The "isoperimetric sandwiches", free boundary problems and approximation by analytic and harmonic functions.

Lecture 3, Bergen, November 3, 2015.

Further directions

I. Extending to $\mathbb{R}^n$, $n \geq 3$.

$\bar{z}$: $\frac{\partial}{\partial \bar{z}}(\bar{z}) = 1$.

$\frac{\partial}{\partial \bar{z}}$ replaced by $\Delta = \sum_1^n \frac{\partial^2}{\partial x_j^2}$

$\bar{z}$ replaced by $|x|^2 = \sum_1^n x_i^2$, $x = (x_1, \dots, x_n)$

$\Delta(|x|^2) = 2n$

Harmonic content: $\Lambda(K) = \text{dist}(|x|^2, H(K))_{C(K)}$

where $K \Subset \mathbb{R}^n$ $H(K) = \{$functions harmonic in a neighborhood of $K\}_{C(K)}$.

Theorem 1 (DK – '86) $\Lambda(K) = 0 \Leftrightarrow H(K) = C(K)$.

This is nontrivial since the Stone-Weierstrass theorem doesn't apply

(Extended to arbitrary 2^d order elliptic (uniformly) operators $\sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_1^n b_i(x) \frac{\partial}{\partial x_i} + h(x)$, $h \leq 0$).

Theorem 2 $\frac{1}{2} R_{harm}^2(K) \leq \Lambda(K) \leq \frac{1}{2} R_{vol}^2(K)$ (1)

where R_{vol} is the volume radius and R_{harm} is the radius of the ball with the same Green's capacity as K. (Capacity measures the ability of the set to "hold" the charge producing bounded potential inside the set vanishing outside)

<u>Corollary 1</u> $R_{harm}(K) \le R_{vol}(K)$. (2)

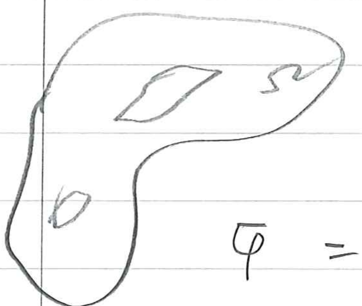
Equality on either side of (1) (hence (2)) occurs iff and only if for smoothly bounded K is a ball.

<u>Remark</u> Note that for $n=2$, $\Lambda(K)$ leads to a different isoperimetric inequality $R_{harm} \le R_{vol}$.
A lower bound depending on Area (K), Per (K) cannot occur because one can construct a "Swiss Cheese" (A. Huber, 1968) with positive area and finite perimeter yet so that $H(K) = C(K)$, i.e. $\Lambda'(K) = 0$.
To get the isoperimetric inequality we need a different "harmonic content", a different "sandwich".

Recall that analytic content for a domain $\Omega \subset \mathbb{C}$ is defined as

$$\lambda(\Omega) = \inf_{\varphi \in A(\Omega)} \| \bar{z} - \varphi \|_1 =$$

$$= \inf_{\varphi \in A(\Omega)} \| \bar{z} - \varphi \|_{C(\bar{\Omega})}$$



$$\varphi = f_1 + i f_2 \,, \quad \frac{\partial f_1}{\partial x_1} = - \frac{\partial f_2}{\partial x_2}$$

$$\frac{\partial f_1}{\partial x_2} = \frac{\partial f_2}{\partial x_1}$$

i.e., denoting by $\vec{f} = (f_1, f_2)$ the vector field $\bar{\varphi}$

we get

$$\operatorname{div} \vec{f} = 0 = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \quad (3)$$

$$\operatorname{curl} \vec{f} = 0 = \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}.$$

This leads to the definition of analytic content in higher dimensions: Define the space $A(\Omega)$ of harmonic vector fields $\vec{f} = (f_1, \dots, f_n)$ in Ω as the set of of all vector fields $\vec{f} \in C^1(\Omega) \cap C(\bar{\Omega})$ s.t.

$$\operatorname{div} \vec{f} = \operatorname{curl} \vec{f} = 0.$$

(Define $B(\Omega) = \{ \nabla \vec{h},\ h \in H(\Omega) \}$.
Note that $B(\Omega) \subset A(\Omega)$, "$\neq$" unless Ω is simply connected)

Set $\|\vec{f}\|_\infty \stackrel{\text{def}}{=} \sup_{x \in \Omega} \left(\sum_{j=1}^{n} f_j^2(x) \right)^{1/2}$.

<u>Definition</u> : $\lambda(\Omega) := \operatorname{dist}_{\| \|_\infty} (\vec{x}, A(\Omega)) =$

$$= \inf_{f \in A(\Omega)} \|\vec{x} - \vec{f}\|_\infty$$

(For a compact set K, let $\Omega_n \downarrow K$ and set $\lambda(K) = \lim_{n \to \infty} \lambda(\Omega_n)$)

<u>Theorem 2 (B. Gustafsson - Dk, 194)</u> Let $\Gamma = \partial \Omega$:

$$\frac{n \operatorname{Vol}(\Omega)}{P(\Gamma)} \le \lambda(\Omega) \le \frac{n^{1+1/n}\, \Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{2n-1}{2n-2}\right)}{2\pi^{\frac{2n-1}{2n}}\, \Gamma\left(\frac{n^2}{2n-2}\right)^{1-\frac{1}{n}}} \operatorname{Vol}(\Omega)^{1/n}$$

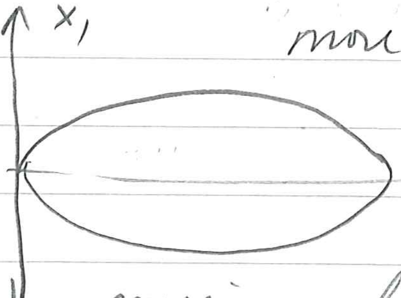
(4)

Exercise: Check that for $n=2$, RHS of (4) becomes $\sqrt{\frac{A(\Omega)}{\pi}} = R_{vol}$, but for all $n \geq 3$ the RHS $> R_{vol}$ (For $n=3$ the isoperimetric inequality is $36\pi V^2 \leq P^3$ while Theorem 2 yields $c\pi V^2 \leq P^3$, $c<36$).

The obstacle comes from the fact that in the Ahlfors-Beurling estimate for the maximum

$$\max_{x\in\Omega} \left\| \nabla \left(\int_\Omega \frac{dV(y)}{|x-y|^{n-2}} \right) \right\|_\infty ,$$

the extremal solids are not balls in all dimensions ≥ 3, although they are axially symmetric algebraic surfaces that are getting more and more tightly sealed to the tangent plane at the maximum point.

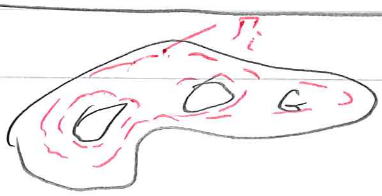


The following conjecture (Gustafsson – DK '94) remains open.

Conjecture 1. $\lambda_n(\Omega) \leq R_{vol}(\Omega)$ for all $n \geq 3$.

II Analytic content in other norms

Smirnov classes of analytic functions.



An analytic function $f(z)$ in a domain G belongs to the class $E^p(G)$, $1 \leq p < \infty$

if there exists a sequence of rectifiable curves $\{\Gamma_j\}$ in G, $\Gamma_j \to \Gamma$:

$$\|f\|^p_{E^p} := \overline{\lim_{j\to\infty}} \int_{\Gamma_j} |f(z)|^p |dz| < \infty$$

E^p are Banach spaces and consist of Cauchy's integrals with $L^p(\Gamma, ds)$ densities.

Set $\lambda_{E^p}(G) := \inf_{\varphi \in E^p(G)} \|\bar{z} - \varphi\|_{E^p}$

<u>Theorem 3</u> (Z. Guadarrama - DK, 2007) Let $p \geq 1$, $q = \frac{p}{p-1}$.

$$\frac{2A(G)}{\sqrt[q]{Per(\Gamma)}} \leq \lambda_{E^p} \leq \sqrt{\frac{A}{\pi}} \, Per(\Gamma)^{1/p}$$

<u>Questions</u> (i) Are disks and annuli the only extremal domains for all λ_{E^p}, $p \geq 1$? (Not known)

(ii) Do the extremal functions (best approximations, BA) to $\bar{z}$ characterise the domain G?

<u>Example</u> (a) If $\varphi \equiv c$ is the best uniform approximation to $\bar{z}$, then G is a disk. Indeed, WLOG $\varphi \equiv 0$, $\therefore$ $|\bar{z} - \varphi| = |\bar{z}| = \lambda = \text{const}$ on Γ $\Rightarrow$ G is a disk.

(b) If $\varphi = \frac{c}{z}$ is the best uniform ($p = \infty$) approximation to $\bar{z}$, then G is an annulus. In that case

$|\bar{z} - \frac{c}{z}| = \lambda$ on Γ $\Rightarrow$ $|z|$ is a local constant on Γ, hence Γ is an annulus.

Theorem 4 (Z. Guadarrama - DK, '07) (1) Let $\Gamma := \partial G$ be real analytic and $p \geq 1$. If the BA to $\bar{z}$ in E^p is a constant, then G is a disk.
(2) If the BA to $\bar{z}$ in E^1 (!) is $\frac{c}{z-a}$, then G is an annulus centered at a.

Conjecture (2) holds for all $p > 1$.

Unknown What are domains with BA to $\bar{z}$ in E^p are rational functions, or polynomials?

Bergman classes of analytic functions.

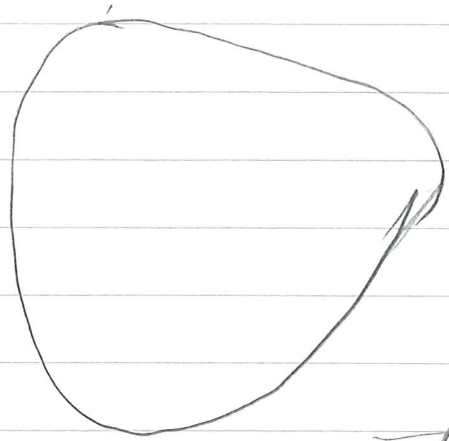
$$A_p = \{ f : \text{anal. in } G : \|f\|_{A^p}^p := \int_G |f|^p dA < \infty \}$$

$$\lambda_{A^p} = \inf_{\varphi \in A^p(G)} \|\bar{z} - \varphi\|_{A^p}.$$

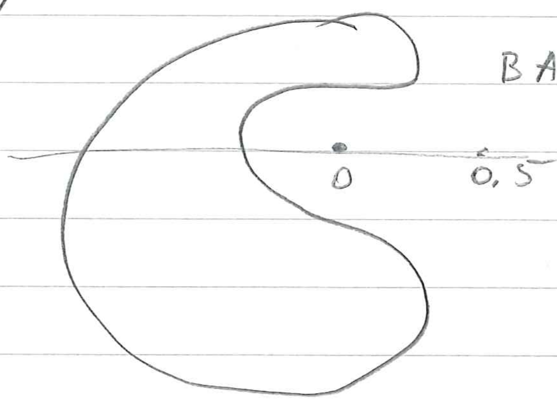
Theorem 4 (Z. G. - DK, '07) (i) If BA to $\bar{z}$ in A^p is a constant, G is a disk, $p \geq 1$.
(ii) If BA to $\bar{z}$ in A^p $p \geq 1$ (!) is $g(z) = \frac{c}{z-a}$, then G is an annulus centered at a.

Moreover, in A^2 we have more;

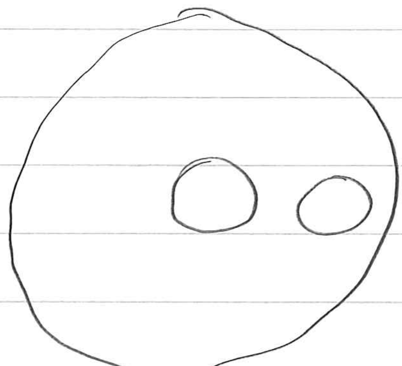
Theorem 5 (M. Fleeman - DK, '15) (i) The BA to $\bar{z}$ in $A^2(G)$ is a polynomial iff G is s.c. and the solution to the Dirichlet problem in G with data $|z|^2$ is a real harmonic polynomial (e.g, G = an ellipse).
(ii) The BA to $\bar{z}$ in A^2 is a rational function iff the solution of the DP with data $|z|^2$ is a real harmonic rational function modulo a logarithmic potential of finitely many point masses (e.g, annuli)



$BA = \frac{3z^2}{10}$



$BA = \frac{1}{3z} + \frac{1}{5(z-\frac{1}{2})}$



$BA = \frac{1}{7z} + \frac{1}{10(z-\frac{1}{2})}$.

Theorem 6 (M. Fleeman - DK '15) (An isoperimetric sandwich) following Olsen - Reguera's result)

$$\sqrt{\rho(G)} \leq \lambda_{A^2(G)} \leq \frac{A(G)}{\sqrt{2\pi}}$$

where $\rho(G)$ = torsional rigidity of G =

$= \sup_{\psi \in C_0^\infty(G)} \left(\frac{2\|\psi\|_{L^1}}{\|\nabla\psi\|_{L^2}} \right)^2$, measures the resilience of the beam of cross-section G to twisting.

Corollary (St. Venant's inequality - conj in 1856, first proved by G. Polya in 1949)

$\rho(G) \leq \frac{A^2(G)}{2\pi}$ and "=" holds only for circles.

Q. What are the sandwiches for A^p?
In which domains the BA to $\bar{z}$ in E^2 is a polynomial, a rational function?

(III) Refining the "isoperimetric sandwiches" to include connectivity of the domain.

(a) Carleman's inequality (1923).

$$\|f\|_{A^2(G)} \le \sqrt{1/4\pi}\ \|f\|_{E^1(G)}$$

<u>Theorem 7</u> (S. Jacobs). If G is n-connected, $n>1$ the constant C_G must be $> 1/4\pi$. There is no uniform bound depending only on n (the latter is obvious). For $n=2$, the "sloppy sandwich" is:

$$\max\left\{\frac{M}{2}, +\frac{\mu^2}{\nu^2}\left(\frac{M}{2}-\frac{1}{\mu}\right)\right\} \le 4\pi C_G^2 < 1+\left(\frac{M}{2}-\frac{1}{\mu}\right),$$

where M = modulus of G, $\mu = \sum_{-\infty}^{\infty} (\cosh kM)^{-2}$, $\nu = \sum_{-\infty}^{\infty} (\cosh kM)^{-1}$.

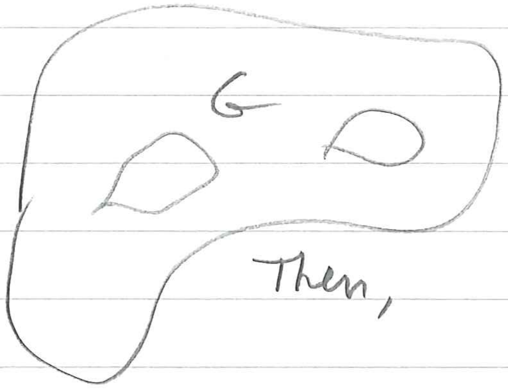
<u>Question</u>: What happens in n-connected domains, $n \ge 3$?

(b) DK - D. Luecking's theorem ('84)

<u>Theorem 8</u> Let G be a n-connected domain with area A, perimeter P, λ is its analytic content, Φ = BA to $\bar{z}$ (in L^∞), F is the extremal function in the (dual problem) of finding

$$\sup_{\substack{F\in E^1(G),\\ \|F\|_{E_1}\le 1}} \left|\int_\Gamma \bar{z}\, F(z)\,dz\right| . \qquad (5)$$

$N_F = \#\{z \in G : F(z) = 0\}$.



Then,

$$\|\Phi'\|^2_{A^2(G)} = \text{Area}(\varphi(G)) \text{ with multiplicity} =$$

$$= -A + \lambda \, \text{Im} \int_{\Gamma} \frac{\overline{F}}{|F|} \, ds - \pi(2-n+N_F)\lambda^2 \qquad (6).$$

<u>Corollary</u> (i) $\|\Phi'\|^2_{A^2} \le \lambda P - A - \pi(2-n+N_F)\lambda^2$

(ii) If $n=1$, then $\|\Phi'\|^2_{A^2} \le \frac{P^2}{4\pi}\left(1 - \frac{4\pi A}{P^2}\right)$

(iii) $4\pi A/P^2 \le 1$, " = " occurs iff G = disk ($\Phi' \equiv$ const).

<u>Questions</u>: Does it extend to other norms?

Can one say anything about G if we know analytic (algebraic) properties of Φ and F?

Can one elaborate more on Cor. (i)?