The "Isoperimetric Sandwiches," Free Boundary Problems and Approximation by Analytic and Harmonic Functions

A Mini Course.

Bergen, 2015, Oct-November

10/20/15

I. The isoperimetric Problem revisited

Among all (simple, closed) curves of a given length P, find the one that surrounds the largest area.

1. Dido's legend. (356 ~ 260 BC)

(A King of Tyre made Dido (his daughter and her brother Pygmalion his joint heirs, who had Dido's husband murdered (for money). Dido flee and arrived to North Africa, where she asked for a piece of land she can surround by an oxhide. By cutting the hide in thin strips she surrounded a hill thus founding Carthage). The Berber king wanted to marry her but she committed suicide instead.

Solution: A circle of the radius $P/2\pi$.

Area of this circle is $A \overset{*}{=} \pi \left(\frac{P}{2\pi}\right)^2$

$= \frac{P^2}{4\pi} \geq A$ – any other Area

So

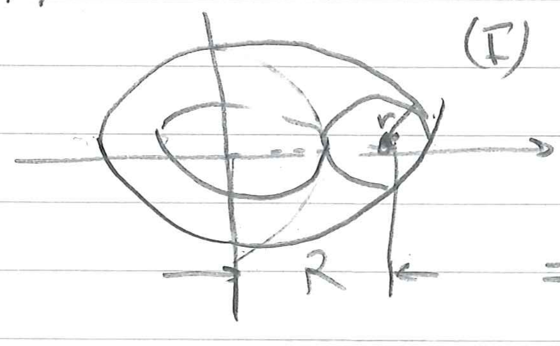
(1) $4\pi A \leq P^2$

the isoperimetric inequality

Greeks definitely knew it

According to Pappus (last of the great Greek geometers) (290 – 350 AD) and Theon of Alexandria Pappus wrote a mathematical collection – 8 books of Greek geometry

Note: Pappus 1st and IInd theorems.



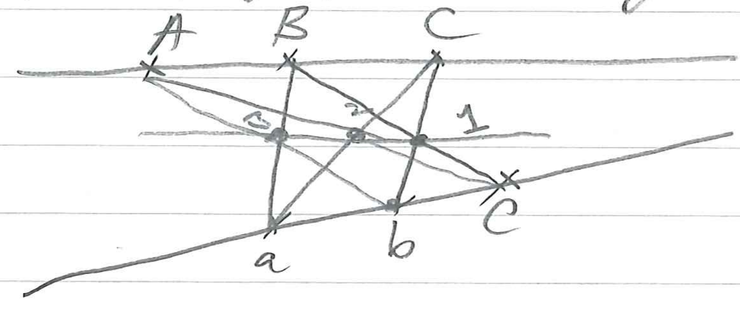
(I) $A = (2\pi r)(2\pi R = 4\pi^2 Rr$

$A = S \cdot d$, d = dist travelled by centroid

(II) $V = \pi r^2 \cdot 2\pi R = 2\pi^2 r^2 R$

$V = A \cdot d$

Pappus' Projection Geometry Theorem

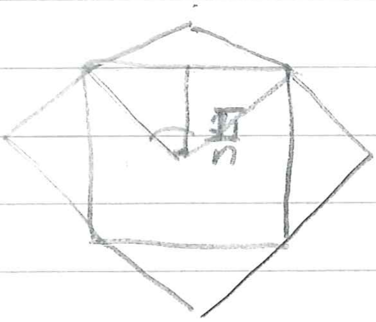


Theon (335 – 405 A.D.)

Father of Hypathia of Alexandria
(~4?? A.D. – 415 A.D. 3/8/315).

Zenderus' Proof (~200 – ~140 BC)
(On isometric figures, invented parabolic mirrors?)

Theorem 1 For regular polygons with same perimeter, more sides imply greater area



Theorem 2 A circle has greater area than any regular polygon with the same perimeter.

Theorem 3 A regular n-gon has greater area than all other n-gons with the same perimeter

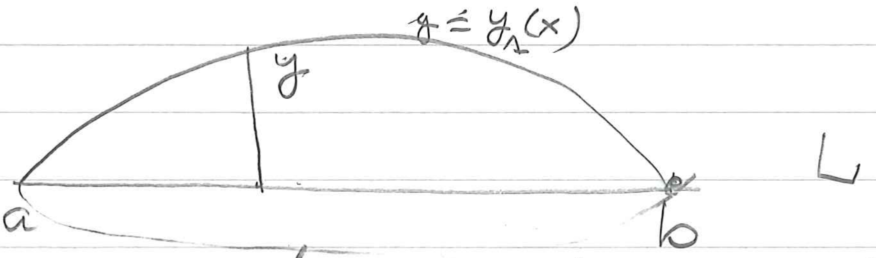
Exercise: Prove Thms. 1–3.

Fatal Flaw: Among all n-gons with given perimeter, there exists one with greatest area

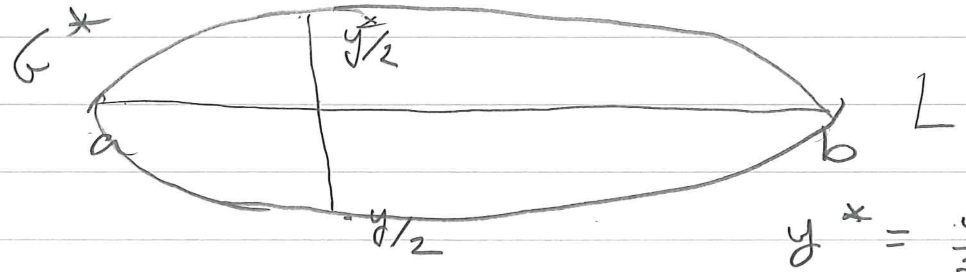
The problem essentially remained dormant until Jakob Steiner (1796 – 1863).

Steiner symmetrization.

Start out with domain G



and symmetrize it wrt horizontal axis L



$$y^* = \frac{y}{2}$$

$A^* = A$ since $A^* = 2\int_a^b \frac{y}{2}\,dx$

$$= \int_a^b y\,dx.$$

$$P^* = 2\int_a^b \left\{1+\left[(y^*)'\right]^2\right\}^{1/2} dx =$$

$$= 2\int_a^b \left[1+\frac{(y')^2}{4}\right]^{1/2} dx = \int_a^b \left[4+(y')^2\right]^{1/2} dx$$

$$P = \int_a^b \left\{\left[1+(y')^2\right]^{1/2}+1\right\} dx.$$

<u>Exercise</u> $P^* \le P$.

Thus, if we fix the area instead and minimize the perimeter, the extremal region must be symmetric wrt any direction, any line through its center of mass.

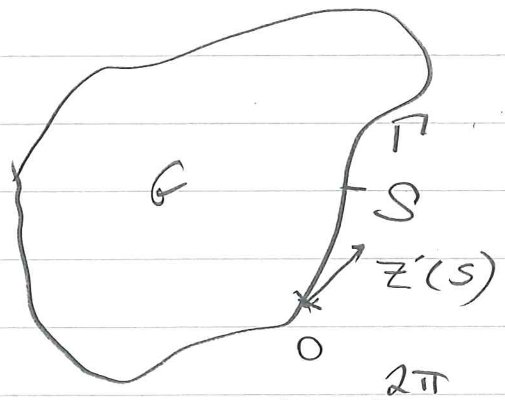
Thus, it is a circle.

Fatal Flaw The same as > 2000 years earlier, why is there an extremal domain

Steiner's proof was completed at the end of 19th century F. Edler (1882), H. A. Schwarz (1884), Carathéodory (1910), Schmidt (1938).

First analytic proof.

A. Hurwitz (1904).



$$z(s) = \sum_{-\infty}^{\infty} c_n e^{ins}$$

$0 \le s \le 2\pi$, $P(\Gamma) = 2\pi$.

"Time = Distance" $\Rightarrow$ velocity $|z'(s)| = 1$

$$1 = \frac{1}{2\pi} \int_0^{2\pi} |z'(s)|^2 \, ds \overset{\text{exercise to}}{=} \sum_{-\infty}^{\infty} n^2 |c_n|^2$$

$$A = \iint_G dx\,dy \underset{\text{Green's formula}}{=} \frac{1}{2} \int_\Gamma x\,dy - y\,dx =$$

$$= \frac{1}{2} \operatorname{Im} \int_\Gamma \overline{z(s)}\, z'(s)\, ds \overset{\text{exercise}}{=} \pi \sum_{n=-\infty}^{\infty} n |c_n|^2$$

$$\le \pi \sum n^2 |c_n|^2 = \pi$$

So, $A \le \pi$ unless

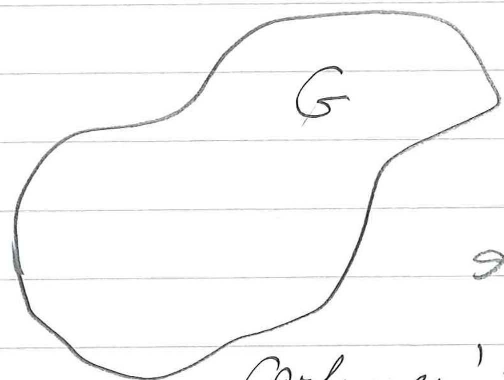
$n = n^2$, i.e. $n = 0$ for all $n \neq 0, 1$, or

$\Gamma = \{ z(s) = c_0 + c_1 e^{is} \}$, a circle.

This is precisely the isoperimitric inequality for the normalization we chose:

$$4\pi A \leq P^2 = (2\pi)^2 = 4\pi^2, \quad A \leq \pi.$$

T. Carleman's Proof (1921)



Let f be analytic in a domain G bounded by a smooth curve Γ.

Carleman's inequality

(2) $$\int_G |f|^2 \, dx\, dy \leq \frac{1}{4\pi} \left(\int |f| \, |dz| \right)^2$$

$f \equiv 1$ gives the isoperimetric inequality (1).

Remark (i) Suffices to have $\int |f|^2 |dz|$ in the RHS, which is trivial (exercise)

(ii) By applying Riemann mapping theorem it suffices to prove (2) for $\mathbb{D}$ (exercise).

(iii) But then (2) is nontrivial

(a) It suffices to prove (2) for $f \neq 0$ in $\mathbb{D}$ since every analytic in $\mathbb{D}$ function factors

$$(3) \quad f = BF = \prod_1^n \frac{z-\alpha_k}{1-\bar{\alpha}_k z} \cdot F(z),$$

where $F \neq 0$ in $\mathbb{D}$ and $|B| \leq 1$ and $|B| = 1$ on $\mathbb{T} := \partial\mathbb{D}$

(b) Now $f \neq 0$, can be written as g^2

Finally, (A^2 = Bergman space)

$$g = \sum a_n z^n$$

$$\|g\|^4_{A^4} = \|g^2\|^2_{A^2} = \Big\| \sum_{n=0}^{\infty} \Big(\sum_{k=0}^{n} a_k a_{n-k}\Big) z^n \Big\|^2$$

$$= \sum_{n=0}^{\infty} \frac{1}{n+1} \Big| \sum_{k=0}^{n} a_k a_{n-k} \Big|^2 \leq \sum_{n=0}^{\infty} \Big(\sum_{k=0}^{n} |a_k| |a_{n-k}|\Big)^2$$

Cauchy-Schwarz

$$= \Big(\sum_{m=0}^{\infty} |a_m|^2\Big)\Big(\sum_{n=0}^{\infty} |a_n|^2\Big) = \Big(\|g\|^2_{H^2}\Big)^2$$

So, $f = g^2$ normalizing area and arclength,

$$(4) \quad \int_{\mathbb{D}} |f|^2 \frac{dA}{\pi} \leq \Big(\frac{1}{2\pi} \int_{\mathbb{T}} |f| \, d\theta\Big)^2,$$

which is (2)

<u>Exercise</u>* (Vucotic, 2003) Find f analytic in $\mathbb{D}$ for which the equality in (4) holds.

In 1967, Carleson's student S. Jacobs tried to extend (2) to