multiply connected domains. Clearly, the constant depends on the geometry of a domain since for \( \{0 < |z| < 1\} \) and \( f(z) = \frac{1}{z} \) with \( p \perp 0 \), the RHS in (2) is while the RHS remain bounded. (What happens if we replace \( H^1 \)-norm by \( H^2 \)-norm in the RHS?)

II. The isoperimetric sandwiches

1. Approximation by rational functions.

Let \( K \) be a compact set in \( \mathbb{C} \).

**Def.** The analytic content \( \lambda(K) \) is defined as

\[
\lambda(K) := \text{dist}(z, R(K)) = \inf \| z - g \|_{C(K)}^2
\]

where \( R(K) \) consists of all rational functions with poles outside \( K \) and their uniform limits on \( K \).

*The Stone-Weierstrass theorem \( \Rightarrow \lambda(K) = 0 \iff R(K) = C(K) \)*

\[
\| f \|_{C(K)} = \max_{z \in K} | f(z) |
\]

By the Cauchy–integral formula, all functions in \( R(K) \) are analytic in \( \text{int} K \).
However, not all continuous on $K$, analytic on $K$ functions belong to $R(K)$. If $\text{co} K = \varnothing$, i.e., $K$ is nowhere dense, the analyticity requirement is void, and the question is when $R(K) = C(K)$?

(i) If $C \setminus K$ is connected, e.g., $K$ is a Jordan arc, the celebrated theorem (1934) of Laurentjev states that 

$$R(K) = P(K) = C(K)$$

in that case. $P(K)$ is the uniform closure of analytic polynomials on $K$.

(ii) In 1951 S. Merzlyan proved that for all $K: C \setminus K$ is connected, all functions analytic in $K$ and continuous on $K$ are uniformly approximable by polynomials.

(iii) However, for $R(K)$ the situation is much more complicated. The celebrated example of Swiss cheese set obtained from the unit disk by removing a sequence of nonoverlapping disks $\{D_i\}$
where radii \( r_j \) satisfy \( \sum r_j < \infty \)
and \( \text{D} \bigcup \{ A_j \} = K \) is nowhere dense,

\[ \begin{array}{c}
\text{Exercise} \\
\text{The measure } \mu = \frac{d\zeta}{2\pi i} \left[ -\sum \frac{d\zeta}{2\pi i} \right] \\
anihilates all \( R(K) \) functions. Hence, \\
\text{by Hahn–Banach theorem, } R(K) \subseteq C(K)
\end{array} \]

One of the first results in studying analytic approximation was the classical
C. Runge’s theorem (1885)

Theorem. If \( f \) extends to be analytic
in an open neighborhood of \( K \), then
\( f \in R(K) \).

The idea for the proof is simple.

\( f(\zeta) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z) \, dz}{z - \zeta}, \quad \zeta \in K \)

where \( \Gamma \) is a nice contour

surrounding \( K \) close enough to \( \text{inside} \)
\( \Gamma \) \( f \) is analytic.
For each \( z \) fixed, \( f(z) \) is a rational function of \( z \) on \( K \),

Riesz's Theorem follows from approximating the integral in (6) by Riemann sums.

Note: If \( K \) has a connected complement then each rational function \( g(z) = \frac{1}{z - z_0} \) \( z \neq K \) can be approximated uniformly on \( K \) by polynomials in \( z \). This is seen by moving the pole \( z \) to \( \infty \).

As we noted before \( \lambda(K) = 0 \Leftrightarrow z \in \mathbb{R}(K) \) and hence, by the Stone-Weierstrass Theorem, \( \mathbb{R}(K) = C(K) \).

The "sandwich"

\[
\frac{\text{Area}(K)}{\text{Perimeter}(K)} \leq \lambda(K) \leq \sqrt{\frac{\text{Area}(K)}{4}},
\]

\[
\frac{2A}{\pi} \leq \lambda(K) \leq \sqrt{\frac{A}{\pi}}
\]

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Moreover, in the RHS equality occurs if \( K \) is a disk union with a set of area zero.
\textbf{Cor 1.} (F. Hartogs - A. Rosenthal)

If \( A(k) = 0 \Rightarrow x = 0 \Rightarrow R(k) = C(k) \)

\textbf{Cor 2.}

\[
\frac{2A}{\pi} \leq \sqrt{\frac{A}{\pi}} \iff 4\pi A \leq 1 \text{ for smooth domains and equality occurs iff } K \text{ is a disk.}
\]

Remark. The RHS \( \sqrt{\frac{A}{\pi}} \) is simply the radius of the disk with the same area as \( K \), so-called the volume radius.

\[2. \quad \text{Proof of Thm 2.1}
\]

\[\text{(a) Cauchy-Green-Koppelman formula.}
\]

Recall \( \frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \).

For any \( g \), say, in \( C^4(G) \), \( G \) is a smoothly bounded domain, we have

\[
g(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{g(z)}{z-w} \, dw - \frac{1}{\pi} \int_{\partial D} \frac{\partial g}{\partial z} \frac{1}{z-w} \, dw.
\]

If \( g \) is analytic in \( G, \frac{\partial g}{\partial z} = 0 \) and (7)
reduces to the Cauchy formula.

To prove (7) exercise from $G$ a disk $D$ and

centred at $z$, and apply Green's formula to

$$\frac{g(z)}{z - \xi} \text{ in } G_\varepsilon = G \setminus D_\varepsilon.$$ Then let $\varepsilon \to 0$.

If we replace $g$, by

$$\left( z - \xi \right) g(z),$$

and (7) becomes

$$0 = -\frac{1}{2\pi i} \int g(z) \, dz - \frac{1}{\pi} \iint \frac{\partial g}{\partial z} \, dx \, dy$$

$$G$$

$n, \quad z = x + iy,$

$$\int g(z) \, dz = 2i \int \frac{\partial g}{\partial z} \, dx \, dy.$$  

This is the complex form of Stokes' formula.

Note that the second statement in (7) follows from (8) by replacing $g$ with

$$\frac{g(z)}{z - \xi} \quad \text{since} \quad \frac{1}{z - \xi} \quad \text{is analytic in } G \setminus \overline{D_\varepsilon}$$

for $z \in G$. 


(ii) \[ P \] of lower bound \[ \text{wlog} \] \[ K = G U R \]

Fix \( h \in R(K) \), analytic in a neighborhood of \( K \). Let \( ds = |dz| \) be the arclength measure on \( \Gamma = \partial K \). \( P = \int ds \). Apply (8) to \( g(z) = \overline{z} - h(z) \), we obtain:

\[ \int \overline{z} - h(z) \, dz = 2i \int 1 \, dx \, dy = 2i A \]

\[ A = \text{Area}(K) \] \[ \text{Eq. (9)} \]

Yet \[ \int \overline{z} - h(z) \, dz \leq \int |\overline{z} - h(z)| \, ds \]

\[ \text{Eq. (10)} \]

Taking the minimum over such \( h \) and using (9) we obtain

\[ 2A \leq \chi(K) P \] \[ \text{Eq. (11)} \]

If \( K \) is not smoothly bounded, (11) still holds as soon as one make sense of the perimeter of \( K \), finite or infinite.

If \( P(K) = \infty \), (11) is trivial. The theory of sets with finite perimeter has been developed in the 50-70s in Geometric
(iii) Upper bound (Alexander's spectral area estimate)

Let $G$ be a smoothly bounded domain containing $K$.

Apply (7) to $g = \overline{z}$:

$$\overline{z} = \frac{1}{2\pi i} \int_{\Gamma \cap G} \frac{\overline{z}}{z-s} \, ds - \frac{1}{\pi} \iint_{G \setminus K} \frac{1}{z-s} \, dxdy$$

(11) \hspace{1cm} s \in K.

Now, let $\phi(s) = \frac{1}{\pi} \iint_{K} \frac{1}{z-s} \, dxdy$.

(11) $\Rightarrow$

$$\overline{z} + \phi(s) = \frac{1}{2\pi i} \int_{\Gamma \cap G} \frac{z}{z-s} \, ds - \frac{1}{\pi} \iint_{G \setminus K} \frac{1}{z-s} \, dxdy$$

(12) \hspace{1cm} G \setminus K

The integral around $\Gamma$ depends analytically on $s$ for $s \in G$, hence, by Runge's theorem, it belongs to $R(K)$.

Claim 4 (Mergelyan's estimate)

$$\frac{1}{\pi} \iint_{G \setminus K} \frac{dxdy}{z-s} \leq \frac{2}{\pi} \left[ \frac{\text{Area } (G \setminus K)}{\pi} \right]^{1/2}$$

when $G \setminus K$. 
Assuming Claim 1, we conclude that
\[ 3 + f(3) \in R(K). \]

Hence, \[ \lambda(K) \leq \| \frac{3 + f(3)}{\sqrt{2}} \|_K. \]

(13)
\[ = \| \frac{f}{\sqrt{K}} \|_K. \]

IV. Ahlfors - Beurling Estimate

Claim 2 (⇒ Claim 1) For any \( K \subseteq \mathbb{C} \)

(14) \[ \max_{\xi \in \mathbb{C}} \left| \frac{1}{\pi} \iint_{K} \frac{dx \, dy}{z - \xi} \right| = \| f^*(K) \|_K \leq \sqrt{\frac{\text{Area}(K)}{\pi}} \]

(claim 2 ⇒ claim 1 since we dropped the factor 2 in the RHS)

Moreover, the equality in (13) occurs iff

the union of

\( K \) is a disk of radius \( \sqrt{\frac{\text{Area}(K)}{\pi}} \) and a closed set of zero area

Note that (13), (14) ⇒ \[ \lambda(K) \leq \sqrt{\frac{\text{Area}(K)}{\pi}}, \]

and the equality occurs iff \( K \) is a disk modulo, perhaps, a set of area zero.

Proof of (14). First, note that if \( K = \phi \)
the closed disk centered at the origin of radius $p > r$, then

$$f_p(\zeta) = \frac{1}{2\pi i} \int_{0}^{2\pi} \frac{1}{re^{i\theta} - \zeta} \, d\theta \oint_{\gamma} \frac{1}{z - \zeta} \, \frac{dz}{z - r}$$

$$= \frac{1}{i5} \left[ \frac{1}{2 - \frac{1}{5}} - \frac{1}{2} \right] d\zeta = \begin{cases} 0, & 151 < r \\ \frac{2i}{5}, & 151 = p \\ 121 = p \\ -\frac{2i}{5}, & 151 > r \end{cases}$$

by the residue calculus.

(The residues at $0$ and $\zeta$ cancel if $151 < r$,
while only the residue at $\zeta = 0$ enters if $151 > r$.) Thus

$$f_p(\zeta) = -\frac{2}{5} \int_{0}^{\min(p,151)} r \, dr \oint_{\gamma} \frac{dz}{z - \zeta} = \begin{cases} -\frac{2i}{5}, & 151 < p \\ \frac{2i}{5}, & 151 = p \\ -\frac{2i}{5}, & 151 > p \end{cases}$$

So, $\left| f_p(\zeta) \right| \leq p = \left[ \frac{\text{Area}(A_p)}{12} \right]^{-1/2}, \zeta \in \mathbb{C}$.

Note that $f(\zeta)$ is analytic off $K$, hence attains its maximum on $K$.

Moreover, (exercise) $f(\zeta)$ is continuous in $\mathbb{C}$ for any $K$ as a convolution of a locally integrable function $\frac{1}{\zeta}$ and a bounded
function \( f_k(z) = \begin{cases} 1, & z \in K \\ 0, & z \notin K \end{cases} \)

Thus, \( f_k(z) \) attains its maximum somewhere on \( K \). Performing a translation we can assume that \( f_k(z) \) attains its maximum at the origin. Furthermore, performing a rotation we can assume that \( f_k(0) > 0 \). Thus

\[
\| f_k \|_{K} = f_k(0) = \frac{1}{\pi \alpha} \int_{K} \text{Re} \left( \frac{1}{z} \right) \, dx \, dy.
\]

Exercise: For any \( c > 0 \), the set \( \{ \text{Re} \left( \frac{1}{z} \right) \geq c \} \) is a disk \( D_c \) centered at \( (\frac{1}{2c}, 0) \) with radius \( \frac{1}{2c} \).

\[
\max_{K \cap D_c} \text{Re} \left( \frac{1}{z} \right) \leq c \leq \min_{K \cap D_c} \text{Re} \left( \frac{1}{z} \right).
\]

Also, choosing \( c \): \( \text{Area}(K) = \text{Area}(D_c) \), we have

\[
\text{Area}(K \cap D_c) = \text{Area}(D_c \cap K),
\]

and hence from (17), (18) it follows that (16) is maximized when \( K \), up to a set of area zero coincides with the disk.
\(-10-\)

\[ \Delta_c \text{ for an appropriate } C, \text{ so that } A_c(k) = \text{Area}(\Delta_c). \] But we have calculated that for the disk \( \Delta_c \)

\[ \frac{\sqrt{A(K)}}{\pi} \]

Since

\[ \frac{\sqrt{A(K)}}{\pi} \leq \frac{\sqrt{\Delta_c}}{\Delta_c} \]

Claim 2 (the Allfors–Bénilian estimate) follows

Remark: The Allfors–Bénilian estimate (14) can be rephrased that among all uniformly charged \( \mathbb{R}^n \) with uniform mass density, the disk produces maximal electrostatic force.

Indeed, the potential of a planar electric field is given by

\[ \Phi(z) = \frac{1}{2\pi} \iint \log \frac{1}{|z-x|} \, dx \, dy \]

and the force (electric field) is given by

\[ \nabla \Phi(z) = \frac{\partial \Phi}{\partial x} \] is 

\[ = -\frac{1}{2\pi} \iint \frac{x \times d\gamma}{|z-x|^3} = -\frac{1}{2\pi} f(s). \]

It turns out that this is no longer true in \( \mathbb{R}^n \), \( n \geq 3 \) where \( \log \frac{1}{|x-\gamma|} \) is replaced by \( |x-\gamma|^{n-2} \) (Newtonian potential), B. Gustafsson, 1991.