A SURVEY OF CERTAIN EXTREMAL PROBLEMS FOR NON-VANISHING ANALYTIC FUNCTIONS

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Abstract. This paper surveys a large class of nonlinear extremal problems in Hardy and Bergman spaces. We discuss the general approach to such problems in Hardy spaces developed by S. Ya. Khavinson in the 1960s, but not well known in the West. We also discuss the major difficulties distinguishing the Bergman space setting and formulate some open problems.

1 INTRODUCTION

Solving extremal problems has been one of the major stimuli for progress in complex analysis, starting with the Schwarz lemma, on to the celebrated problems of Carathéodory-Fejér, Kakeya, Landau, etc.., (see the historical notes in [14], pp. 51-54 and pp. 110-112), and finally to general linear problems in Hardy spaces. Since the introduction of methods of functional analysis (the Hahn-Banach theorem) in the study of linear extremal problems in analytic function spaces by S. Ya. Khavinson in 1949 ([13]) and, independently, by Rogosinski and Shapiro in 1953 ([25]), the theory of extremal problems in Hardy spaces has achieved a significant level of elegance and clarity (cf. [7], Ch. 8).

Recently, substantial progress has occurred in the twin theory of linear extremal problems in Bergman spaces (see [12, 8, 9] and the references cited there). In this brief survey we are mostly concerned with the problems that are not covered by the elegant umbrella of clean and simple methods of functional analysis, namely, nonlinear extremal problems. More precisely, we consider here some well-known basic extremal problems such as finding the maximum value of a simple linear functional, but posed for *non-vanishing* functions in either Hardy or Bergman spaces.

The latter set of functions is obviously non-convex, and accordingly, new methods are required to solve problems in this new setting. A celebrated example of a problem that is still far from being solved is the Krzyż conjecture for bounded non-vanishing analytic functions. Namely, if we consider the family $\mathcal{F}$ consisting of all non-vanishing, bounded analytic functions $f(z) = \sum_{k=0}^{\infty} a_k z^k$ such that $|f(z)| \leq 1$ for $|z| < 1$, the Krzyż conjecture states that, for $m \geq 1$,

$$\max\{|a_m| : \ f \in \mathcal{F}\} = \frac{2}{e}.$$

This conjecture has been proven only for $1 \leq m \leq 5$ (see [10, 11, 16, 18, 17, 20, 21, 22, 23, 24, 26, 27, 31, 32, 30, 33, 35]). At the same time, if we considered the *linear* analogue of this question by removing the condition that $f(z) \neq 0$ in $\mathbb{D}$, then the problem is trivial and the extremal functions $f^*(z) = e^{i\alpha} z^m$ give the value 1 for

[1]The work of the first author was supported, in part, by the University of South Florida Internal Awards Program under Grant No. RO61636.

[2]The second author gratefully acknowledges partial support from the National Science Foundation under the grant DMS-0701873.

the maximum. S. Ya. Khavinson developed, in the early '60s, a general approach to problems for non-vanishing functions in Hardy spaces that allowed him, if not to solve the problem explicitly, to at least obtain the particular form of extremal functions. Yet, he did not publish it until the 1970s. Moreover, the latter work was not translated into English until 1986 (see [14]). Under his guidance, his former student, V. Terpigoreva, quickly extended his results to more general Orlicz-Hardy spaces in the paper [37] following her thesis [36]. She published a complete version with proofs in 1970 (see [38]). This perhaps partly explains why Khavinson postponed publication of his less general results until their inclusion in his monograph ([14]) that unfortunately was never published in Russian in book form.

Some of S. Ya. Khavinson's results (but not the general method) were rediscovered in the 70s and 80s by western authors (see [11, 32]). Yet, the attack on extremal problems for non-vanishing functions in Bergman spaces has only just begun (see [1, 2, 3, 4, 5]), and still, the simplest problems remain unsolved.

The layout of this survey is as follows. In Section 2, we outline S. Ya. Khavinson's theory for Hardy spaces. In Section 3, we illustrate the general theory by discussing some particular examples in Hardy spaces. Section 4 contains the discussion of the Bergman space case. There we focus on the simplest problems that are still unresolved; in particular, we explain in detail where S. Ya. Khavinson's arguments that work so smoothly for Hardy spaces run into a wall in the Bergman space context. We finish with several observations and conjectures for the Bergman spaces problem that we hope will attract more researchers to this field.

Acknowledgement. The authors are grateful to Peter Duren for many helpful suggestions improving the exposition.

2 GENERAL THEORY FOR HARDY SPACES

Let us begin by discussing the general theory of coefficient type extremal problems for non-vanishing functions in Hardy spaces. This discussion is based on the work of S. Ya. Khavinson in [14]. (The results there were originally obtained in the mid 60s, yet the original version of [14] was only published in Russian in 1981.)

We shall be looking at a general extremal problem of the following type: given $\tau_0, \tau_1, \ldots, \tau_m \in \mathbb{C}$, find

$$\sup_{f \in H_0^p} \operatorname{Re} \left\{ \sum_{k=0}^{m} \tau_k \frac{f^{(k)}(0)}{k!} \right\} , \tag{2.1}$$

where H_0^p is the set of non-vanishing functions in the unit ball of H^p. That is,

$$\begin{aligned} H_0^p \quad &:= \quad \{f : f \text{ is analytic and non-vanishing in } \mathbb{D}, \\ &\qquad \|f\|_p^p := \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p \, dt \le 1\}. \end{aligned}$$

Let's define the coefficient region $A_m(H_0^p) \subset \mathbb{C}^{m+1}$ to be the set of points $\vec{c} =< c_0, c_1, \ldots, c_m >$ such that $f(z) = \sum_{k=0}^{m} c_k z^k + \ldots$ for some $f \in H_0^p$. Notice that if

$$f^*(z) = \sum_{k=0}^{m} c_k^* z^k + \ldots$$

is a solution to a problem of type (2.1) then

$$\vec{c}^{\,*} =< c_0^*, c_1^*, \ldots, c_m^* >

is a boundary point of $A_m(H_0^p)$. Thus, studying the boundary of $A_m(H_0^p)$ in $\mathbb{C}^{m+1}$ gives information about f^*. Unfortunately, the closure of the coefficient space $\overline{A_m}(H_0^p) = A_m(H_0^p) \cup \{0\}$ is *not* a convex set because H_0^p is not convex, and therefore describing its boundary points is not a straightforward task. However, every $f \in H_0^p$ can be written as $f(z) = \exp(q(z))$; if we call Q_p^* the class of logarithms $q(z)$ of functions in H_0^p, then Q_p^* is now a convex class. The coefficient set $A_m(Q_p^*)$ corresponding to the set of first $m+1$ coefficients of all elements of Q_p^* is therefore also convex. Moreover, it is not difficult to show that $A_m(Q_p^*)$ is a closed, proper subset of $\mathbb{C}^{m+1}$ with non-empty interior, and that there is a homeomorphism between $\overline{A_m}(H_0^p)$ and $A_m(Q_p^*)$; therefore, finite boundary points of $A_m(Q_p^*)$ correspond to non-zero boundary points of $A_m(H_0^p)$. With this relationship in mind, let us study the boundary points of $A_m(Q_p^*)$.

Let

$$\vec{a}^* =< a_0^*, a_1^*, \ldots, a_m^* >$$

be a boundary point of $A_m(Q_p^*)$. This means that there exists a supporting hyperplane passing through that point: that is, there exist constants $d \in \mathbb{R}$ and $\gamma_0, \gamma_1, \ldots, \gamma_m \in \mathbb{C}$ such that

$$\mathrm{Re}\left(\sum_{k=0}^{m} \gamma_k a_k\right) \leq d$$

for every $\vec{a} \in A_m(Q_p^*)$ and

$$\mathrm{Re}\left(\sum_{k=0}^{m} \gamma_k a_k^*\right) = d.$$

In other words, we are interested in finding, given $\gamma_0, \gamma_1, \ldots, \gamma_m \in \mathbb{C}$ fixed,

$$\lambda_p^* = \sup\left\{\mathrm{Re}\sum_{k=0}^{m} \gamma_k a_k : q(z) = \sum_{k=0}^{\infty} a_k z^k \in Q_p^*\right\}. \tag{2.2}$$

Now examine the structure of such functions q a little more closely.

It is well-known (see [7]) that every function $f \in H_0^p$ has non-tangential limits (almost everywhere on the unit circle $\mathbb{T}$) $f(e^{it}) \in L^p([0, 2\pi])$, and f can be written as

$$f(z) = \exp\left(q(z)\right), \tag{2.3}$$

where

$$q(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\left(\log|f(e^{it})|\, dt + d\mu(t)\right), \tag{2.4}$$

with μ a negative singular measure on $\mathbb{T}$. If we write

$$S(t) := \log|f(e^{it})|^p,$$

then $S \in L^1[0, 2\pi]$ and

$$\frac{1}{2\pi}\int_0^{2\pi} e^{S(t)}\, dt \leq 1. \tag{2.5}$$

Then q can be written

$$q(z) = \frac{1}{2\pi p}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\left(S(t)\, dt + d\mu(t)\right), \tag{2.6}$$

where μ (relabeled) is a negative singular measure on $\mathbb{T}$. Conversely, any function $S \in L^1[0, 2\pi]$ satisfying (2.5) and any negative singular measure μ correspond to a function $f \in H_0^p$ via (2.3) and (2.6).

Let us define the class σ to be the set of absolutely continuous measures of the form $S(t)\,dt$, where $S \in L^1[0, 2\pi]$ satisfies the normalization (2.5), and the class Σ to be the set of measures $S(t)\,dt + d\mu(t)$, where $S \in \sigma$ and μ is a negative singular measure on $\mathbb{T}$. Of course $\sigma \subset \Sigma$. We have already defined the class Q_p^* as the set of functions q that have the representation (2.6), where $S\,dt + d\mu \in \Sigma$; the class Q_p will be defined to be the class of functions q that have the representation

$$q(z) = \frac{1}{2\pi p} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} S(t)\,dt, \tag{2.7}$$

where $S(t)\,dt \in \sigma$. In other words, the class Q_p^* is the set of logarithms of functions in H_0^p, and Q_p is the set of logarithms of the outer parts of functions in H_0^p.

Now, let us get back to Problem (2.2): consider $q(z) = \sum_{k=0}^{\infty} a_k z^k$ in Q_p or Q_p^*. Fix $\gamma_0, \gamma_1, \ldots, \gamma_m \in \mathbb{C}$, and consider the extremal problem of finding

$$\lambda_p = \sup_{q \in Q_p} \operatorname{Re}\left(\sum_{k=0}^{m} \gamma_k a_k\right) \tag{2.8}$$

(or the corresponding problem of finding λ_p^* for Q_p^*).

A simple calculation shows that

$$\operatorname{Re}\left(\sum_{k=0}^{m} \gamma_k a_k\right) = \frac{1}{2\pi p} \int_0^{2\pi} \alpha(t) S(t)\,dt, \tag{2.9}$$

where

$$\alpha(t) := \operatorname{Re}\left(\gamma_0 + 2\sum_{k=1}^{m} \gamma_k e^{-ikt}\right)$$

and S "represents" q via (2.7), that is, $S(t) = p \operatorname{Re} q(e^{it})$.

Now if $\alpha(t)$ is continuous on the interval $[0, 2\pi]$, it is not hard to see that the supremum

$$\sup_{S \in \sigma} \int_0^{2\pi} S(t)\alpha(t)\,dt \tag{2.10}$$

is finite if and only if $\alpha(t) \geq 0$ on $[0, 2\pi]$. Indeed, if $\alpha(t) \geq 0$, then

$$\int_0^{2\pi} S(t)\alpha(t)\,dt \leq \int_0^{2\pi} e^{S(t)}\alpha(t)\,dt < \infty,$$

since α is continuous on $[0, 2\pi]$ and S satisfies the normalization (2.5). On the other hand, suppose there were an interval I and $\epsilon > 0$ such that $\alpha(t) < -\epsilon$ for $t \in I$. Then we could construct a sequence of functions S_N equal to $-N$ on that interval I and 0 elsewhere. The measures $S_N(t)\,dt$ certainly lie in the class σ, and the integrals

$$\int_0^{2\pi} S_N(t)\alpha(t)\,dt \to \infty.$$

Similarly,

$$\sup_{\nu \in \Sigma} \int_0^{2\pi} \alpha(t) d\nu(t) < \infty$$

if and only if $\alpha(t) \geq 0$ on $[0, 2\pi]$. The following lemma, based on a surprising but simple inequality, is the key in the solution of this problem.

Lemma 2.1. *Let $\alpha(t) \geq 0$ on $[0, 2\pi]$. Then*

$$\begin{aligned} \sup_{S\in\sigma} \int_0^{2\pi} \alpha(t)S(t)\,dt &= \sup_{\nu\in\Sigma} \int_0^{2\pi} \alpha(t)d\nu(t) && (2.11) \\ &= \int_0^{2\pi} \alpha(t)\ln\frac{\alpha(t)}{A}\,dt, && (2.12) \end{aligned}$$

where $A = \frac{1}{2\pi}\int_0^{2\pi} \alpha(t)\,dt$.

Proof. Since μ is a non-positive measure and $\alpha(t) \geq 0$, for any $S \in \sigma$,

$$\int_0^{2\pi} \alpha(t)(S(t)\,dt + d\mu(t)) \leq \int_0^{2\pi} \alpha(t)S(t)\,dt,$$

and equality will only hold when μ is a measure that has support where $\alpha(t) = 0$. Therefore, (2.11) is clear.

Notice that it is enough to consider $S \in \sigma$ such that

$$\frac{1}{2\pi}\int_0^{2\pi} e^{S(t)}\,dt = 1.$$

Now, the following simple inequality can be checked directly, for any $u, v > 0$:

$$u\ln u - u \geq u\ln v - v. \qquad (2.13)$$

Moreover, if $u \neq v$, then the inequality is strict. Applying (2.13) to $u = \alpha(t)/A$ and $v = e^{S(t)}$ gives

$$\frac{\alpha(t)}{A}\ln\left(\frac{\alpha(t)}{A}\right) - \frac{\alpha(t)}{A} \geq \frac{\alpha(t)}{A}\ln\left(e^{S(t)}\right) - e^{S(t)}.$$

Integrating both sides and simplifying then gives

$$\frac{1}{2\pi}\int_0^{2\pi} \alpha(t)\ln\left(\frac{\alpha(t)}{A}\right)\,dt \geq \frac{1}{2\pi}\int_0^{2\pi} \alpha(t)S(t)\,dt,$$

since

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{\alpha(t)}{A}\,dt = \frac{1}{2\pi}\int_0^{2\pi} e^{S(t)}\,dt = 1.$$

If $\ln(\frac{\alpha(t)}{A}) \in L^1[0, 2\pi]$, then the extremal problem is solved uniquely for $S(t) = \ln(\frac{\alpha(t)}{A})$. Otherwise, a standard approximation of $\ln(\frac{\alpha(t)}{A})$ by, say, step functions, shows that

$$\sup_{S\in\sigma}\int_0^{2\pi} S(t)\alpha(t)\,dt = \int_0^{2\pi} \alpha(t)\ln\frac{\alpha(t)}{A}\,dt,$$

although the extremal problem has no solution in σ or Σ. ◻

The supremum in the problem (2.8)

$$\lambda_p = \sup_{q\in Q_p} \mathrm{Re}\left(\sum_{k=0}^{m} \gamma_k a_k\right)$$

will be finite, therefore, if and only if $\alpha(t) \geq 0$. Since $\alpha(t)$ is a trigonometric polynomial of order m, by the Fejér-Riesz theorem ([15, 34]), this implies that $\alpha(t) = |P(e^{it})|^2$, where $P(z)$ is an analytic polynomial of degree m. Writing

$$P(z) = C \prod_{k=1}^{m} (1 - \bar{\alpha}_k z),$$

where $\alpha_k \in \overline{\mathbb{D}}$ and C is a positive constant, we see that

$$\ln(\alpha(t)) = 2 \ln C + 2 \sum_{k=1}^{m} \ln |1 - \bar{\alpha}_k e^{it}|.$$

Therefore $S(t) = \ln(\frac{\alpha(t)}{A})$, where $A = \mathrm{Re}(\gamma_0)$, and so

$$q(z) = \frac{2}{p} \ln(\frac{C}{A}) + \frac{2}{p} \sum_{k=1}^{m} \ln(1 - \bar{\alpha}_k z) \tag{2.14}$$

is the unique extremal function that solves (2.8).

In addition, if we consider the problem for λ_p^* of finding

$$\lambda_p^* = \sup \left\{ \mathrm{Re} \sum_{k=0}^{m} \gamma_k a_k : q(z) = \sum_{k=0}^{\infty} a_k z^k \in Q_p^* \right\},$$

the corresponding singular measure may only have mass at the zeros of $\alpha(t)$, and so extremal solutions to the problem for λ_p^* are given by functions

$$q^*(z) = q(z) - \sum_{|\alpha_k|=1} \lambda_k \frac{\alpha_k + z}{\alpha_k - z}, \tag{2.15}$$

where q is as in (2.14), and $\lambda_k \geq 0$ are arbitrary.

In other words, what we have shown so far is that if

$$\vec{a}^* =< a_0^*, a_1^*, \ldots, a_m^* >$$

is a boundary point of $A_m(Q_p^*)$, then any function $q^*(z) \in Q_p^*$ that corresponds to that boundary point (i.e., $q^*(z) = \sum_{k=0}^m a_k^* z^k + \ldots$), must have the general form (2.15). In fact, it turns out that there is a *unique* function $q^* \in Q_p^*$ that corresponds to a given boundary point $\vec{a}^*$. This follows from the fact that q^* is determined by its representing measure $\ln(\frac{\alpha(t)}{A})\, dt + d\mu(t)$, where $\alpha(t)$ and A are uniquely determined, and μ is a singular measure with at most m atoms on the circle. But the first $m+1$ coefficients of q^* are determined, which forces the measure μ to be determined as well. (For full details, see [14, p. 91].) Finally, by appealing to the homeomorphism between $A_m(Q_p^*)$ and $\overline{A_m}(H_0^p)$ via $f(z) = \exp(q(z))$, we arrive at the following theorem.

Theorem 2.2. *([14]) To each non-zero boundary point*

$$\vec{c}^* =< c_0^*, c_1^* \ldots, c_m^* >\in \overline{A_m}(H_0^p),$$

there corresponds a unique function $f \in H_0^p$ such that

$$f(z) = \sum_{k=0}^{m} c_k^* z^k + \ldots.$$

This function has the form

$$f(z) = C \prod_{k=1}^{m} (1 - \bar{\alpha}_k z)^{\frac{2}{p}} \exp\left(- \sum_{|\alpha_k|=1} \lambda_k \frac{\alpha_k + z}{\alpha_k - z} \right), \tag{2.16}$$

where $C > 0$ is a constant such that

$$\frac{1}{2\pi} \int_0^{2\pi} |f(e^{it})|^p \, dt = 1,$$

$|\alpha_k| \leq 1,$ *and* $\lambda_k \geq 0.$

Remark. The above reasoning applies to any extremal problem involving Taylor coefficients of non-vanishing functions, and its solution will be unique whenever the extremal function in the problem determines a boundary point in $A_m(H_0^p)$; *e.g.*, as in minimal interpolation problems of finding $\inf \|f\|_p$ for nonvanishing functions $f \in H^p$ with prescribed initial coefficients $c_0, c_1, \ldots, c_m$. The same type of argument also applies to interpolation problems where the origin is replaced by arbitrary points in the unit disk.

3 SOME SPECIFIC PROBLEMS IN HARDY SPACES

Let us now consider some specific examples of problems of type (2.1)

$$\sup_{f \in H_0^p} \operatorname{Re} \left\{ \sum_{k=0}^{m} \tau_k \frac{f^{(k)}(0)}{k!} \right\}.$$

One of the simplest problems of this type is that of finding, for a positive integer m,

$$\sup_{f \in H_0^p} \operatorname{Re} \left\{ \frac{f^{(m)}(0)}{m!} \right\}. \tag{3.1}$$

Surprisingly, for $p > 1$, the exact form of the extremal function is unknown for $m \geq 3$. By Theorem 2.2, we know that the extremal solution is the $2/p$-th power of a polynomial of degree m times a singular function with atomic masses at at most m points on the circle. Those points on the circle can only occur where the polynomial has roots on the circle.

In particular, if we look at the "limiting case", that is, when $p = \infty$, we see that the solution to

$$\sup_{f \in H^\infty} \left\{ \operatorname{Re} \frac{f^{(m)}(0)}{m!} : \|f\|_\infty \leq 1 \, , \, f \text{ non-vanishing in } \mathbb{D} \right\} \tag{3.2}$$

must be a singular function with at most m atomic masses on the circle. That the value of that supremum is $2/e$ and occurs when the singular function has masses at the roots of unity is the famous Krzyż conjecture, which remains an open problem for $m > 5$. It is known to be true for $m = 1, 2$, and 3 (see [11] and the discussion there), $m = 4$ (see [35], and also [33]), and $m = 5$ ([30]). We also refer the reader to [18, 17, 20, 21, 22, 23, 24, 26, 27] for further developments. Significantly, Horowitz ([10]) showed that the supremum in (3.2) is strictly less than 1.

While studying the Krzyż conjecture and proving the $m=3$ case, the authors in [11] conjectured that the solution to Problem (3.1) for non-vanishing functions in H^p, $p>1$, is

$$\sup_{f\in H^p_0} \operatorname{Re}\left\{\frac{f^{(m)}(0)}{m!}\right\} = \left(\frac{2}{e}\right)^{\frac{1}{q}}, \tag{3.3}$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $m\geq 1$. The conjectured extremals are

$$f^*(z) = \left(\frac{(1+z^m)^2}{2}\right)^{\frac{1}{p}} \left(\exp\frac{z^m-1}{z^m+1}\right)^{\frac{1}{q}}.$$

The authors note that for $p=1$, the functions

$$f_m(z) = \frac{(1+z^m)^2}{2}$$

are extremal for Problem (3.1) and, for all $m\geq 1$,

$$\sup_{f\in H^1_0} \operatorname{Re}\left\{\frac{f^{(m)}(0)}{m!}\right\} = 1. \tag{3.4}$$

They also remark that uniqueness of extremals fails badly here: any function

$$f(z) = C\prod_{j=1}^{m}(z-\alpha_j)(1-\bar{\alpha}_j z),$$

where $|\alpha_j|=1$ and C is chosen so that $\|f\|_{H^1}=1$ is also extremal. Note that all of these extremals are of the general form (2.16) from Section 2. Conjecture 3.3 for non-vanishing Hardy space functions was shown to be true for $m=1$ in [6] and for $m=2$ in [32].

While studying explicit solutions to *linear* extremal problems in H^p, the authors in [5] considered the related problem of finding, for $p\geq 1$, $m\geq 1$, and $0<c<1$ fixed,

$$\max_{f\in H^p_0}\left\{\operatorname{Re} f^{(m)}(0)/m!:\ f(0)=c\right\}. \tag{3.5}$$

They were only able to solve this problem explicitly for $m=1$. The extremals depend on the value of c : if $0<c<2^{-1/p}$, then the extremal function has a singular part and is equal to

$$f^*(z) = 2^{-\frac{1}{p}}(1+z)^{\frac{2}{p}}\exp\left(-\mu_0\frac{-1+z}{-1-z}\right),$$

where $\mu_0 = -\log(2^{\frac{1}{p}}c)$, while if $2^{-1/p}\leq c<1$, then the extremal has no singular part and is equal to

$$f^*(z) = \left(c^{\frac{p}{2}} + z\sqrt{1-c^p}\right)^{\frac{2}{p}}.$$

By varying c and finding the corresponding maximum, the authors gave another proof of Conjecture (3.3) for $m=1$. Problem (3.5) is equivalent to an interpolation problem for non-vanishing functions, of finding, for $c_0,\dots,c_m$ fixed,

$$\inf\{\|f\|_p:\ f(0)=c_0,\dots,\frac{f^{(m)}(0)}{m!}=c_m,\ f\in H^p,\ f \text{ non-vanishing }\}.$$

If we fix for instance, $m = 1$, $p = 2$, and $c < 0$, then the following infimum

$$\inf\left\{\|f\|_p : \ f(0) = 1, f'(0) = c, \ f \in H^p, \ f \text{ non-vanishing } \right\} \tag{3.6}$$

is attained by the trivial solution $f^*(z) = 1 + cz$ if $|c| \leq 1$, otherwise by

$$f^*(z) = C(1+z)\exp\left(-\mu_0 \frac{1+z}{1-z}\right),$$

where C and μ_0 are determined by the interpolating conditions. Notice that, as in the general form of the extremal in (2.16), the point mass of the singular part occurs exactly at the zero of the outer part of f^*, making the extremal function continuous in the closed unit disk. This example will be discussed in more detail in the Bergman space setting in the following section.

Finally, we note that in [31], the author proved a "Horowitz type" result for the H^p space problem (3.1), namely that for each $1 < p < \infty$, there exists a bound $C_p < 1$ such that

$$\sup_{f \in H_0^p} \operatorname{Re}\left\{\frac{f^{(m)}(0)}{m!}\right\} \leq C_p.$$

4 DISCUSSION OF THE BERGMAN SPACE SETTING

Now let's discuss where and why the approach discussed in Section 2 fails in the context of Bergman spaces. So, for $0 < p < \infty$, let

$$A^p = \left\{ f \text{ analytic in } \mathbb{D} : \left(\int_{\mathbb{D}} |f(z)|^p \, dA(z)\right)^{\frac{1}{p}} =: \|f\|_{A^p} < \infty \right\},$$

where dA denotes normalized area measure in the unit disk $\mathbb{D}$. We will write A_0^p for the set of A^p functions that are non-vanishing. (We do not restrict A_0^p to functions of norm less than or equal to 1, here, as we did in the Hardy space case.)

For an account of the modern state of Bergman space theory, we refer the reader to the recent monographs [8, 9] on the subject.

We consider the following "model" extremal problem for the Taylor coefficients of non-vanishing functions: find

$$\inf\left\{\int_{\mathbb{D}} |f|^p \, dA : \quad f \in A_0^p, \ f^{(j)}(0) = c_j, \, 0 \leq j \leq m\right\}, \tag{4.1}$$

where the c_j are given non-zero complex numbers. (See [1] for a detailed discussion.)

First, note that since for any $g \in A_0^p$, the function $f := g^{\frac{p}{2}} \in A_0^2$, it suffices to carry out the analysis of (4.1) for $p = 2$ alone. Thus, from now on, $p = 2$. Also, without loss of generality, we can assume that $c_0 = 1$.

As was noted in [1], the solution to (4.1) always exists and is unique. Moreover, by writing every $f \in A_0^p$ as $f = e^q$, q analytic in $\mathbb{D}$, we note that we can rewrite (4.1) as

$$\inf\left\{\|e^q\|_{A^2} : \quad q \text{ is analytic in } \mathbb{D}, \ q^{(j)}(0) = a_j \, , \, 0 \leq j \leq m\right\}. \tag{4.2}$$

By considering the "cut-off" problem (4.2) with q running over the class P_n of polynomials of degree $n \geq m$ and showing that the exponentials $e^{p_n^*}$ of extremal polynomials p_n^* in (4.2) have uniformly bounded H^2 norms, then taking a limit as $n \to \infty$, it was shown in [1], following the ideas in [29, 12], that the extremal function $f^* = e^{q^*}$ for (4.1) must actually be in the Hardy class H^2. In particular, the extremal function f^* has well-defined boundary values. Furthermore, by using a more delicate variation stemming from the seminal work in [2, 3, 4] on the so-called minimal area problem, i.e., the problem of finding, for b fixed,

$$\inf\left\{\int_{\mathbb{D}} |F'|^2\, dA : F(0)=0, F'(0)=1, F''(0)=b,\ F \text{ univalent in } \mathbb{D}\right\}, \tag{4.3}$$

it was shown in [1] that the extremal f^* is in fact bounded in $\mathbb{D}$. It was conjectured in [1] that the extremal function f^* has the form (for $p=2$):

$$f^*(z) = C\prod_{j=1}^{m}(1-\bar{\alpha}_j z)\,\exp\left(\sum_{j=1}^{k}\lambda_j \frac{e^{i\theta_j}+z}{e^{i\theta_j}-z}\right), \tag{4.4}$$

where $|\alpha_j| \le 1$, $j=1,\dots,m$, $k \le m$, and $\lambda_j \le 0$ and $C>0$ are constants. Thus, the conjectured extremal function f^* in the Bergman space context has a form similar to that of the extremal in non-vanishing Hardy spaces. Yet, even for $m=1$, that is, for a problem of type (4.3) for *locally univalent* functions, the above form (4.4) has not yet been proved. Moreover, if the conjectured form (4.4) is correct, it was shown in [1] that the extremal f^* for (4.1), $m=1$, is

$$f^*(z) = C(z-1-\mu_0)e^{-\mu_0\frac{1+z}{1-z}}, \tag{4.5}$$

where the constants C and μ_0 depend on the data c_0 and c_1. (Note that here we are considering the non-trivial case $|c_0| < |c_1|$. If $|c_0| \ge |c_1|$, then the above problem is trivially solved by $f^*(z) = c_0 + c_1 z$.) If so, this will provide the first known example of a *discontinuous* solution to these very simple extremal problems, something that never happens for similar interpolation problems with finitely many contraints in Hardy spaces.

Let us sketch here the main differences between the Bergman and Hardy spaces problem to illustrate why the approach that works so well for Hardy spaces runs into a snag in the Bergman space context. Since the extremal functions for (4.2) belong to H^2 (and even H^∞), $q^*(z) = \log f^*$, $f^* \in A_0^2$ is representable by the Schwarz integral

$$q^*(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\, d\nu^*(t), \tag{4.6}$$

where

$$d\nu^*(t) = \log\rho^*(t)\,dt + d\mu^*(t) \tag{4.7}$$

$$\rho^* \ge 0, \rho^*, \log\rho^* \in L^1(\mathbb{T}), d\mu^* \le 0 \text{ and singular.} \tag{4.8}$$

Consider subsets B_r of the balls of radius $r>0$ in A_0^2 :

$$B_r := \{f = e^q, \|f\|_{A^2} \le r\},$$

where q is of the form (4.6). Consider the map $\Lambda : B_r \to \mathbb{C}^{m+1}$ defined

$$\Lambda(f) = < q^{(j)}(0) >_{j=0}^{m}\ .$$

Clearly, Λ maps the set of functions representable by the measures

$$\Sigma_r := \{\nu : d\nu(t) = \log \rho(t)\, dt + d\mu(t), \rho, \mu \text{ satisfying (4.8)}, \|e^q\|_{A^2} \le r\}$$

into $\mathbb{C}^{m+1}$. Following S. Ya. Khavinson's scheme from Section 2, it is straightforward to show that the image $A_r := \Lambda(\Sigma_r)$ is a closed, convex, proper subset of $\mathbb{C}^{m+1}$ with non-empty interior. If we denote by $\vec{a} =< a_0, \ldots, a_m >$ the vector of coordinate data in (4.2), then the value of the infimum in (4.2) equals

$$r_0 = \inf\{r > 0 : \vec{a} \in A_r\}.$$

Hence, the extremal function f^* corresponds to the extremal measure $\nu^* \in \Sigma_{r_0}$ for which $\Lambda(\nu^*)$ is in the boundary of A_{r_0}. Thus, as in the previous situation, we are interested in describing the boundary of A_{r_0}.

To simplify notation, let us suppose $r_0 = 1$ and omit the index r altogether. If $\vec{w} =< w_0, w_1, \ldots, w_m >$ is a finite boundary point of A, there exists a hyperplane

$$H: \ \mathrm{Re}\left(\sum_{j=0}^{m} a_j z_j\right) = d$$

in $\mathbb{C}^{m+1}$ passing through $\vec{w}$ such that for all $z \in A$,

$$\mathrm{Re}\left(\sum_{j=0}^{m} a_j z_j\right) \le d, \tag{4.9}$$

while

$$\mathrm{Re}\left(\sum_{j=0}^{m} a_j w_j\right) = d.$$

Let $\nu^* = \Lambda^{-1}(\vec{w}) \in \Sigma$. From (4.6), (4.7), and (4.8), we infer that the above inequality (4.9) becomes

$$\frac{1}{2\pi}\int_{\mathbb{T}} P(e^{i\theta}) d\nu(\theta) \le d \tag{4.10}$$

for all $\nu \in \Sigma$, where

$$P(e^{i\theta}) = \mathrm{Re}\left(\sum_{j=0}^{m} a_j e^{ij\theta}\right)$$

is a trigonometric polynomial of degree m. Again, as before, we note that for d in (4.10) to be finite for all measures $\nu \in \Sigma$, it is imperative that $P \ge 0$ on $\mathbb{T}$. In view of the Fejér-Riesz Theorem (see [15], or [34], Chapter I, Sect. 1.2),

$$P(e^{i\theta}) = a \prod_{j=0}^{n} |1 - \alpha_j e^{i\theta}|^2,$$

where $|\alpha_j| \le 1$ and $a > 0$ are constants. Indeed, if $P < 0$ on an arc E of the circle, we could choose $d\nu = s d\theta$ with s negative and large in absolute value on E and fixed on the rest of the circle, thus making the left hand side of (4.10) arbitrarily large and positive while keeping the constraints on the set Σ intact. Now we run into a wall. In order to establish the atomicity of the singular part μ^* of the extremal measure

ν^* in (4.10), we must come up with a variation of ν^* that increases (4.10) without taking us out of the set Σ, i.e., without violating the norm restriction on $\|e^q\|$. For the Hardy space norm, dropping the singular inner factor altogether does not change the norm at all, which is certainly not the case in the Bergman space. If P has at least one zero $e^{it_0} = \alpha_0$ on the unit circle, we would immediately be able to verify that the singular part μ^* in the extremal measure ν^* in (4.10) is atomic. Simply note that if μ^* put mass on a set $E \subset \mathbb{T}$ where $P > 0$, we could replace μ^* by $\mu_1 = \mu^* - \mu^*|_E$ while compensating with a large negative point mass at e^{it_0} not to increase the norm $\|e^q\|$. This would make the integral in (4.10) larger (the mass at e^{it_0} does not change its value), contradicting the extremality of ν^*.

It was shown in [1] (even for a slightly more general problem than in (4.1)) that if the singular part μ^* in the representation (4.6) of $q^* = \log f^*$ is supported on a Carleson set, then the outer factor of f^* is indeed a polynomial as in the conjectured form (4.4). The proof is rather intricate and relies on the Korenblum-Roberts description of H^2 functions that are cyclic for the Bergman shift.

Here is the gist. A standard variational argument (see [1, 12]) shows that if f^* is an extremal function in (4.1), then f^* satisfies the following "orthogonality" conditions

$$\int_{\mathbb{D}} |f^*(z)|^2 z^{m+k+1}\, dA(z) = 0, \quad k = 0, 1, 2, \dots \tag{4.11}$$

Now we are trying to show that the outer part of f^* is a polynomial of degree at most m. In view of the Fejér-Riesz theorem cited above, it certainly suffices to show that

$$\int_{\mathbb{T}} |f^*(z)|^2 e^{i(m+k+1)t}\, dt = 0, \quad k = 0, 1, 2, \dots \tag{4.12}$$

Replacing dt by $iz d\bar{z}$, applying the complex form of Green's theorem to (4.12), and using (4.11), we arrive at

$$\begin{aligned} i\int_{\mathbb{T}} |f^*(z)|^2 z^{m+2+k} d\bar{z} &= 2(m+2+k)\int_{\mathbb{D}} |f^*(z)|^2 z^{m+1+k}\, dA(z) \\ &+ 2\int_{\mathbb{D}} \overline{f^*(z)} f^{*\prime}(z) z^{m+2+k}\, dA(z) \\ &= 2\int_{\mathbb{D}} \frac{f^{*\prime}(z)}{f^*(z)} |f^*(z)|^2 z^{m+k+2}\, dA(z). \end{aligned}$$

Now, since we know that $f^* \in H^\infty$ (see [1]), the condition (4.12) would follow at once from (4.11) provided that $(\log f^*)' = (f^*)'/(f^*)$ is integrable in $\mathbb{D}$ (with respect to the weight $|f^*|^2$). However, to show the latter is a major technical difficulty. In general, if we write $(\log f^*)'$ as

$$(q^*)'(z) = \left(\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z}\, d\nu^*(t) \right)',$$

no conclusion regarding the integrability of q^* can a priori be made. Assuming that the singular part $d\mu^*$ of the extremal measure $d\nu^*$ in (4.6) is supported on a Carleson set essentially allows us to make the above argument rigorous (see [1], also see similar arguments in [12]). Unfortunately, we still do not know how to circumvent this assumption on the measure.

5 FURTHER CONJECTURES

In the setting of the linear version of the problem (4.1) (i.e., without the non-vanishing restriction), the use of duality and powerful results from PDE allowed the authors in [12] to make the argument at the end of the previous section legitimate. Lack of linearity, and therefore, the unavailability of duality present a formidable obstruction.

(i) Pertinent to the discussion in Section 4, we can expect the singular part μ^* of the extremal measure ν^* in (4.6) to be non-trivial. The question is, where should we expect the atoms of the singular measure to be located? The example in [1] for Problem (4.1) with $m = 1$ shows that it may be possible that the extremal functions in Bergman spaces need not be continuous in the closed disk, so the atoms are not expected to be at the zeros of $P \geq 0$.

The following conjecture was offered in [1].

Conjecture 5.1. *If $P > 0$ on the unit circle $\mathbb{T}$, then the singular part μ^* of the extremal measure ν^* is supported on the set of local minimum points of P on $\mathbb{T}$.*

In other words, the singular inner part of the extremal function f^* for Problem (4.1) is atomic with atoms located at the local minima of P on $\mathbb{T}$. The conjecture is intuitive in the sense that in order to maximize the integral

$$\int_{\mathbb{T}} P\, d\mu,$$

we are best off if we concentrate all the negative contributions from the singular factor at the points where P is smallest.

(ii) What can be said in relation to a Krzyż type conjecture in the context of Bergman spaces? In [28] (also, cf. [19, 39, 40]), Ryabych solved the problem of finding, for $p \geq 1$ and $m \geq 1$,

$$\max \operatorname{Re} \left\{ f^{(m)}(0)/m! : \ \|f\|_{A^p} \leq 1 \right\} \tag{5.1}$$

and showed that the extremal function is

$$f^*(z) = \left(\frac{mp+2}{2} \right)^{\frac{1}{p}} z^m,$$

thus making the maximum in (5.1) equal to $(\frac{mp+2}{2})^{\frac{1}{p}}$. Of course, the latter quantity tends to 1 (the H^∞ case) when $p \to \infty$.

What are we to expect in Problem (5.1), but set in A_0^p rather than A^p? Let us consider that problem for $m = 1, p = 2$. If the conjecture in [1] regarding the general form of the extremal function is true, then

$$f^*(z) = C(z - 1 - \mu) \exp\left(\mu \frac{1+z}{z-1} \right), \tag{5.2}$$

where $\mu > 0$ and $C = C(\mu)$ is a constant such that $\|f^*\|_{A^p} = 1$. If we denote by F^* the antiderivative of f^* such that $F^*(1) = 0$, then $F^*(z) = \frac{C}{2}(z-1)^2 \exp\left(\mu \frac{1+z}{z-1}\right)$. Using the complex form of Green's theorem together with the fact that

$$\left| \exp\left(\mu \frac{1+z}{z-1} \right) \right| = 1 \ \text{ a.e. on } \mathbb{T},$$

we calculate

$$\begin{aligned}
\int_{\mathbb{D}} |f^*(z)|^2\, dA &= \frac{i}{2\pi}\int_{\mathbb{T}} F^*(z)\overline{f^*(z)}\, d\bar{z} \\
&= \frac{i}{2\pi}\int_{\mathbb{T}} F^*(z)\overline{f^*(z)}(-i)\bar{z}\, d\theta \\
&= \frac{1}{2\pi}\int_{\mathbb{T}} \frac{C}{2}(z-1)^2 \exp\left(\mu\frac{1+z}{z-1}\right) C(\bar{z}-1-\mu)\overline{\exp\left(\mu\frac{1+z}{z-1}\right)}\,\bar{z}\, d\theta \\
&= \frac{C^2}{2}\frac{1}{2\pi}\int_{\mathbb{T}} (z^2-2z+1)(\bar{z}-(1+\mu))\,\bar{z}\, d\theta \\
&= \frac{C^2}{2}(3+2\mu).
\end{aligned}$$

Since $\int_{\mathbb{D}} |f^*(z)|^2\, dA = 1$, we get that $C = \sqrt{\frac{2}{3+2\mu}}$, where $\mu > 0$. Substituting C into (5.2), we obtain

$$(f^*)'(0) = \sqrt{\frac{2}{3+2\mu}}\, e^{-\mu}(1+2\mu+2\mu^2). \tag{5.3}$$

It is not hard to see that this function of μ, when $\mu > 0$, is maximized when $\mu = 1$. We thus obtain

$$\max\left\{\mathrm{Re} f'(0) :\ \|f\|_{A^2} \le 1\,, f \text{ non-vanishing in } \mathbb{D}\right\} = \sqrt{2}\,\frac{\sqrt{5}}{e}. \tag{5.4}$$

It is also natural then to expect that an extremal function for any m in the problem

$$\max\{\mathrm{Re} f^{(m)}(0) :\ \|f\|_{A^2} \le 1\,, f \text{ non-vanishing in } \mathbb{D}\} \tag{5.5}$$

would be $c_m f^*(z^m)$, where f^* is the extremal for (5.4)and c_m is the normalizing constant. From this, we leap to the following rather bold conjecture, although present evidence in its favor is not abundant.

Conjecture 5.2.

$$\limsup_{m\to\infty} \frac{\max\left\{\mathrm{Re}\, f^{(m)}(0)/m! :\ \|f\|_{A^2} \le 1\,,\ f \text{ non-vanishing in } \mathbb{D}\right\}}{\sqrt{m}} \le \frac{\sqrt{5}}{e}.$$

Since we still have not been able to completely solve the above problem for $m = 1$, it is really premature to speculate about the precise asymptotics of the maximum above. However, it may be plausible to obtain “Horowitz type” results for A^p functions even without solving this problem explicitly. Thus, we suggest the following, more realistic problem. Let $1 < p < \infty$, and define

$$\Lambda_m = \Lambda_{m,p} = \max\{\mathrm{Re}\, f^{(m)}(0)/m! :\ \|f\|_{A^p} \le 1\,, f \text{ non-vanishing in } \mathbb{D}\}.$$

Conjecture 5.3.

$$\limsup_{m\to\infty} \frac{\Lambda_m}{\left(\frac{mp+2}{2}\right)^{\frac{1}{p}}} < 1.$$

Denote by λ_m the analog of Λ_m in the H^p_0-context. A priori, of course, $\Lambda_m \ge \lambda_m$.

Question. What are the asymptotics of Λ_m? Is $\Lambda_m \sim \lambda_m\, m^{1/p}$?

We think that perhaps with the advances in the theory of Bergman spaces in the last decade, the time has come for a thorough study of these fundamental extremal problems.

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