$v(w) = \frac{1}{2\pi} \int_{\partial K} \frac{\partial}{\partial n_\xi} g(w, \xi) d\mu(\xi),$$

where $g(w, \xi)$ is the Green function of K and $d\mu(\xi) = \lambda_1 \delta_{w_{q_1}} + \dots + \lambda_m \delta_{w_{q_s}}$ for some $\lambda_{q_1}, \dots, \lambda_{q_s} \in \mathbb{R}$. Therefore

$$v(w) = \lambda_{q_1} \frac{\partial g(w, w_{q_1})}{\partial n_{w_{q_1}}} + \dots + \lambda_{q_s} \frac{\partial g(w, w_{q_s})}{\partial n_{w_{q_s}}}.$$

$\lambda_{q_1}, \dots, \lambda_{q_s}$ are all not equal to zero and $v(w)$ must have a single valued conjugate $u(w)$. Since $g(w, w_q)$ has a logarithmic pole at each w_q then near each arc l_i containing w_q , $|v(w)| \sim \frac{|\lambda_q|}{|w_q - w|}$. Therefore $|f \circ \varphi(w)| \sim \frac{|\lambda_q|}{|w_q - w|}$ near each arc l_i containing w_q . In particular, when $\alpha = \alpha_{q_1}$ then near the arc of ∂K containing w_{q_1} , since $\lambda_{q_1} \neq 0$, we obtain from (5.2) that

$$|f \circ \varphi(w)| |\varphi'(w)|^{\pi/\alpha} \geq C |w - w_{q_1}|^{1 - \pi/\alpha - 1} = C |w - w_{q_1}|^{-\pi/\alpha}$$

for some constant C . This implies that $f(\varphi(w))[\varphi'(w)]^{\pi/\alpha} \notin H^{\alpha/\pi}(K)$ and therefore $f(z) \notin E^{\alpha/\pi}(G)$. This is a contradiction. $\square$

The following theorem shows that the latter result is essentially sharp.

Theorem 5.4. *Let G be an n -connected domain in $\mathbb{C}$ bounded by the curves $\gamma_1, \dots, \gamma_n$ which are real analytic except at the points $z_1, \dots, z_m \in \Gamma$, $m \geq n$, where there are corners with interior angles $\alpha_1, \dots, \alpha_m$. If $\pi < \alpha_j \leq 2\pi$ for all $j = 1, \dots, m$ then for all $p < \frac{\alpha}{\pi}$ where $\alpha = \min\{\alpha_1, \dots, \alpha_m\}$, there exists a non-constant $f(z) \in E^p(G)$ with real boundary values a.e.*

Proof. Let $w_j \in \partial K$, $j = 1, \dots, m$ be the same as in the proof of Theorem 5.3. Consider the harmonic function

$$v(w) = \frac{1}{2\pi} \int_{\partial K} \frac{\partial}{\partial n_\xi} g(w, \xi) d\mu(\xi),$$

where $g(w, \xi)$ is the Green function of K and $d\mu(\xi) = \lambda_1 \delta_{w_1} + \dots + \lambda_m \delta_{w_m}$ for some $\lambda_1, \dots, \lambda_m \in \mathbb{R}$. Then

$$v(w) = \lambda_1 \frac{\partial g(w, w_1)}{\partial n_{w_1}} + \dots + \lambda_m \frac{\partial g(w, w_m)}{\partial n_{w_m}}$$

and $v(w) = 0$ on $\partial K \setminus \{w_1, \dots, w_m\}$. Also, near each arc of ∂K containing w_j $|v(w)| \asymp \frac{\lambda_j}{|w_j - w|}$. By Proposition 4.1 $\lambda_1, \dots, \lambda_m$ can be chosen so that $v \not\equiv 0$ and has a single valued harmonic conjugate u . Let $F = u + iv$. Then F is an analytic function on K that is real valued a.e. and $|F(w)| \asymp \frac{\lambda_j}{|w_j - w|}$ near each arc of ∂K containing w_j . Let, as before, $\varphi : K \rightarrow G$ be the conformal mapping of the n -connected circular domain K onto G and $\varphi(w_j) = z_j$. Then as in the proof of Theorem 5.3,

$$|\varphi'(w)| \leq C_j |w - w_j|^{\alpha_j/\pi - 1}$$

near each arc of ∂K containing w_j . Let $f = F \circ \varphi^{-1}$. Then

$$|f \circ \varphi(w)| |\varphi'(w)|^{1/p} \leq C_j |w - w_j|^{\frac{\alpha_j}{p\pi} - 1/p - 1}$$

near each arc of ∂K containing w_j . Then $f(\varphi(w)) [\varphi'(w)]^{1/p} \in H^p(K)$ when $p(\frac{\alpha_j}{p\pi} - 1/p - 1) > -1$. (Indeed, it is immediately seen that F belongs to the Smirnov class $N^+(K)$, cf. [1], [3], [5], while $\varphi' \in H^1(K) \subset N^+(K)$ since ∂G is rectifiable. Thus, by the Polybarinova theorem, for $(f \circ \varphi)(\varphi')^{1/p}$ to be in $H^p(K)$ we only need to check that $|f \circ \varphi|^p |\varphi'| \in L^1(\partial K)$.) This means that $\alpha_j/\pi - 1 - p > -1$ for all j . That is $p < \alpha_j/\pi$ for all j . Therefore $f(z) \in E^p(G)$ for all $p < \alpha/\pi$. $\square$

Remark 5.5. The hypothesis of Theorem 5.4 is sufficient but not necessary. Under certain conditions of symmetry, it is possible to have fewer than n corners and still find an analytic function of class E^p with real boundary values.

Example 5.6. Consider the triply connected domain bounded by the unit circle, a circle of radius $1/4$ centered at $-1/2$, and the closed curve parametrized by the function $(1/4 \sin^5(t) + 1/2, 1/2 \cos(t))$, $0 \leq t \leq 2\pi$, cf. Fig. 5.1.

This domain has two cusps on the boundary that are symmetric about the real axis and there is an interior angle of 2π at each point of the cusps. There exists a conformal mapping from a circular domain similar to the one defined in Example 4.3 having the same symmetries with respect to the real and imaginary axes onto G . Using the same argument as in Example 4.3 and the proof of Theorem 5.4, one can construct an analytic function in G that is in Smirnov class E^p for all $p < 2$.

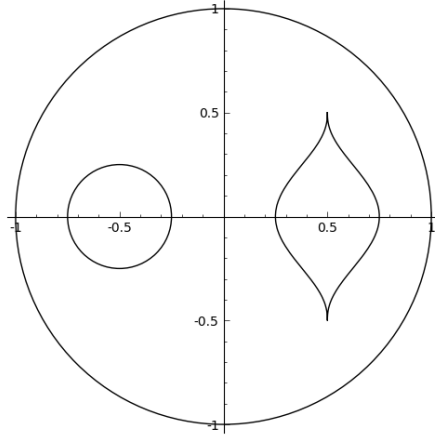


Fig. 5.1 Symmetry example.

6 Neuwirth-Newman's theorem.

Recall that $f \in H^p(G)$ if and only if $f \circ T \in H^p(\mathbb{D})$, where $T : \mathbb{D} \rightarrow G$ is the uniformization map (cf. [3], Sec. 3.4 and [5], Sec. 1 Theorems 1.1 and 1.2 and references [2], [11], [18] there). Thus if $f \in H^{1/2}(G)$ and $f(\zeta) \geq 0$ a.e. with respect to the harmonic measure on Γ , then $f \circ T$ satisfies the hypothesis of the Neuwirth-Newman Theorem [11] and, therefore, is a constant.

In [2] we have noted at the end that the Neuwirth-Newman argument extends essentially word for word to E^p classes in simply connected domains and renders the following:

Theorem 6.1. *Let G be a simply connected Smirnov domain with rectifiable boundary Γ . Let $p_0 \geq 1$ be defined as the smallest $p \geq 1$ such that $f \in E^p(G)$ and f has real boundary values a.e. on Γ imply that f is a constant. Then all $f \in E^{p_0/2}$ such that $f \geq 0$ on Γ are constants.*

Although we believe the same statement holds for finitely connected domains, we have been unable to prove it. More precisely, the difficulty in directly extending the Neuwirth-Newman argument to multiply connected domains is as follows. From [5], Theorem 4.5, it follows that, $f = QBSF^2$, where B is the generalized Blaschke product, S a singular inner function, F^2 an outer factor, $F \in E^{p_0}$, Q is an invertible bounded analytic function, and $|Q|$ is a local constant on Γ . The problem is that $|B|$ and $|S|$ are local constants a.e. on the boundary of G . Hence, the assumption that $f \geq 0$ a.e. on Γ only yields on Γ that $f = QBSF^2 = |QBS|F\bar{F}$ and, accordingly,

$$QBSF = |QBS|\bar{F} \text{ a.e. on } \Gamma. \quad (6.1)$$

Unfortunately, (6.1) alone does not seem to imply that $\bar{F}$ and hence $F + \bar{F} = 2\operatorname{Re}(F)$ are analytic functions in $E^{p_0}(G)$. Indeed, already in the annulus $G = \{r < |z| < R\}$ the function $f(z) = \frac{1}{\bar{z}}$ differs only by constant multiples from an analytic function z :

$$\frac{1}{\bar{z}} \Big|_{|z|=r} = r^2 z, \quad \frac{1}{\bar{z}} \Big|_{|z|=R} = R^2 z.$$

Thus, extending Theorem 6.1 to finitely connected Smirnov domains remains an open question. The following example illustrates why we think Theorem 6.1 should extend to multiply connected domains. For the sake of clarity, we present the example for doubly connected domains although it readily extends to domains of higher connectivity.

Theorem 6.2. *Let G be a doubly connected domain with boundary Γ that is real analytic except at the points $z_1, \dots, z_m$, $m \geq 2$, where there are inward pointing cusps. Then every $f \in E^1(G)$ with non-negative boundary values is a constant.*

Remark 6.3. In view of our main results, Theorems 5.3 and 5.4 ($\alpha = 2\pi$ for the inward cusps), $p_0 = 2$ in this case.

Proof. Suppose that there exists such an $f \in E^1(G)$. Let γ_1 denote the curve that contains z_1 . Let $\varphi : A \rightarrow G$ be the conformal map of the annulus $A = \{z : 1 < |z| < R\}$ onto G and $\phi_1(w)$ be the conformal mapping of the domain bounded by the unit circle and containing A onto the domain bounded by γ_1 and containing G . Assume that $\varphi(1) = z_1$ and that φ maps the unit circle onto γ_1 . Since there is an inward pointing cusp at z_1 , the interior angle is 2π , so by [12], Ch. 3, Sec. 4, near 1, $\phi_1'(w) = (w-1)g(w)$ where $g(w)$ is bounded away from 0 and ∞ . This implies that by Lemma 5.2 there exist constants c_1 and c_2 such that near 1

$$c_1|w-1| \leq |\phi_1'| \leq c_2|w-1|. \quad (6.2)$$

Let U be the intersection of a neighborhood of 1 and A such that U does not contain any of the pre-images of $z_2, \dots, z_m$. Since $f(\varphi(w))\phi_1'(w) \in H^1(A)$ then by (6.2) $f(\varphi(w-1)) \in H^1(U)$. Let

$$h(w) = f(\varphi(w))(w-1) \left(\frac{1}{w} - 1 \right) = f(\varphi(w)) \frac{(1-w)^2}{-w}.$$

Then $h \in H^1(U)$ and, on $\eta = \partial U \cap \partial A$, we have $h(w) = f(\varphi(w))|1-w|^2 \geq 0$. So by the generalized Schwarz reflection principle h can be analytically continued across η . This implies that $|f(\varphi(w))| = O\left(\frac{1}{(w-1)^2}\right)$ in U . Then by Corollary 2.4

$$f(\varphi(w)) = \lambda_1 iR(w, 1) + \lambda_2 iR'(w, 1) + a(w),$$

where $R(w, 1)$ is the Schwarz kernel of U and $a(w)$ is an analytic function that is real valued on η and analytic across. $Re(\lambda_1 iR(w, 1))$ takes arbitrarily large positive and negative values in a neighborhood of 1 on η . Thus, since $f(\varphi(w)) \geq 0$ on η , $\lambda_2 \neq 0$. Therefore,

$$|f(\varphi(w))(1-w)| \geq \frac{\text{const.}}{|1-w|} \text{ near } 1.$$

Hence $f(\varphi(w))(1-w)$ is not in H^1 . Hence, $\lambda_1 = \lambda_2 = 0$. The same argument applies to $z_2, \dots, z_m$. Hence $f(\varphi(w))$ is bounded on A and therefore, is a constant. $\square$

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