THE SEARCH FOR SINGULARITIES OF SOLUTIONS TO THE DIRICHLET PROBLEM: RECENT DEVELOPMENTS

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ABSTRACT. This is a survey article based on an invited talk delivered by the first author at the CRM workshop on Hilbert Spaces of Analytic Functions held at CRM, Université de Montreal, December 8-12, 2008.

1. THE MAIN QUESTION

Let Ω be a smoothly bounded domain in $\mathbb{R}^n$. Consider the Dirichlet Problem (DP) in Ω of finding the function u, say, $\in C^2(\Omega) \bigcap C(\overline{\Omega})$ and satisfying

$$\left\{ \begin{array}{l} \Delta u = 0 \\ u|_{\Gamma} = v \end{array} \right. , \tag{1.1}$$

where $\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$ and $\Gamma := \partial\Omega, v \in C(\Gamma)$. It is well known since the early 20th century from works of Poincare, C. Neumann, Hilbert, and Fredholm that the solution u exists and is unique. Also, since u is harmonic in Ω, hence real-analytic there, no singularities can appear in Ω. Moreover, assuming $\Gamma := \partial\Omega$ to consist of real-analytic hypersurfaces, the more recent and difficult results on "elliptic regularity" assure us that if the data v is real-analytic in a neighborhood of $\overline{\Omega}$ then u extends as a real-analytic function across $\partial\Omega$ into an open neighborhood Ω' of $\overline{\Omega}$. In two dimensions, this can be done using the reflection principle. In higher dimensions, the boundary can be biholomorphically "flattened", but this leads to a general elliptic operator for which the reflection principle does not apply. Instead, analyticity must be shown by directly verifying convergence of the power series representing the solution through difficult estimates on the derivatives (see [14]).

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Question Suppose the data v is a restriction to Γ of a "very good" function, say an entire function of variables $x_1, x_2, ..., x_n$. In other words, the data presents no reasons whatsoever for the solution u of (1.1) to develop singularities.

(i) Can we then assert that all solutions u of (1.1) with entire data $v(x)$ are also entire?

(ii) If singularities do occur, they must be caused by geometry of Γ interacting with the differential operator Δ. Can we then find data v_0 that would force the worst possible scenario to occur? More precisely, for any entire data v, the set of possible singularities of the solution u of (1.1) is a subset of the singularity set of u_0, the solution of (1.1) with data v_0.

2. The Cauchy Problem

An inspiration to this program launched by H. S. Shapiro and the first author in [22] comes from reasonable success with a similar program in the mid 1980's regarding the analytic Cauchy Problem (CP) for elliptic operators, in particular, the Laplace operator. For the latter, we are seeking a function u with $\Delta u = 0$ near Γ and satisfying the initial conditions

$$\left\{ \begin{array}{l} (u-v)|_\Gamma = 0 \\ \nabla(u-v)|_\Gamma = 0 \end{array} \right. , \tag{2.1}$$

where v is assumed to be real-analytic in a neighborhood of Γ. Suppose as before that the data v is a "good" function (e.g. a polynomial or an entire function). In that context, the techniques developed by J. Leray [26] in the 1950's (and jointly with L. Gårding and T. Kotake [15]) together with the works of P. Ebenfelt [11], G. Johnsson [18], and, independently, by B. Sternin and V. Shatalov [33] in Russia and their school produced a more or less satisfactory understanding of the situation. To mention briefly, the answer (for the CP) to question (i) in two dimensions is essentially "never" unless Γ is a line while for (ii) the data mining all possible singularities of solutions to the CP with entire data is $v_0 = |x|^2 = \sum x_j^2$ (see [21], [19], [34], and [20] and references therein).

3. The Dirichlet problem: When does entire data imply entire solution?

Let us raise Question (i) again for the Dirichlet Problem: Does (real) entire data v imply entire solution u of (1.1)?

In this section and the next, $\mathbb{P}$ will denote the space of polynomials and $\mathbb{P}_N$ the space of polynomials of degree $\leq N$. The following pretty fact goes back to the 19th century and can be associated with the names of E. Heine, G. Lamé, M. Ferrers, and probably many others (cf. [20]). The proof is from [22] (cf. [2], [3]).

Proposition 3.1. *If* $\Omega := \{x : \sum \frac{x_j^2}{a_j^2} - 1 < 0, a_1 > ... > a_n > 0\}$ *is an ellipsoid, then any DP with a polynomial data of degree* N *has a polynomial solution of degree* $\leq N$.

Proof. Let $q(x) = \sum \frac{x_j^2}{a_j^2} - 1$ be the defining function for $\Gamma := \partial\Omega$. The (linear) map $T : P \to \Delta(qP)$ sends the finite-dimensional space $\mathbb{P}_N$ into itself. T is injective (by the maximum principle) and, therefore, surjective. Hence, for any P, $degP \geq 2$ we can find P_0, deg $P_0 \leq$ deg $P - 2$. $TP_0 = \Delta(qP_0) = \Delta P$. $u = P - qP_0$ is then the desired solution. □

The following result was proved in [22].

Theorem 3.2. *Any solution to DP (1.1) in an ellipsoid* Ω *with entire data is also entire.*

Later on, D. Armitage sharpened the result by showing that the order and the type of the data are carried over, more or less, to the solution [1]. The following conjecture has also been formulated in [22].

Conjecture 3.3. *Ellipsoids are the only bounded domains in* $\mathbb{R}^n$ *for which Theorem 3.2 holds. i.e. ellipsoids are the only domains in which entire data implies entire solution for the DP (1.1).*

In 2005 H. Render [30] proved this conjecture for all *algebraically* bounded domains Ω defined as bounded components of $\{\phi(x) < 0, \phi \in \mathbb{P}_N\}$ such that $\{\phi(x) = 0\}$ is a *bounded* set in $\mathbb{R}^n$ or, equivalently, the senior homogeneous part $\phi_N(x)$ of ϕ is elliptic, i.e., $|\phi_N(x)| \geq C|x|^N$ for some constant C. For $n = 2$, an easier version of this result was settled in 2001 by M. Chamberland and D. Siegel [6]. At the beginning of the next section we will outline their argument, which establishes similar results as Render's for the following modified conjecture.

Conjecture 3.4. *Ellipsoids are the only surfaces for which polynomial data implies polynomial solution.*

Remark: We will return to Render's Theorem below. For now let us note that, unfortunately, it already tells us nothing even in 2 dimensions for many perturbations of a unit disk, e.g., $\Omega := \{x \in \mathbb{R}^2 : x^2 + y^2 - 1 + \varepsilon h(x, y) < 0\}$ where, say, h is a harmonic polynomial of degree > 2.

4. When does polynomial data imply polynomial solution?

Let $\gamma = \{\phi(x) = 0\}$ be a bounded, irreducible algebraic curve in $\mathbb{R}^2$. If the DP posed on γ has polynomial solution whenever the data is a polynomial, then as Chamberland and Siegel observed, (a) γ is an ellipse or (b) there exists data $f \in \mathbb{P}$ such that the solution $u \in \mathbb{P}$ of DP has deg $u >$ deg f.

In case (b) $u - f|_\gamma = 0$ implies that ϕ divides $u - f$ by Hilbert's Nullstelensatz, and, since deg $u = M >$ deg f, $u_M = \phi_k g_l$ where ϕ_k and u_M are the senior homogeneous terms of ϕ and u respectively. The senior term of u must have the form $u_M = az^M + b\bar{z}^M$ since u_M is harmonic. Hence, u_M factors into linear factors and so must ϕ_k. Hence γ is unbounded. This gives the following result [6].

Theorem 4.1. *Suppose deg $\phi > 2$ and ϕ is square-free. If the Dirichlet problem posed on $\{\phi = 0\}$ has a polynomial solution for each polynomial data, then the senior part of ϕ, which we denote by ϕ_N, of order N, factors into real linear terms, namely,*

$$\phi_N = \prod_{j=0}^{n} (a_j x - b_j y),$$

where a_j, b_j are some real constants and the angles between the lines $a_j x - b_j y = 0$, for all j, are rational multiples of π.

This theorem settles Conjecture 3.4 for bounded domains $\Omega \subseteq \{\phi(x) < 0\}$ such that the set $\{\phi(x) = 0\}$ is bounded in $\mathbb{R}^2$. However, the theorem leaves open simple cases such as $x^2 + y^2 - 1 + \varepsilon(x^3 - 3xy^2)$.

Example: The curve $y(y-x)(y+x) - x = 0$ (see figure 1) satisfies the necessary condition imposed by the theorem. Moreover, any quadratic data can be matched on it by a harmonic polynomial. For instance, $u = xy(y^2 - x^2)$ solves the interpolation problem (it is misleading to say "Dirichlet" problem, since there is no bounded component) with data $v(x, y) = x^2$. On the other hand, one can show (non-trivially) that the data x^3 does not have polynomial solution.

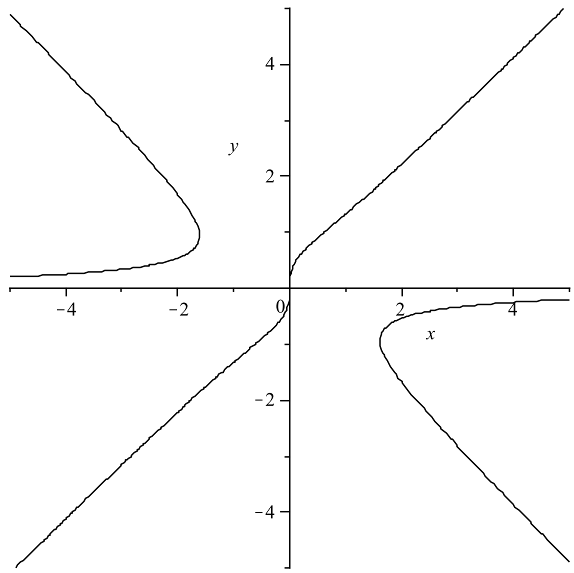

FIGURE 1. A cubic on which any quadratic data can be matched by a harmonic polynomial.

5. DIRICHLET'S PROBLEM AND ORTHOGONAL POLYNOMIALS

Most recently, N. Stylianopoulos and the first author showed that if for a polynomial data there always exists a polynomial solution of the DP (1.1), with an additional constraint on the degree of the solution in terms of the degree of the data (see below), then Ω is an ellipse [23]. This result draws on the 2007 paper of M. Putinar and N. Stylianopoulos [29] that found a simple but surprising connection between Conjecture 3.4 in $\mathbb{R}^2$ and (Bergman) orthogonal polynomials, i.e. polynomials orthogonal with respect to the inner product $\langle p, q \rangle_\Omega := \int_\Omega p\bar{q} dA$, where dA is the area measure. To understand this connection let us consider the following properties:

(1) There exists k such that for a polynomial data of degree n there always exists a polynomial solution of the DP (1.1) posed on Ω of degree $\leq n + k$.
(2) There exists N such that for all m, n, the solution of (1.1) with data $\bar{z}^m z^n$ is a harmonic polynomial of degree $\leq (N-1)m + n$ in z and of degree $\leq (N-1)n + m$ in $\bar{z}$.
(3) There exists N such that orthogonal polynomials $\{p_n\}$ of degree n on Ω satisfy a (finite) $(N+1)$-recurrence relation, i.e.

$$zp_n = a_{n+1,n}p_{n+1} + a_{n,n}p_n + \dots + a_{n-N+1}p_{n-N+1},$$

where $a_{n-j,n}$ are constants depending on n.

(4) The Bergman orthogonal polynomials of Ω satisfy a finite-term recurrence relation, i.e., for every fixed $k > 0$, there exists an $N(k) > 0$, such that $a_{k,n} = \langle zp_n, p_k \rangle = 0$, $n \geq N(k)$.

(5) Conjecture 3.4 holds for Ω.

Putinar and Stylianopoulos noticed that with the additional minor assumption that polynomials are dense in $L^2_a(\Omega)$, properties (4) and (5) are equivalent. Thus, they obtained as a corollary (by way of Theorem 4.1 from the previous section) that the only bounded algebraic sets satisfying property (4) are ellipses. We also have (1) $\Rightarrow$ (2), (2) $\Leftrightarrow$ (3), and (3) $\Rightarrow$ (4). Stylianopoulos and the first author used the equivalence of properties (2) and (3) to prove the following theorem which has an immediate corollary.

Theorem 5.1. *Suppose $\partial\Omega$ is C^2-smooth, and orthogonal polynomials on Ω satisfy a (finite) $(N+1)$-recurrence relation, in other words property (3) is satisfied. Then, $N = 2$ and Ω is an ellipse.*

Corollary 5.2. *Suppose $\partial\Omega$ is a C^2-smooth domain for which there exists N such that for all m, n, the solution of (1.1) with data $\bar{z}^m z^n$ is a harmonic polynomial of degree $\leq (N-1)m + n$ in z and of degree $\leq (N-1)n + m$ in $\bar{z}$. Then $N = 2$ and Ω is an ellipse.*

Sketch of proof. First, one notes that all the coefficients in the recurrence relation are bounded. Divide both sides of the recurrence relation above by p_n and take the limit of an appropriate subsequence as $n \to \infty$. Known results on asymptotics of orthogonal polynomials (see [35]) give $\lim\limits_{n\to\infty} \frac{p_{n+1}}{p_n} = \Phi(z)$ on compact subsets of $\overline{\mathbb{C}} \setminus \overline{\Omega}$, where $\Phi(z)$ is the conformal map of the exterior of Ω to the exterior of the unit disc. This leads to a *finite* Laurent expansion at ∞ for $\Psi(w) = \Phi^{-1}(w)$. Thus, $\Psi(w)$ is a rational function, so $\tilde{\Omega} := \overline{\mathbb{C}} \setminus \overline{\Omega}$ is an unbounded quadrature domain, and the Schwarz function (cf. [7], [37]) of $\partial\Omega$, $S(z)$ ($= \bar{z}$ on $\partial\Omega$) has a meromorphic extension to $\tilde{\Omega}$. Suppose, for the sake of brevity and to fix the ideas, for example, that $S(z) = cz^d + \sum_{j=1}^{M} \frac{c_j}{z-z_j} + f(z)$, where $f \in H^\infty(\tilde{\Omega})$, and $z_j \in \tilde{\Omega}$. Since our hypothesis is equivalent to Ω satisfying property (2) discussed above, the data $\bar{z}P(z) = \bar{z}\prod_{j=1}^{n}(z - z_j)$ has polynomial solution, $g(z) + \overline{h(z)}$ to the DP. On Γ we can replace $\bar{z}$ with $S(z)$. Write $\overline{h(z)} = h^\#(\bar{z})$, where $h^\#$ is a polynomial whose coefficients are complex conjugates of their counterparts in h. We have on Γ

$$S(z)P(z) = g(z) + h^\#(S(z)), \tag{5.1}$$

which is actually true off Γ since both sides of the equation are analytic. Near z_j, the left-hand-side of this equation tends to a finite limit (since $S(z)P(z)$ is analytic in $\tilde{\Omega} \setminus \infty$!) while the right-hand-side tends to ∞ unless the coefficient c_j is zero. Thus,

$$S(z) = cz^d + f(z). \tag{5.2}$$

Using property (2) again with data $|z|^2 = z\bar{z}$ we can infer that $d = 1$. Hence, $\tilde{\Omega}$ is a null quadrature domain. Sakai's theorem [32] implies now that Ω is an ellipse. $\square$

Remark: It is well-known that families of orthogonal polynomials on the line satisfy a 3-term recurrence relation. P. Duren in 1965 [8] already noted that in $\mathbb{C}$ the only domains with real-analytic boundaries in which polynomials orthogonal with respect to arc-length on the boundary satisfy 3-term recurrence relations are ellipses. L. Lempert [25] constructed peculiar examples of C^∞ non-algebraic Jordan domains in which no finite recurrence relation for Bergman polynomials holds. Theorem 5.1 shows that actually this is true for *all* C^2-smooth domains except ellipses.

6. Looking for singularities of the solutions to the Dirichlet Problem

Once again, inspired by known results in the similar quest for solutions to the Cauchy problem, one could expect, e.g., that the solutions to the DP (1.1) exhibit behavior similar to those of the CP (2.1). In particular, it seemed natural to suggest that the singularities of the solutions to the DP outside Ω are somehow associated with the singularities of the Schwarz potential (function) of $\partial\Omega$ which does indeed completely determine $\partial\Omega$ (cf. [21], [37]). It turned out that singularities of solutions of the DP are way more complicated than those of the CP. Already in 1992 in his thesis, P. Ebenfelt showed [9] that the solution of the following "innocent" DP in $\Omega := \{x^4 + y^4 - 1 < 0\}$ (the "TV-screen")

$$\begin{cases} \Delta u = 0 \\ u|_{\partial\Omega} = x^2 + y^2 \end{cases} \tag{6.1}$$

has an infinite discrete set of singularities (of course, symmetric with respect to 90° rotation) sitting on the coordinate axes and running to ∞ (see figure 2).

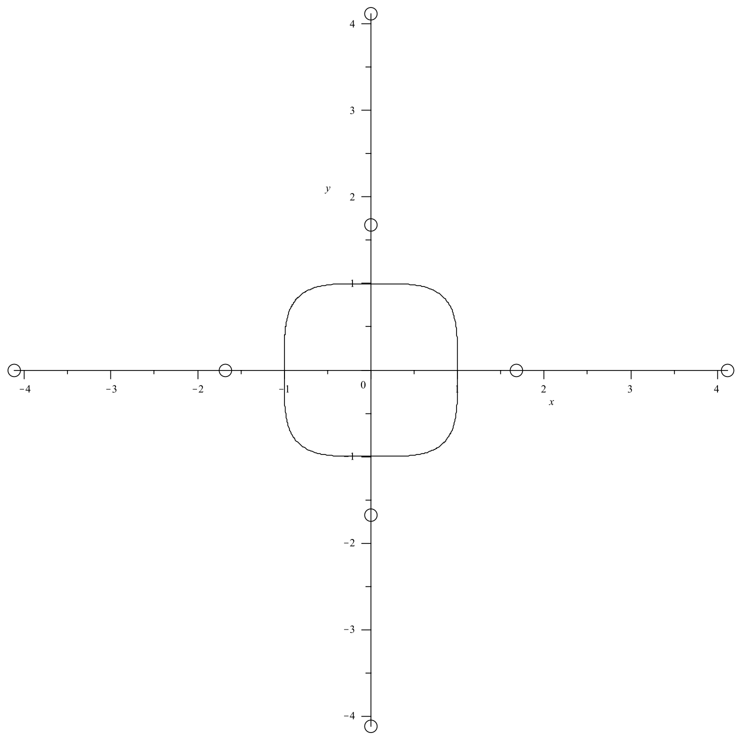

FIGURE 2. A plot of the "TV screen" $\{x^4 + y^4 = 1\}$ along with the first eight singularities (plotted as circles) encountered by analytic continuation of the solution to DP (6.1).

To see the difference between analytic continuation of solutions to CP and DP, note that for the former

$$\frac{\partial u}{\partial z}|_{\Gamma:=\partial\Omega} = v_z(z,\bar{z}) = v_z(z, S(z)), \tag{6.2}$$

and since $\frac{\partial u}{\partial z}$ is analytic, (6.2) allows u_z to be continued everywhere together with v and $S(z)$, the Schwarz function of $\partial\Omega$. For the DP we have on Γ

$$u(z,\bar{z}) = v(z,\bar{z}) \tag{6.3}$$

for $u = f + \bar{g}$ where f and g are analytic in Ω. Hence, (6.3) becomes

$$f(z) + \overline{g(\overline{S(z)})} = v(z, S(z)). \tag{6.4}$$

Now, $v(z, S(z))$ does indeed (for entire v) extend to any domain free of singularities of $S(z)$, but (6.4), even when v is real-valued so that $g = f$, presents a very non-trivial functional equation supported by a rather mysterious piece of information that f is analytic in Ω. (6.4) however gives an insight as to how to capture the DP-solution's singularities by considering the DP as part of a Goursat problem in $\mathbb{C}^2$ (or $\mathbb{C}^n$ in general). The latter Goursat problem can be posed as follows (cf. [36]). Given a complex-analytic variety $\hat{\Gamma}$ in $\mathbb{C}^n$, ($\hat{\Gamma} \bigcap \mathbb{R}^n = \Gamma := \partial\Omega$), find $u : \sum_{j=1}^n \frac{\partial^2}{\partial z_j^2} u = 0$ near $\hat{\Gamma}$ (and also in $\Omega \subset \mathbb{R}^n$) so that $u|_{\hat{\Gamma}} = v$, where v is, say, an entire function of n complex variables. Thus, if $\hat{\Gamma} := \{\phi(z) = 0\}$, where ϕ is, say, an irreducible polynomial, we can, e.g., ponder the following extension of Conjecture 3.3:

Question: For which polynomials ϕ can every entire function v be split (Fischer decomposition) as $v = u + \phi h$, where $\Delta u = 0$ and u, h are entire functions (cf. [13], [36])?

7. Render's breakthrough

Trying to establish Conjecture 3.3 H. Render [30] has made the following ingenious step. He introduced the *real* version of the Fischer space norm

$$\langle f, g \rangle = \int_{\mathbb{R}^n} f \bar{g} e^{-|x|^2} dx, \tag{7.1}$$

where f and g are polynomials. Originally, the Fischer norm (introduced by E. Fischer [13]) requires the integration to be carried over all of $\mathbb{C}^n$ and has the property that multiplication by monomials is adjoint to differentiation with the corresponding multi-index (e.g., multiplication by $(\sum_{j=1}^n x_j^2)$ is adjoint to the differential operator Δ). This property is only partially preserved for the real Fischer norm. More precisely [30],

$$\langle \Delta f, g \rangle = \langle f, \Delta g \rangle + 2(deg(f) - deg(g))\langle f, g \rangle \tag{7.2}$$

for homogeneous f, g.

Suppose u solves the DP with data $|x|^2$ on $\partial\Omega \subseteq \{P = 0 : deg(P) = 2k, k > 1\}$. Then $u - |x|^2 = Pq$ for analytic q, and thus $\Delta^k(Pq) = 0$. Using (7.2), this (non-trivially) implies that the real Fischer product

$\langle (Pq)_{m+2k}, q_m \rangle$ between all homogeneous parts of degree $m+2k$ and m of Pq and q, respectively, is zero. By a tour de force argument, Render used this along with an added assumption on the senior term of P (see below) to obtain estimates *from below* for the decay of the norms of homogeneous parts of q. This, in turn yields an if-and-only-if criterion for convergence in the real ball of radius R of the series for the solution $u = \sum_{m=0}^{\infty} u_m$, u_m homogeneous of degree m. Let us state Render's main theorem.

Theorem 7.1. *Let P be an irreducible polynomial of degree $2k$, $k > 1$. Suppose P is elliptic, i.e. the senior term P_{2k} of P satisfies $P_{2k}(x) \geq c_P|x|^{2k}$, for some constant c_P. Let ϕ be real analytic in $\{|x| < R\}$, and $\Delta^k(P\phi) = 0$ (at least in a neighborhood of the origin). Then, $R \leq C(P,n) < +\infty$, where C is a constant depending on the polynomial P and the dimension of the ambient space.*

Remark: The assumption in the theorem that P is elliptic is equivalent to the condition that the set $\{P = 0\}$ is bounded in $\mathbb{R}^n$.

Corollary 7.2. *Assume $\partial\Omega$ is contained in the set $\{P = 0\}$, a bounded algebraic set in $\mathbb{R}^n$. Then, if a solution of the DP (1.1) with data $|x|^2$ is entire, Ω must be an ellipsoid.*

Proof. Suppose not, so $\deg(P) = 2k > 2$, and the following (Fischer decomposition) holds: $|x|^2 = P\phi + u$, $\Delta u = 0$. Hence, $\Delta^k(P\phi) = 0$ and ϕ cannot be analytically continued beyond a *finite* ball of radius $R = C(P) < \infty$, a contradiction. □

Caution: We want to stress again that, unfortunately, the theorem still tells us nothing for say small perturbations of the circle by a non-elliptic term of degree ≥ 3, e.g., $x^2 + y^2 - 1 + \varepsilon(x^3 - 3xy^2)$.

8. Back to $\mathbb{R}^2$: lightning bolts

Return to the $\mathbb{R}^2$ setting and consider as before our boundary $\partial\Omega$ of a domain Ω as (part of) an intersection of an analytic Riemann surface $\hat{\Gamma}$ in $\mathbb{C}^2$ with $\mathbb{R}^2$. Roughly speaking if say $\partial\Omega$ is a subset of the algebraic curve $\Gamma := \{(x,y) : \phi(x,y) = 0\}$, where ϕ is an irreducible polynomial, then $\hat{\Gamma} = \{(X,Y) \in \mathbb{C}^2 : \phi(X,Y) = 0\}$. Now look at the Dirichlet problem again in the context of the Goursat problem: Given, say, a polynomial data P, find f, g holomorphic functions of one variable near $\hat{\Gamma}$ (a piece of $\hat{\Gamma}$ containing $\partial\Omega \subseteq \hat{\Gamma} \bigcap \mathbb{R}^2$) such that

$$u = f(z) + g(w)|_{\hat{\Gamma}} = P(z,w), \tag{8.1}$$

where we have made the linear change of variables $z = X + iY$, $w = X - iY$ (so $\bar{w} = z$ on $\mathbb{R}^2 = \{(X,Y)\colon\ X, Y \text{ are both real}\}$). Obviously, $\Delta u = 4\frac{\partial^2}{\partial z \partial w} = 0$ and u matches P on $\partial\Omega$. Thus, the DP in $\mathbb{R}^2$ has become an interpolation problem in $\mathbb{C}^2$ of matching a polynomial on an algebraic variety by a sum of holomorphic functions in each variable separately. Suppose that for all polynomials P the solutions u of (8.1) extend as analytic functions to a ball $B_\Omega = \{|z|^2 + |w|^2 < R_\Omega\}$ in $\mathbb{C}^2$. Then, if $\hat{\Gamma} \bigcap B_\Omega$ is path connected, we can interpolate every polynomial $P(z,w)$ on $\hat{\Gamma} \bigcap B_\Omega$ by a holomorphic function of the form $f(z) + g(w)$. Now suppose we can produce a compactly supported measure μ on $\hat{\Gamma} \bigcap B_\Omega$ which annihilates *all* functions of the form $f(z) + g(w)$, f, g holomorphic in B_Ω and at the same time does not annihilate all polynomials $P(z,w)$. This would force the solution u of (8.1) to have a singularity in the ball B_Ω in $\mathbb{C}^2$. Then, invoking a theorem of Hayman [17] (see also [20]), we would be able to assert that u cannot be extended as a real-analytic function to the *real* disk B_R in $\mathbb{R}^2$ containing Ω and of radius $\geq \sqrt{2}R$. An example of such annihilating measure supported by the vertices of a "quadrilateral" was independently observed by E. Study [38], H. Lewy [27], and L. Hansen and H. S. Shapiro [16]. Indeed, assign alternating values ± 1 for the measure supported at the four points $p_0 := (z_1, w_1)$, $q_0 := (z_1, w_2)$, $p_1 := (z_2, w_2)$, and $q_1 := (z_2, w_1)$. Then $\int (f+g) d\mu = f(z_1) + g(w_1) - f(z_1) - g(w_2) + f(z_2) + g(w_2) - f(z_2) - g(w_1) = 0$ for all holomorphic functions f and g of one variable. This is an example of a closed lightning bolt (LB) with four vertices. Clearly, the idea can be extended to any even number of vertices.

Definition. *A complex closed lightning bolt (LB) of length $2(n+1)$ is a finite set of points (vertices) $p_0, q_0, p_1, q_1, ..., p_n, q_n, p_{n+1}, q_{n+1}$ such that $p_0 = p_{n+1}$, and each complex line connecting p_j to q_j or q_j to p_{j+1} has either z or w coordinate fixed and they alternate. i.e. if we arrived at p_j with w coordinate fixed then we follow to q_j with z fixed etc.*

For "real" domains lightning bolts were introduced by Arnold and Kolmogorov in the 1950s to study Hilbert's 13th problem (see [24] and the references therein).

The following theorem has been proved in [4] (see also [5]).

Theorem 8.1. *Let Ω be a bounded simply connected domain in $\mathbb{C} \cong \mathbb{R}^2$ such that the Riemann map $\phi : \Omega \to \mathbb{D} = \{|z| < 1\}$ is algebraic. Then all solutions of the DP with polynomial data have only algebraic singularities only at branch points of ϕ with the branching order of the former dividing the branching order of the latter iff ϕ^{-1} is a rational*

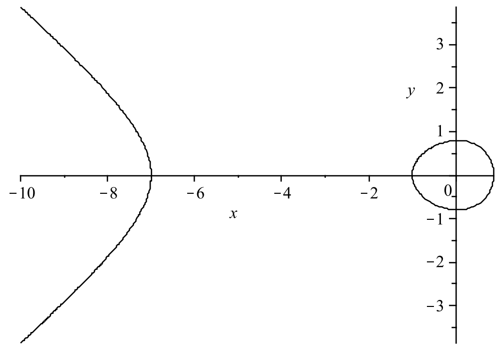

FIGURE 3. A Maple plot of the cubic $8x(x^2-y^2)+57x^2+77y^2-49=0$, showing the bounded component and one unbounded component (there are two other unbounded components further away).

function. This in turn is known to be equivalent to Ω being a quadrature domain.

Idea of proof: The hypotheses imply that the solution $u = f+\bar{g}$ extends as a single-valued meromorphic function into a $\mathbb{C}^2$-neighborhood of $\hat{\Gamma}$. By another theorem of [4], one can find (unless ϕ^{-1} is rational) a continual family of closed LBs on $\hat{\Gamma}$ of bounded length avoiding the poles of u. Hence, the measure with alternating values ± 1 on the vertices of any of these LBs annihilates all solutions $u = f(z) + g(w)$ holomorphic on $\hat{\Gamma}$, but does not, of course, annihilate all polynomials of z, w. Therefore, ϕ^{-1} must be rational, i.e. Ω is a quadrature domain [36]. □

The second author [28] has recently constructed some other examples of LBs on complexified boundaries of planar domains which do not satisfy the hypothesis of Render's theorem. The LBs validate Conjecture 3.3 and produce an estimate regarding how far into the complement $\mathbb{C} \setminus \overline{\Omega}$ the singularities may develop. For instance, the complexification of the cubic, $8x(x^2 - y^2) + 57x^2 + 77y^2 - 49 = 0$ has a lightning bolt with six vertices in the (non-physical) plane where z and w are real, i.e., x is real and y is imaginary (see figure 3 for a plot of the cubic in the plane where x and y are real and see figure 4 for the "non-physical" slice including the lightning bolt). If the solution with appropriate cubic data is analytically continued in the direction of the closest unbounded

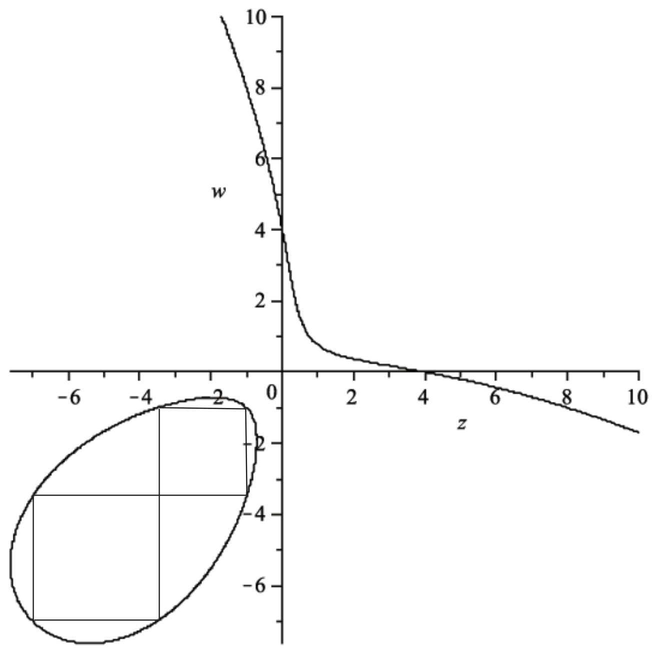

FIGURE 4. A lightning bolt with six vertices on the cubic $2(z+w)(z^2+w^2)+67zw-5(z^2+w^2)=49$ in the non-physical plane with z and w real, i.e. x real and y imaginary.

component of the curve defining $\partial\Omega$, it will have to develop a singularity before it can be forced to match the data on that component.

9. CONCLUDING REMARKS, FURTHER QUESTIONS

In two dimensions one of the main results in [4] yields that disks are the only domains for which all solutions of the DP with rational (in x, y) data v are rational. The fact that in a disk every DP with rational data has a rational solution was observed in a senior thesis of T. Fergusson at U. of Richmond [31]. On the other hand, algebraic data may lead to a transcendental solution even in disks (see [10], also cf. [12]). In dimensions 3 and higher, rational data on the sphere (e.g., $v = \frac{1}{x_1 - a}$, $|a| > 1$) yields transcendental solutions of (1.1), although we have not been able to estimate the location of singularities precisely (cf. [10]).

It is still not clear on an intuitive level why ellipsoids play such a distinguished role in providing "excellent" solutions to DP with "excellent" data. A very similar question, important for applications, (which actually inspired the program launched in [22] on singularities of the solutions to the DP) goes back to Raleigh and concerns singularities of solutions of the Helmholtz equation ($[\Delta - \lambda^2]u = 0$, $\lambda \in \mathbb{R}$) instead. (The minus sign will guarantee that the maximum principle holds and, consequently, ensures uniqueness of solutions of the DP.) To the best of our knowledge, this topic remains virtually unexplored.

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