multiply connected domains. Clearly, the constant depends on the geometry of a domain since for $\{\rho < |z| < 1\}$ and $f(z) = \frac{1}{z}$ with $\rho \downarrow 0$, LHS in (2) $\uparrow \infty$ while the RHS remain bounded. (What happens if we replace H^1-norm by H^2-norm in the RHS?

II. The isoperimitric sandwiches

1. Approximation by rational functions.

Let K be a compact set in $\mathbb{C}$.

Def. The analytic content $\lambda(K)$ is defined as

$$(5) \qquad \lambda(K) := \operatorname{dist}(\bar{z}, R(K)) = \inf_{g \in R(K)} \|\bar{z} - g\|_{C(K)},$$

where $R(K)$ consists of all rational functions with poles outside K and their uniform limits on K.

The Stone-Weierstrass theorem $\Rightarrow \lambda(K) = 0$ iff $R(K) = C(K)$

$\left(\|f\|_{C(K)} = \|f\|_K = \max_{z \in K} |f(z)| \right)$

By the Cauchy integral formula all functions in $R(K)$ are analytic in $\mathring{K}$ = interior of K

However, not all continuous on K, analytic on $\mathring{K}$ functions belong to $R(K)$.

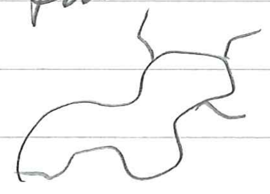
If $\mathring{K} = \emptyset$, i.e., K is nowhere dense, the analyticity requirement is void and the question is when $R(K) = C(K)$?

(i) If $\mathbb{C} \setminus K$ is connected, e.g., K is a Jordan arc, the celebrated theorem (1934) of Lavrentjev states that

$$R(K) = P(K) = C(K)$$

in that case, $P(K)$ is the uniform closure of analytic polynomials on K.

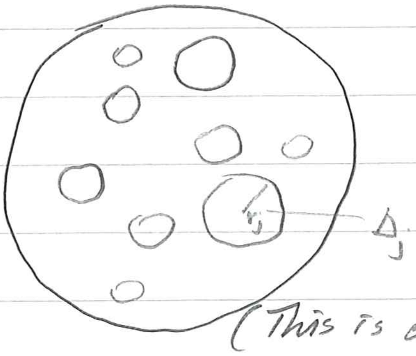
(ii) In 1951 S. Mergelyan proved that for all K : $\mathbb{C} \setminus K$ is connected, all functions analytic in $\mathring{K}$ and continuous on K are uniformly approximable by polynomials.



(iii) However, for $R(K)$ the situation is much more complicated

The celebrated example of "Swiss cheese" set obtained from the unit disk by removing a sequence of nonoverlapping disks $\{\Delta_i\}$

where radii r_j : satisfy $\sum r_j < \infty$.
and $\mathbb{D} \setminus \bigcup_1^\infty \Delta_j \zeta_j =: K$ is nowhere dense,



serves as example of
a compact K: $\mathring{K} = \emptyset$,
but $R(K) \neq C(K)$
why?
(This is celebrated S. Mergelyan ('51) – A. Roth ('38)) (example)

<u>Exercise</u> The measure $\mu = \frac{dz}{2\pi i}\Big|_{\mathbb{T}} - \sum_1^\infty \frac{dz}{2\pi i}\Big|_{\partial\Delta_j}$

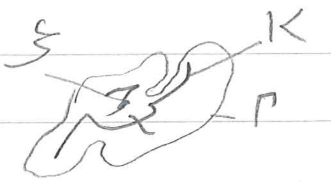
annihilates all $R(K)$ functions. Hence,
by the Hahn-Banach theorem, $R(K) \neq C(K)$.

(iv) One of the first results in studying analytic approximation was the classical C. Runge's theorem (1885)

<u>Theorem</u> If f extends to be analytic in an open neighborhood of K, then $f \in R(K)$.

The idea for the proof is simple.

$$(6) \qquad f(\zeta) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)\,dz}{z - \zeta}, \quad \zeta \in K$$



where Γ is a nice contour surrounding K close enough so inside Γ f is analytic since for each $z \in \Gamma$ fixed, $f(z)/_{z-\zeta}$ is a rational function of ζ on K, Runge's theorem follows from approximating the integral in (6) by Riemann sums.

Note: If K has a connected complement then each rational function $g(\zeta) = \frac{1}{z-\zeta}$, $z \notin K$, can be approximated uniformly on K by polynomials in ζ. This is seen by moving the pole z to ∞.

As we noted before $\lambda(K) = 0 \Leftrightarrow \bar{z} \in R(K)$ and hence, by the Stone-Weierstrass theorem, $R(K) = C(K)$.

The "sandwich"

Theorem 2.1. $\frac{2\,\mathrm{Area}(K)}{\mathrm{Perimeter}(K)} \le \lambda(K) \le \sqrt{\frac{\mathrm{Area}(K)}{\pi}}$,

$$\frac{2A}{\Pi} \le \lambda(K) \le \sqrt{\frac{A}{\pi}}$$

Khavinson '82

H. Alexander '73
Gamelin-Khavinson 83-86

Moreover, in the RHS equality occurs iff K is a disk union with a set of area zero.

Cor 1 (F. Hartogs -- A. Rosenthal)
If $A(K)=0 \Rightarrow \lambda = 0 \Rightarrow R(K) = C(K)$.

Cor 2. $\frac{2A}{P} \le \sqrt{\frac{A}{\pi}} \iff 4\pi A \le P^2$
for smooth domains
and equality occurs iff K is a disk

Remark: The RHS $\sqrt{\frac{A}{\pi}}$ is simply the radius of the disk with the same area as K, so-called the volume radius.

2. Proof of Thm 2.1

(i) Cauchy - Green - Pompeiu formula.

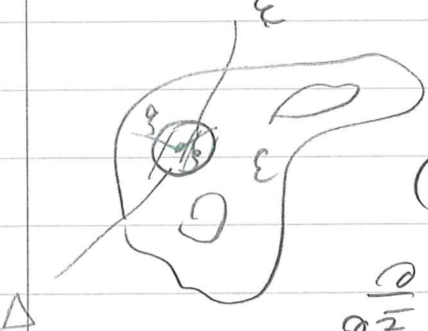
Recall $\frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right)$.

For any g, say, in $C^1(\bar G)$, G is a smoothly bounded domain, we have

(7)
$$g(\zeta) = \frac{1}{2\pi i} \int_{\Gamma = \partial G} \frac{g(z)}{z-\zeta}\, dz - \frac{1}{\pi} \iint_G \frac{\partial g}{\partial \bar z} \frac{1}{z-\zeta}\, dA, \quad \zeta \in G$$
$$0 = \quad " \quad\quad " , \quad \zeta \in \mathbb{C} \setminus \bar G.$$

If g is analytic in G, $\frac{\partial g}{\partial \bar z} = 0$ and (7) reduces to the Cauchy formula.

To prove (7) excise from G a disk Δ centred at ζ, of radius ε, and apply Green's formula to $\frac{g(z)}{z-\zeta}$ in $G_\varepsilon = G \setminus \Delta_\varepsilon$. Then let $\varepsilon \downarrow 0$.



If we replace g, by

$$(z-\zeta)\, g(z)$$

$$\frac{\partial}{\partial \bar z}\left[(z-\zeta)g(z)\right] = (z-\zeta)\frac{\partial g}{\partial \bar z},$$

and (7) becomes

$$0 = \frac{1}{2\pi i}\int_\Gamma g(z)\,dz - \frac{1}{\pi}\iint_G \frac{\partial g}{\partial \bar z}\,dx\,dy$$

or, $z = x+iy$,

$$(8) \qquad \int_\Gamma g(z)\,dz = 2i\int_G \frac{\partial g}{\partial \bar z}\,dx\,dy.$$

This is the complex form of Stokes' formula.
Note that the second statement in (7) follows from (8) by replacing g with $\frac{g(z)}{z-\zeta}$ since $\frac{1}{z-\zeta}$ is analytic in $\bar G$ for $\zeta \notin \bar G$.

(ii) <u>Pf of lower bound</u> WLOG $K = G \cup \Gamma$

Fix $h \in R(K)$, analytic in a neighborhood of K. Let $ds = |dz|$ be the arclength measure on $\Gamma = \partial K$, $P = \int_\Gamma ds$. Apply (8) to $g(z) = \bar{z} - h(z)$, we obtain:

$$(9) \quad \int_\Gamma [\bar{z} - h(z)]\, dz = 2i \int_G 1 \cdot dx\, dy = 2i A,$$

$$A = \text{Area}(K).$$

Yet

$$\left| \int_\Gamma [\bar{z} - h(z)]\, dz \right| \le \int_\Gamma |\bar{z} - h(z)|\, ds$$

$$(10) \quad \le \max_{z \in \Gamma} |\bar{z} - h(z)| \int_\Gamma ds = \|\bar{z} - h\|_K P.$$

Taking the infimum over such h and using (9) we obtain

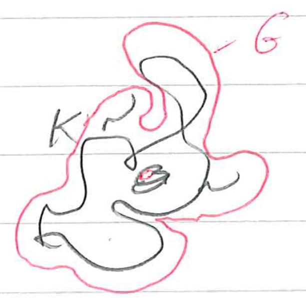
$$2A \le \lambda(K) P. \qquad (11)$$

<u>If</u> K is not smoothly bounded, (11) still holds as soon as we make sense of the perimeter of K, finite or infinite. If $P(K) = \infty$, (11) is trivial. The theory of sets with finite perimeter has been developed in the 50-70s in Geometric

measure theory (mostly, by W. Flemings, H. Federer and E. Di Georgi).

(iii) Upper Bound (Alexander's spectral area estimate)

Let G be a smoothly bounded domain containing K.



Apply (7) to $g = \bar{z}$:

$$\bar{\xi} = \frac{1}{2\pi i} \int_{\Gamma = \partial G} \frac{\bar{z}\, dz}{z - \xi} dz - \frac{1}{\pi} \iint_G \frac{1}{z-\xi} dx dy$$

(11) $\xi \in K$.

Now, let $f(\xi) = \frac{1}{\pi} \iint_K \frac{1}{z-\xi} dx dy$.

(11) $\Rightarrow$

$$(12) \quad \bar{\xi} + f(\xi) = \frac{1}{2\pi i} \int_{\Gamma = \partial G} \frac{\bar{z}\, dz}{z - \xi} dz - \frac{1}{\pi} \iint_{G \setminus K} \frac{1}{z-\xi} dx dy.$$

The integral around Γ depends analytically on ξ for $\xi \in \mathring{G}$, hence, by Runge's theorem it belongs to $R(K)$.

<u>Claim 1</u> (Mergelyan's estimate)

$$\frac{1}{\pi} \left| \int_{G \setminus K} \frac{dx\, dy}{z - \xi} \right| \le \frac{2}{\pi} \left[\frac{\text{Area}(G \setminus K)}{\pi} \right]^{1/2} \downarrow 0$$

when $G \downarrow K$.

Assuming Claim 1, we conclude that
$\bar{\xi} + f(\xi) \in R(K)$.

Hence, $\lambda(K) \le \| \bar{z} - (\bar{z} + f(z)) \|_K =$

(13) $= \|f\|_K$

(IV) **Ahlfors - Beurling Estimate**

Claim 2 ($\Rightarrow$ Claim 1) For any $K \Subset \mathbb{C}$

(14) $$\max_{\xi \in \mathbb{C}} \left| \frac{1}{\pi} \iint_K \frac{dx\,dy}{z - \xi} \right| = \|f(\xi)\|_K \le$$
$$\le \sqrt{\frac{Area(K)}{\pi}}$$

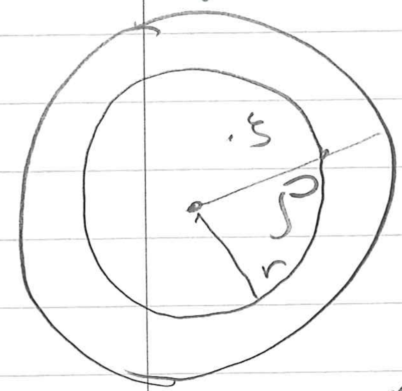
(Claim 2 $\Rightarrow$ Claim 1 since we dropped the factor 2 in the RHS).

Moreover the equality in (13) occurs iff K is a union of a disk of radius $\sqrt{\frac{A(K)}{\pi}}$ and a closed set of zero area

Note that (13), (14) $\Rightarrow \lambda(K) \le \sqrt{\frac{Area(K)}{\pi}}$
and the equality occurs iff K is a disk modulo, perhaps, a set of area zero.

Proof of (14). First, note that if $K = \Delta$

the closed disk centered at the origin of radius $\rho > 0$, then



$$f_{\Delta_\rho}(\xi) = \frac{1}{\pi}\int_0^\rho \left\{\int_0^{2\pi} \frac{1}{re^{i\theta}-\xi}\, d\theta\right\} r\,dr$$

$$\int_0^{2\pi}\frac{d\theta}{re^{i\theta}-\xi} = \int_{|z|=r} \frac{1}{z-\xi}\,\frac{dz}{iz} =$$

$$= \frac{1}{i\xi}\int_{|z|=\rho}\left[\frac{1}{z-\xi}-\frac{1}{z}\right]dz = \begin{cases}0, & |\xi|<r,\\ -\frac{2\pi}{\xi}, & |\xi|>r\end{cases}$$

by the residue calculus.
(The residues at 0 and ξ cancel if $|\xi|<r$, while only the residue at $z=0$ enters if $|\xi|>r$). Thus

$$(15)\quad f_{\Delta_\rho}(\xi) = -\frac{2}{\xi}\int_0^{\min(\rho,|\xi|)} r\,dr = \begin{cases}-\bar{\xi}, & |\xi|\le\rho\\ -\frac{\rho^2}{\xi}, & |\xi|>\rho\end{cases}$$

So,
$$|f_{\Delta_\rho}(\xi)| \le \rho = \left[\frac{\text{Area}(\Delta_\rho)}{\pi}\right]^{1/2}, \quad \xi\in\mathbb{C}.$$

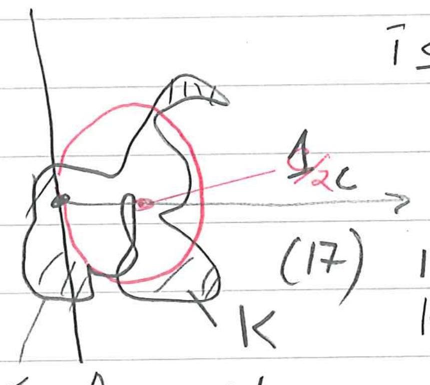
Note that $f(\xi)$ is analytic off K, hence attains its maximum on K.

Moreover, (exercise) $f(\xi)$ is continuous in $\mathbb{C}$ for any K as a convolution of a locally integrable function $\frac{1}{z}$ and a bounded function $\chi_K(z) = \begin{cases} 1, & z \in K \\ 0, & z \notin K \end{cases}$.

Thus, $|f_K(\zeta)|$ attains its maximum somewhere on K. Performing a translation we can assume that $f_K(\zeta)$ attains its maximum at the origin. Furthermore, performing a rotation we can assume that $f_K(0) > 0$. Thus

$$(16) \quad \|f_K\|_K = f_K(0) = \frac{1}{\pi} \iint_K \mathrm{Re}\,\frac{1}{z}\,dx\,dy.$$

<u>Exercise:</u> For any $c > 0$, the set $\{\mathrm{Re}\,\frac{1}{z} \geq c\}$ is a disk Δ_c centered at $(\frac{1}{2c}, 0)$ with radius $1/2c$.



$$(17) \quad \max_{K \setminus \Delta_c} \mathrm{Re}\,\frac{1}{z} \leq c \leq \min_{\Delta_c \setminus K} \mathrm{Re}\,\frac{1}{z}$$

Also, choosing c: $\mathrm{Area}(K) = \mathrm{Area}(\Delta_c)$ we have (18) $\mathrm{Area}(K \setminus \Delta_c) = \mathrm{Area}(\Delta_c \setminus K)$, and hence from (17), (18) it follows that (16) is maximized when K, up to a set of area zero coincides with the disk

Δ_C for an appropriate C, so that Area$(K)=$ Area(Δ_c). But we have calculated that for the disk Δ_C:

$$\|f_{\Delta_C}\|_{\Delta_C} = \text{radius} = \sqrt{\frac{A(K)}{\pi}}$$

Since

$$\|f_K\|_K \leq \|f_{\Delta_C}\|_{\Delta_C}$$

Claim 2 (the Ahlfors-Beurling estimate) follows

<u>Remark</u> The Ahlfors Beurling estimate (14) can be rephrased that among all uniformly charged (or, with uniform mass density) the disk produces maximal electrostatic force.

Indeed, the potential of a planar electric field with uniform charge distribution in a plate G is given by

$$u(\zeta) = \frac{1}{2\pi}\iint_G \log\frac{1}{|z-\zeta|}\,dx\,dy,$$

and the force (electric field) is given by

$$\nabla u(\zeta) = 2\frac{\partial}{\partial\zeta}u(\zeta) = -\frac{1}{2\pi}\iint_G \frac{dx\,dy}{z-\zeta} = -\frac{1}{2}\overline{f_G(\zeta)}.$$

It turns out, that this is no longer true in $\mathbb{R}^n$ $n\geq 3$ where $\log\frac{1}{|z-\zeta|}$ is replaced by $|z-\zeta|^{2-n}$ (Newtonian potential), B. Gustafsson - DK, '94.