"The isoperimetric sandwiches", free boundary problems and approximation by analytic and harmonic functions

Lecture 2, Bergen, October 27, 2015.

Recall the "sandwich":

$$K \Subset \mathbb{C}, \quad \lambda(K) := \mathrm{dist}_{C(K)}(\bar{z}, R(K))$$

$$\frac{2A(K)}{P(K)} \leq \lambda(K) \leq \sqrt{\frac{A(K)}{\pi}}, \quad (1)$$

where $P(K)$ = Perimeter of ∂K (possibly, ∞) and $A(K)$ = area of K (possibly, 0).

We also observed that "=" in the RHS holds iff $K = \text{disk} \cup E$, $A(E)=0$.

<u>Question</u> (AK, ~'82). For which K does the equality hold in the LHS of (1)?

<u>Example 1.</u> If $K = \{|z-a| \leq R\}$, then

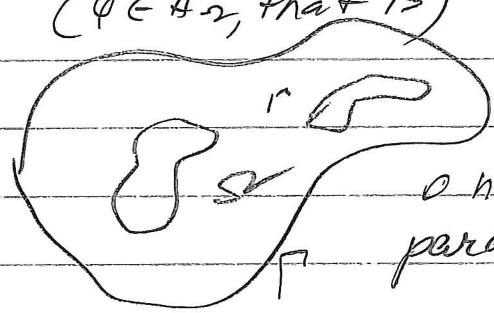
$$\frac{2A}{P} = \frac{2\pi R^2}{2\pi R} = R = \sqrt{\frac{A}{\pi}},$$ hence, of course, equality holds in both inequalities.

Q. Are disks the only examples?

Theorem (DK, ~'86). Let $K = \overline{\Omega}$, where Ω is a smoothly bounded, finitely connected domain. TFAE

(i) $\lambda = 2A/P$

(ii) $\exists\, \varphi$ analytic in Ω, continuous in $\overline{\Omega}$ ($\varphi \in A_\Omega$, that is)



s.t.

$$\overline{z}(s) - i\lambda \frac{d\overline{z}}{ds} = \varphi(z(s))$$

on $\Gamma = \partial\Omega$, s is the arclength parameter.

(iii) $$\frac{1}{A(\Omega)} \int_\Omega f\, dA = \frac{1}{P} \int_\Gamma f\, ds,$$

for all $f \in A_\Omega := \{ f$ analytic in Ω continuous in $\overline{\Omega} \}$.

Discussion:

Example (1) (iii) holds for disks

(exercise: write down $f(z) = \sum_{-\infty}^{\infty} c_n z^n$, as Laurent's series and integrate in polar coordinates to get c_0 for both sides.

(3.) Note that if Γ contains a circular arc then Ω is a disk or an annulus. Indeed, suppose, for simplicity that Γ contains an arc γ centered at the origin and of radius R. Then, on γ we can write

$$z(s) = Re^{is/R},$$ where s is the arclength. Then, on γ, we have

$$\bar{z}(s) - i\lambda \bar{\dot{z}}(s) = \frac{R^2 - R\lambda}{z(s)} = \frac{c}{z} = \varphi(z)$$ (2).

But since $\varphi \in A(\Omega)$, (2) holds everywhere in Ω. If $c = 0$, then $\varphi = 0$, $|z| = \text{const} = \lambda$ and Γ is a disk centered at the origin. If $c \neq 0$,

$$i\lambda z \bar{\dot{z}}(s) = |z|^2 - c,$$ or

$$\operatorname{Re}\left(z \bar{\dot{z}}(s)\right) = \frac{1}{2}\frac{d|z|^2}{ds} = 0.$$

$\therefore$ $|z(s)|^2$ is a local (!) constant on Γ, so Ω is an annulus.

Conjecture 1 (DK - '83-86) $\lambda = 2A/P$ iff Ω is a disk, or an annulus.

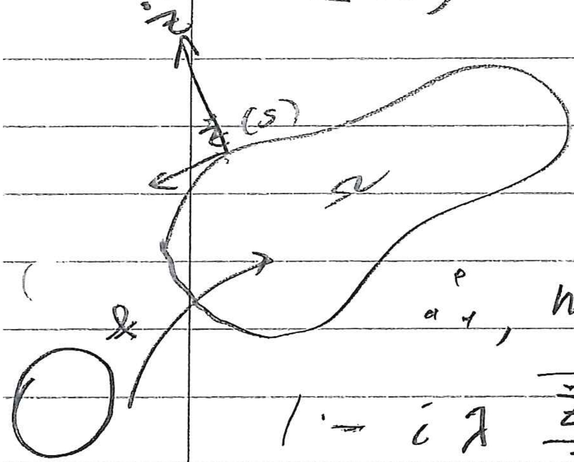
True. (A. Abanov, CB, DK, R. Teodorescu - 2011-2015, preprint will be available shortly)

3a).

Let us illustrate the usefulness of Thm 1 by showing that if Ω is simply-connected, then it must be a disk (D.K - B. Gustafsson, '8?)

Consider (ii)

$$\bar{Z}(s) - i\lambda \frac{d\bar{z}}{ds} = \varphi(z(s)) \qquad (*).$$



Differentiate wrt s

$$\dot{\bar{Z}} - i\lambda\, \overline{\ddot{Z}} = \varphi'(z(s))\cdot \dot{z} \qquad (**)$$

, multiplying by $\dot{z}$ ($|\dot{z}|=1$!)

$$1 - i\lambda \frac{\overline{\ddot{Z}}}{\overline{\dot{Z}}} = \varphi'(z(s))(\dot{z})^2 \qquad (**)$$

$\ddot{z} \perp \dot{z}$, so $i\frac{\ddot{Z}}{\dot{Z}} = i\ddot{Z}/\dot{Z}$ is real valued.

Translating via Riemann mapping theorem to the disk, we get

$\varphi'(f(w))(f'(w))^2 w^2$ is real valued on $\mathbb{T}$.

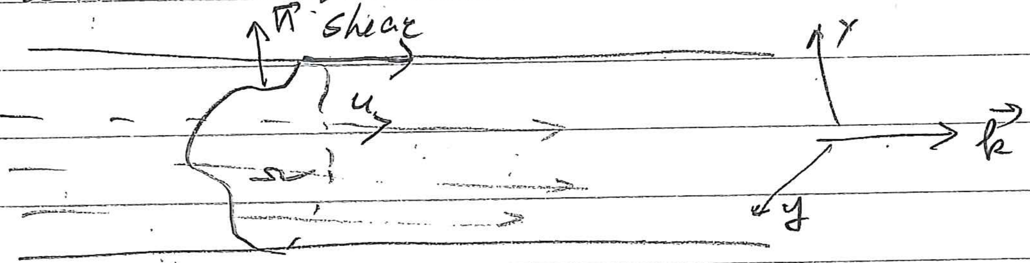
Hence it is a constant, infact, $\equiv 0$. Thus,

$\varphi' \equiv 0$, $\varphi \equiv$ const, $(*) \Rightarrow$ Γ is a disk of radius λ. #

Before proving the Thm. let us discuss some applications to free boundary problems.

(a) Laminary flow of the viscous fluid.

Suppose we have a viscous incompressible fluid flowing through a pipe with cross-section Ω. It is Newtonian fluid, i.e



the shear stress, that is, the stress exerted on the pipewall is proportional to the velocity gradient. The constant of proportionality is known as viscosity μ. Intuitively, the viscosity measures how "sticky" the fluid is. A viscous fluid is, therfore, one s.t $\mu \neq 0$, e.g, oil or tar. A fluid is incompressible means the density of the fluid is constant along its flow lines this yields conservation of mass, which implies that the divergence of the velocity vector $\vec{v}$ is zero, $\operatorname{div} \vec{v} = \nabla \cdot \vec{v} = 0$. Assuming that the flow is laminary, i.e, the fluid flows in lines // to the pipe walls ($\vec{z}$-axis) and the flow is steady $\frac{d\vec{v}}{dt} = 0$ (t = time). Hence, $\vec{v} = v(x,y)\vec{k}$

The Navier-Stokes equations in this simplified context can be written as

$$(3) \qquad \Delta v = -\frac{1}{\mu}\frac{\partial p}{\partial z}.$$

where $p = p(x, y, z)$ is the pressure and $p = p(z)$ since the flow is laminar. Since $\frac{\partial^2 p}{\partial z^2} = 0$, the RHS of (3) is a constant.

Thus, scaling, (3) reduces to the BV problem

$$\Delta v = 1 \text{ in } \Omega, \quad v = 0 \text{ on } \partial\Omega \qquad (4)$$

(The pipe stays).

The force exerted by the fluid on the pipe is given by ($\vec{n}$ is an outward unit normal):

$$(5) \qquad \vec{F} = \left(p - \frac{4}{3}\mu \nabla\cdot v\right)\vec{n} + \mu\left(\vec{n}\times(\nabla\times\vec{v})\right)$$

Since $\vec{v} = v\vec{k}$, $\nabla\cdot\vec{v} = 0$, we have

$$(6) \qquad \vec{F} = p\vec{n} - \mu\frac{\partial v}{\partial n}\vec{k},$$

Thus, the quantity $\mu\frac{\partial v}{\partial n}$ represents the "shear stress" on the wall (as expected).

Serrin's Theorem ('71). The shear stress at each point on the wall is the same iff the cross-section of the pipe is a disk.

Mathematically, rescaling

If the overdetermined boundary value problem

$$\Delta v = 1 \quad \text{in } \Omega$$
$$(7) \quad v = 0 \quad \text{on } \Gamma = \partial\Omega$$
$$v_n = \text{const} \quad \text{on } \Gamma = \partial\Omega$$

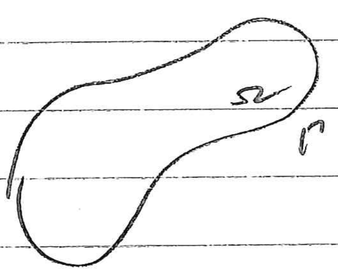
has a smooth solution.

What does it have to do with our Theorem?

~~Look~~ at the condition (iii). If Ω is simply connected, taking the real parts gives

$$\frac{1}{A}\int u \, dA = \frac{1}{P}\int_\Gamma u \, ds \quad (8)$$

for all u harmonic in Ω. Consider v, the solution of the Dirichlet problem



$$\Delta v = 1 \quad \text{in } \Omega \quad (9)$$
$$v = 0 \quad \text{on } \Gamma.$$

Take any harmonic function u (test-function) then using Green's formula and (8) we obtain

$$\int_\Gamma u \, v_n \, ds \overset{\text{Green}}{\underset{(9)}{=}} \int_\Omega u \, dA \overset{(8)}{=} \frac{A}{P}\int u \, ds.$$

Since u was an arbitrary test-function $v_n = A/P$ for v in (9) if (8) holds

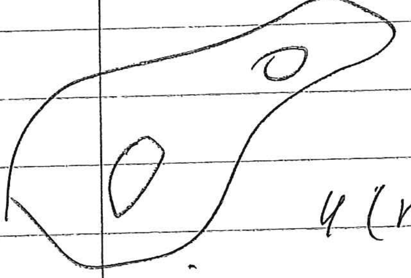
Thus, Serrin's theorem (that is difficult) [illegible] our Conjecture for simply connected domains.

<u>Remark</u> The Serrin's type free boundary problems are exceedingly difficult. The following (still) open problem," free boundary problem, is strikingly similar.

Suppose there exists throughout Ω a smooth solution of the following problem:

$$(10) \qquad \Delta u + \lambda u = 0 \quad \text{on } \Omega, \quad \lambda > 0,$$
$$u = c, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma$$

Does Ω have to be a disk?



In $\mathbb{D}$, e.g., the function

$$u(r) = J_0(\sqrt{\lambda}\, r), \quad \lambda = \mu^2, \ \mu$$

is any zero of $J_1(r)$ furnishes a nontrivial example. ($J_\alpha(t) = \sum_0^\infty \frac{(-1)^k t^{\alpha+2k}}{2^{\alpha+2k} k!\, \Gamma(\alpha+k+1)}$ are Bessel functions. Surprisingly (10) is equivalent to the following, Pompeiu's Problem.

Are disks the only domains in $\mathbb{R}^2$ so that there exists $\not\equiv 0$ function $f \in C(\mathbb{R}^2)$:

$$\int_{\alpha(\Omega)} f\, dA = 0 \quad \text{for all euclidean}$$

motions α.

Conjecture 1 is equivalent in this context to the Conjecture 2

Conjecture 2 Let Ω be a finitely connected domain, if for $n = \#$ (connected components of $\Gamma = \partial\Omega$) ≥ 2 the Overdetermined Boundary Value Problem

$$\Delta v = 1 \text{ in } \Omega,$$
$$(11) \quad \frac{\partial v}{\partial n} = \frac{A}{P} \text{ on } \Gamma$$
$$v\Big|_{\gamma_j} = c_j, \quad 1 \leq j \leq n$$

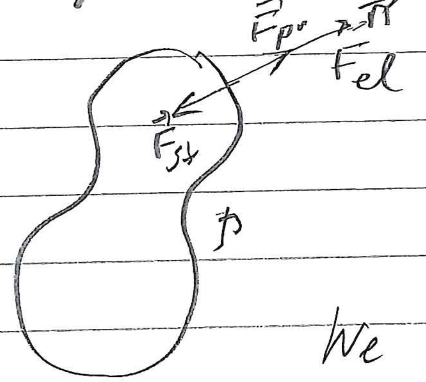
has a smooth solution in Ω, then Ω must be an annulus

A Banov - CB - DK - Teodorescu $\Rightarrow$ Conj. 2.

Until now, the efforts of Serrin, Alessandrini (1992), Reichel ('96), Willms, Gladwell & D. Siegel ('94), Sirakov, L. Payne and H. Weinberger ('8[illegible]), Philippin and Ragoub ('95) only verified Conj. 2 under various additional provisions that v doesn't achieve local extrema inside Ω. Of course, (11) $\Rightarrow$ v achieves its minima inside Ω.

(b) Shapes of electrified droplets.

Consider a droplet of perfectly conducting fluid in the plane with given electrostatic potential U, there are three forces acting on the boundary of the droplet: $\vec{F}_{el}$, the electrostatic force, and the pressure $\vec{F}_{pr}$; both are trying to tear the droplet apart and the force due to surface tension $\vec{F}_{sf}$ trying to keep the droplet from ruin



We want to find the free boundary of the charged droplet in equilibrium.

$$\vec{F}_{el} \sim |\vec{E}|^2 \vec{n}\, ds, \quad \vec{E} = \nabla U, \text{ the electric field}$$

If we set $\Phi = U + iV$ to be the analytic potential relevant to U,

$$\vec{E} = \underbrace{2\frac{\partial}{\partial \bar{z}}}_{\nabla}\left(\frac{\Phi + \bar{\Phi}}{2}\right) = \overline{\frac{\partial \Phi}{\partial z}}$$

Assuming Γ is "super smooth", real-analytic so it is parametrized by the Schwarz function: $\Gamma = \{\bar{z} = S(z)\}$

(Ex. $\Gamma = \{|z-a| = R\}$, $S(z) = \frac{R^2}{z-a}$, $\Gamma = \mathbb{R}$, $S(z) = z$, Γ = ellipse – exercise).

Now, $1 = \frac{d\bar{z}}{ds}\frac{dz}{ds} = S'(z)\left(\frac{dz}{ds}\right)^2$, so the normal $\vec{n} = -i \frac{dz}{|dz|} = -\frac{i}{\sqrt{S'(z)}}$, so

$$\vec{F}_{el} \sim -\frac{i}{\sqrt{S'(z)}} |\partial \Phi|^2 ds .$$

while on Γ $\vec{E} \perp \Gamma$
$\vec{E} = \overline{\partial \Phi} = |\partial \Phi| \left(-\frac{i}{\sqrt{S'(z)}}\right)$.

$$|\partial \Phi| = i \sqrt{S'}\, \overline{\partial \Phi} = -i/\sqrt{S'}\, \partial \Phi ,$$

as $|S'| = 1$ on Γ. Summarizing,

$$(12) \quad \vec{F}_{el} \sim -\frac{i}{\sqrt{S'}} \left(-\frac{1}{S'}\right)(\partial \Phi)^2 ds = \frac{i}{(S')^{3/2}} (\partial \Phi)^2 ds$$

The surface tension is proportional to the curvature, that is $\vec{F}_{sf} \sim \frac{d\vec{T}}{ds} ds$, where $\vec{T} = \frac{dz}{ds} =$

$$= \frac{1}{\sqrt{S'}} , \quad \frac{d}{ds} = \frac{d}{dz}\frac{dz}{ds} = \frac{1}{\sqrt{S'}} \frac{d}{dz} \quad \text{we}$$

find

$$(13) \quad \vec{F}_{sf} \sim \frac{ds}{\sqrt{S'}} \frac{d}{dz} \left(\frac{1}{\sqrt{S'}}\right) .$$

Finally, the pressure (14) $\vec{F}_{pr} \sim \vec{n}\, ds = -\frac{i}{\sqrt{S'}} ds$.

The equilibrium then becomes

$$c_1 \left(\frac{\partial \Phi}{\partial z}\right)^2 \frac{i}{(S')^{3/2}} ds + c_2 \frac{d}{dz}\left(\frac{1}{\sqrt{S'}}\right) ds + c_3 \frac{-i}{\sqrt{S'}} ds = 0 ,$$

or

$$c_1 \left(\frac{\partial \Phi}{\partial z}\right)^2 \cancel{\frac{\cdot}{(S')^{?}}} + i c_2 S' \frac{d}{dz}\left(\frac{1}{\sqrt{S'}}\right) - c_3 S' = 0$$

Noticing, $S' \frac{d}{dz}\left(\frac{1}{\sqrt{S'}}\right) = \frac{-1}{2\sqrt{S'}} \frac{dS'}{dz}$, while

$$\frac{d\sqrt{S'}}{dz} = \frac{1}{2\sqrt{S'}}\frac{dS'}{dz}\,),$$ we obtain

(14). $c_1\left(\frac{\partial\phi}{\partial z}\right)^2 + i c_2 \frac{d}{dz}(\sqrt{S'}) - c_3 S' = 0$.

Set $F(z) = c_1 \int \left(\frac{\partial\phi}{\partial z}\right)^2 dz$. Then (14) becomes, after integration,

$$F(z) + i c_2 \sqrt{S'} - c_3 S(z) = 0.$$

Dividing by c_3, renaming $F(z)/c_3 = \varphi(z)$, $c_2/c_3 = \lambda$ and using $S(z) = \bar{z}$ on Γ and $\sqrt{S'(z)} = \frac{1}{\frac{dz}{ds}} = \frac{d\bar{z}}{ds}$ we arrive at

(15) $$\bar{z}(s) - i\lambda \frac{d\bar{z}}{ds} = \varphi(z),$$

which is precisely (ii) of Thm. 1.

Remarks (1) If the potential U has a point charge at z_0, then $\varphi(z) \sim \frac{c}{z - z_0}$ near z_0, that is it has a pole.

(2) Usually, for a physical droplet, the fluid is assumed to be incompressible. Then either the area is assumed to be fixed, or the area, the pressure and the temperature are connected by the "equation of state". In particular, for an incompressible fluid, the pressure has to be adjusted each time the area is fixed. If we amend the problem with this requirement, the physical picture is the following. Consider a plane

with a system of charges on it, we throw a droplet of fluid onto the plane and see where it comes to rest and what shape it will have. For example, if there is only one charge, the charge will induce a dipole moment on the droplet, and the dipole will move to "swallow" the charge. Then, there will be no charge outside and the charge inside will redistribute itself over the surface, while at ∞ we still have $\varphi \sim \frac{c}{z}$. Hence, in this case, (15) becomes

$$\bar{z}(s) - i\lambda \frac{d\bar{z}}{ds} = \frac{c}{z}, \qquad (16)$$

or

$$S(z) - i\lambda\sqrt{S'(z)} = \frac{c}{z}.$$

If we denote $u(z) = \sqrt{S'(z)}$ and differentiate wrt z, (16) becomes

$$u^2 - i\lambda u' = -\frac{c}{z^2}, \qquad (17)$$

the Ricatti equation and the solution (it is a first order ODE!) $u = \frac{\text{Const}}{z}$ is easily found, giving $S' = \frac{\text{const}}{z}$, giving Γ to be a circle centered at the origin.

Note that the "physical" solution yields the same result without any calculations, merely by noticing that c/z is radially symmetric ($U = \log|z|$), and, hence, the problem must have radially symmetric solution, i.e. a circle or an annulus.

(3) Let us again look at (15) where $\varphi(z) = \text{const} \int \left(\frac{\partial \Phi}{\partial z}\right)^2 dz$, $\Phi = U + iV$ is the analytic potential. Then (15) induces an extra condition on the problem, namely, that $\sqrt{\varphi'}$ is a single-valued function. In general, if $U(z) = \int \log|z-\zeta|\, d\mu(\zeta)$ is an arbitrary potential of a charge distribution μ, then

$$\sqrt{\varphi'} = \text{const} \frac{\partial \Phi}{\partial z} = \int \frac{d\mu(\zeta)}{\zeta - z},$$

a single-valued function. We shall call the solution to the problem (15) with $\sqrt{\varphi'}$ single-valued a physical droplet versus a mathematical droplet if not.

(4) The overdetermined ~~boundary~~ problem (15) is very restrictive, as we had noted earlier, if the free boundary Γ contains a circular arc, Ω must be a disk, or an annulus.

(5) Finally, we mention a slightly more general free boundary problem

(18) $p\bar{z} - i\tau \frac{d\bar{z}}{ds} = F(z)$,

where F is a given analytic, or meromorphic function was studied by Garabedian, Macleod '64-'65, and recently by D. K. - A. Solynin - D. Vassilev (2005) (K-S-S, '05). In particular, choosing pressure $p = 0$, and F analytic in $\mathbb{C} \setminus \Omega$ with a simple pole at ∞ gives rise to an interesting family of algebraic droplets. Only one (!) of them is a physical droplet (found by Macleod).

Physically $p \approx 0$ (i.e, small) correspond also to free boundary of air bubble in a fluid flow. For $\tilde{\iota} = 0$, of course, there are no "physical" droplets, mathematical droplets are ellipses, though. For $p=0$, F analytic in $\bar{\mathbb{C}} \setminus \Omega$ the free boundary is a circle if it is smooth, yet there are "monstrous" (non Smirnov) curves for which $\frac{\bar{d}z}{ds} = g(z)$ a.e. on Γ, g analytic in Ω, and $\neq 0$ there and $\frac{1}{g}$ is bounded. This goes back to (2002) P. Ebenfelt - DK - H. S. Shapiro. Surprisingly, it came up in the study of the first eigenvalues of the spectrum of the single layer potential (~~A~~ simple question is for which curves the equilibrium charge distribution has constant density).

(6) (a) Curiousity Viscous fluids look like solids

~~QUEENSland University, set in 1927, 8 drops~~ ~~(asphalt)~~ 7-13 years for drop to form.

Dublin, Trinity College (set in 1944), was finally caught on camera in 2013

(b). Electrowetting applications An electric force is applied at the interface of a droplet of conducting fluid and a solid: digital cameras, camera phones, home security systems). In 2003, at Philips Research people created a fluid lens: two non-mixing fluids, one conducting and one not. The layer between (the meniscus) acts as a lens. Changing the electric field

Sketch of the proof of Thm. 1 (p. 2)

Let φ be the best approximation to $\bar{z}$ in Ω.

$$\frac{2A}{P} = \lambda = \|\bar{z} - \varphi\| \geq \frac{1}{P}\int_{\Gamma} |\bar{z} - \varphi|\, ds \geq \frac{1}{P}\left|\int (\bar{z} - \varphi)\, dz\right| = \frac{2A}{P}$$

Since we have "=" at each step, we must have

$$(19) \quad e^{i\alpha}(\bar{z} - \varphi)\, dz = \lambda\, ds \text{ on } \Gamma.$$

Integrating, applying Green'-Stokes, yields $e^{i\alpha} = -i$, so, from (19)

$$(20) \quad \bar{z} - i\lambda \frac{ds}{dz} = \varphi \quad \text{on } \Gamma,$$

thus (i) $\Rightarrow$ (ii)

(ii) $\Rightarrow$ (iii) follows at once by Stokes' formula (8) in Lecture 1.

To show that (iii) $\Rightarrow$ (ii), note that $\forall f \in A$,

$$\frac{1}{A}\int_{\Omega} f\, dA = \frac{1}{A}\int_{\Omega} \frac{\partial}{\partial \bar{z}}(\bar{z} f)\, dA \overset{\text{Stokes'}}{=} \frac{1}{2iA}\int_{\Gamma} \bar{z} f\, dz \overset{(iii)}{=} \frac{1}{P}\int_{\Gamma} f \frac{ds}{dz}\, dz. \quad (21)$$

Thus, $\left(\bar{z} - i\lambda \frac{d\bar{z}}{ds}\right)\Big|_{\Gamma}$ annihilates all analytic functions and (20) follows from F. & M. Riesz Thm.

(ii) (as in (20)) $\Rightarrow (\bar{z} - \varphi)(-i)\, dz = \lambda\, ds > 0$ on Γ

and $|\bar{z} - \varphi| = \lambda$. Hence, φ is the best approximation to $\bar{z}$ and Green's formula $\Rightarrow \lambda = 2A/P$.