AN ASYMPTOTIC GAUSS-LUCAS THEOREM

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Abstract. In this note we extend the Gauss-Lucas theorem on the zeros of the derivative of a univariate polynomial to the case of sequences of univariate polynomials whose almost all zeros lie in a given convex bounded domain in $\mathbb{C}$.

1. Introduction

The celebrated Gauss-Lucas theorem claims that for any univariate polynomial $P(z)$ with complex coefficients, all roots of $P'(z)$ belong to the convex hull of the roots of $P(z)$, see Theorem 6.1 of [5]. Many generalizations have been obtained over the years see, e.g., [1, 2, 7] and references therein.

In the present note, motivated by problems in potential theory in $\mathbb{C}$, we extend the Gauss-Lucas theorem to sequences of polynomials of increasing degrees whose almost all zeros lie in a given convex bounded domain in $\mathbb{C}$. Namely, given a convex bounded domain $\Omega \subset \mathbb{C}$, let $\{p_n(z)\}_{n=0}^{\infty}$ be a sequence of univariate polynomials with the degrees $\deg p_n = m_n$ such that $\lim_{n\to\infty} m_n = +\infty$. Assume that $\lim_{n\to\infty} \frac{\sharp_n(\Omega)}{m_n} = 1$, where $\sharp_n(\Omega)$ is the number of zeros of p_n lying in Ω (counted with multiplicities).

Problem 1. *Following the above notation we now ask whether there exists* $\lim_{n\to\infty} \frac{\sharp'_n(\Omega)}{m_n-1}$, *where* $\sharp'_n(\Omega)$ *denotes the number of zeros of* $p'_n(z)$ *lying in* Ω*?*

It turns out that the answer to Problem 1 formulated verbatim as above , is, in general, negative.

Example 1. Let O be the open square $(-2, 2)\times(-4i, 0)$. If $T_n(z) := \cos(n \arccos z)$ is the n-th Chebyshev polynomial of the first kind, then the derivative of $(z-i)T_n(z)$ has all its zeros in the upper half plane. Therefore, if we replace z by $z + ia_n$ for some sufficiently small a_n (i.e., shift all zeros downward by a_n), then we obtain polynomials of degrees $n+1$ with n zeros in O, but whose derivatives have no zeros

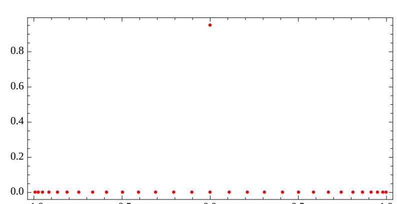

Figure 1. Zeros of $((z-i)T_{30}(z))'$

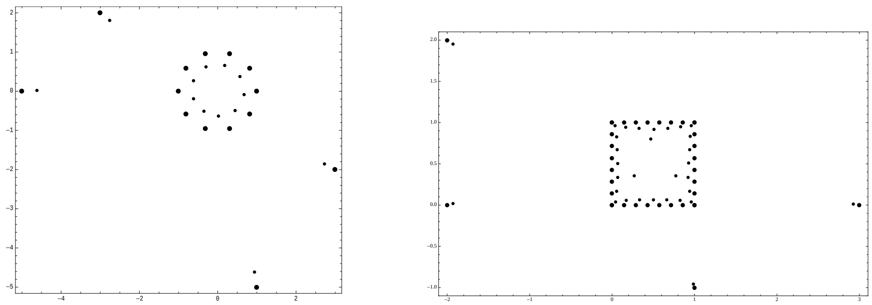

FIGURE 2. Zeros of P (larger dots) and P' (smaller dots) for $P = (z^{10} - 1)(z - 3 + 2I)(z + 3 - 2I)(z - 1 + 5I)(z + 5)$ (left) and $P = Q(z-3)(z+2-2I)(z-1+I)(z+2)$ where Q has 7 uniformly placed zeros on each side of the unit square (right).

in O. Choosing a_n appropriately for each n, we get a sequence of polynomials with all but one zeros in O whose derivatives have no zeros in O.

Strict convexity (e.g., as in the case of the open unit disk) will not be of much help either. Just replace above z by $M_n z$ with some large M_n and then make a vertical translation so that after all these operations the image of $[-1, 1]$ becomes a tiny secant segment of the unit circle. (This example was suggested to the third author by Professor V. Totik.)

However with slightly weaker requirements Problem 1 has a positive answer.

Theorem 1. *Given a polynomial sequence $\{p_n(z)\}$ as above and any $\epsilon > 0$,*

$$\lim_{n\to\infty} \frac{\sharp'_n(\Omega_\epsilon)}{m_n - 1} = 1,$$

where $\sharp'_n(\Omega_\epsilon)$ is the number of zeros of $p'_n(z)$ lying in Ω_ϵ, and Ω_ϵ is the open ϵ-neighborhood of $\Omega \subset \mathbb{C}$.

Two illustrations of Theorem 1 are given in Figure 2.

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2. PROOF

We first prove Theorem 1 in the case when Ω is a disk. Fix $\epsilon > 0$. Let $\mathbb{D} = \{|z| < 1\}$ be the open unit disk, $p_n(z) = \prod_{k=1}^{m_n}(z - \alpha_k)$, $\lim m_n = \infty$. Let us factor p_n as follows:

(1)

$$p_n = q_n r_n = \prod_{k=1}^{k_n}(z - a_k) \prod_{k=k_{n+1}}^{m_n}(z - b_k), \quad |b_k| > 1+\epsilon, \quad |a_k| < 1+\epsilon, \quad \text{with} \quad \lim_{n\to\infty} \frac{k_n}{m_n} = 1.$$

Denote by $\mathfrak{Z}'_n := \sharp\{z : |z| < 1 + \epsilon : p'_n(z) = 0\}$. We want to show that $\lim_{n\to\infty} \frac{\mathfrak{Z}'_n}{m_n - 1} = 1$.

Let

(2) $$\hat{\mu}_n(z) := \frac{1}{m_n}\sum_{k=1}^{n}\frac{1}{z-\alpha_k} = \frac{1}{m_n}\frac{p_n'}{p_n},$$

denote the Cauchy transform of the root-counting probability measure μ_n of p_n. Note that

(3) $$\hat{\mu}_n(z) = \frac{1}{m_n}\left(\frac{q_n'}{q_n}+\frac{r_n'}{r_n}\right) = \frac{1}{m_n}(k_n\hat{\nu}_n + (m_n-k_n)\hat{\psi}_n),$$

where ν_n and ψ_n are the root-counting measures of q_n and r_n respectively. All zeros of q_n' lie in the unit disk D by the Gauss-Lucas theorem. Also (1) implies that

(4) $$\frac{(m_n-k_n)}{m_n}||\psi_n|| \to 0.$$

Formula (4) implies that for all $p : 1 \le p < 2$, we have

(5) $$\left|\left|\frac{1}{m_n}\frac{r_n'}{r_m}\right|\right|_{L^p(dA)} \to 0$$

on compact subsets of $\mathbb{C}$. Here, $dA = \frac{1}{\pi}dxdy$ denote the normalized area measure.

Equation (5) follows from a trivial observation. Let μ be a Borel measure with a compact support. Then for any compact set $K \subset \mathbb{C}$, and any $p : 1 \le p < 2$, we have

(6) $$||\hat{\mu}(z)||^p_{L^p(K,dA)} \le C(p,K)||\mu||.$$

Indeed, $\hat{\mu} = \int \frac{d\mu(\xi)}{\xi - z}$, hence

(7) $$\int_K |\hat{\mu}|^p dA \le ||\mu|| \int_{|\xi|<R} \frac{1}{|\xi|^p} dA \le C||\mu||,$$

where R is chosen so that $\forall \xi \in K$ the disk of radius R centered at ξ contains K. The integral in (7) converges for all $p < 2$ and (6) follows, hence does (5).

Thus, we have from (5) the following corollary.

Corollary 1. . *For any fixed $R > 1+\epsilon$, and any $p : 1 \le p < 2$, for almost all $r : 1+\epsilon < r < R$, we have*

(8) $$\lim_{n\to\infty}\frac{1}{m_n}\int_{|z|=r}\frac{|r_n'|^p}{|r_n|^p}ds_r = 0,$$

where ds_r is the arclength measure on $\{z : |z| = r\}$.

Thus, from (3), we now obtain

Corollary 2.

$$\lim_{n\to\infty}\frac{1}{m_n}\int_{|z|=r}\left|\frac{p_n'}{p_n} - \frac{q_n'}{q_n}\right|^p ds_r = 0$$

for almost all $r : 1+\epsilon < r < R$ and $p : 1 \le p < 2$.

However
$$\frac{1}{m_n}\left(\frac{p_n'}{p_n}-\frac{q_n'}{q_n}\right)=\frac{1}{m_n}\left(\sum_{k=k_n+1}^{m_n}\frac{1}{z-b_k}\right)$$
by (3), and hence is analytic inside $\{|z|<1+\epsilon\}$ since $|b_k|>1+\epsilon$.

Therefore, from standard results on Hardy spaces H^p in the disk, cf. [3],we conclude that

$$\frac{1}{m_n}\left(\frac{p_n'}{p_n}-\frac{q_n'}{q_n}\right)\to 0 \tag{9}$$

uniformly in the closed disk $\overline{D}=\{|z|\le 1+\epsilon$.

Since $\frac{1}{m_n}\frac{q_n'}{q_n}$ vanishes at k_n-1 points in D by Gauss-Lucas theorem, invoking Hurwitz's theorem, we conclude that

$$\lim_{n\to\infty}\frac{1}{m_n}\left[\sharp(z:|z|<1+\epsilon:p_n'(z)=0)-(k_n-1)\right]=0. \tag{10}$$

Since, by assumption, $\lim_{n\to\infty}\frac{k_n-1}{m_n}=\lim_{n\to\infty}\frac{k_n}{m_n}=1$, we arrive at
$$\lim_{n\to\infty}\frac{\mathfrak{Z}_{\mathfrak{n}}'}{m_n}=\lim_{n\to\infty}\frac{\mathfrak{Z}_{\mathfrak{n}}'}{m_n-1}=1,$$
which settles Theorem 1 in the case of a disk.

To finish the proof for the general case of an arbitrary convex domain Ω observe that we only used some properties of a disk to get a convenient foliation of a neighborhood of the unit disk by concentric circles and when applying Gauss-Lucas theorem. Both these facts are readily available for an arbitrary bounded convex domain. Finally, the Hardy spaces of analytic functions in the disk are replaced by the Smirnov classes E^p of functions representable by Cauchy integrals with L^p-densities (with respect to arclength). In the domains with piecewise smooth boundaries, e.g., convex domains, the latter behave in the very same manner as Hardy spaces – cf. [3]. □

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