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On a uniqueness property of harmonic functions

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To Walter Hayman with admiration on the occasion of his 80th birthday.

Abstract. We investigate the problem of uniqueness for functions u harmonic in a domain Ω and vanishing on some parts of the intersection (not necessarily connected) of Ω with a line m. It turns out that for some configurations u must vanish on the whole intersection of m and Ω, but this is not always the case. Generalizations to solutions of more general analytic elliptic equations are discussed as well.

Keywords. Analytic continuation, harmonic functions, cells of harmonicity, Schwarz reflection principle.

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1. Introductory Remarks

The following simple question was posed to the second author by N. Nadirashvili more than a decade ago — cf. [6, p. 2]. Consider the spherical shell $\Omega := \{x \in \mathbb{R}^3 : r < |x| < R\}$ and let u be a harmonic function in Ω that vanishes on the segment $(-R, -r)$ of, say, the x_1-axis. Does u also vanish on the segment (r, R), the remaining part of the intersection of the x_1-axis with Ω?

It is instructive to keep in mind the following simple example of a situation where a similar question has a negative answer.

Example 1. Let $\Omega = \left\{z : r < |z| < R, -\frac{\pi}{4} < \arg z < \frac{5\pi}{4}\right\}$. The function $u(z) = \arg z$ is harmonic in Ω, vanishes on the interval of the real axis (r, R) but is equal to π on $(-R, -r)$.

To fix the ideas for further discussion, let us consider the same question in the annulus $\Omega : \{z = x + iy : r < |z| < R\}$ in the plane. Define the (harmonic) function $v(z)$ by

$$v(z) := u(z) + u(\bar{z}). \tag{1.1}$$

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Since Ω is symmetric about the x-axis, (1.1) is well-defined. Moreover, in view of the Schwarz Reflection Principle (cf. [3, 6, 8]) and the hypothesis, v vanishes identically on a small disk centered somewhere on the segment $(-R, -r)$. But v is harmonic and hence real-analytic throughout Ω, so it must vanish identically in Ω. In particular, we have $v(x) = 2u(x) = 0$ on (r, R), so $u|_{(r,R)} = 0$ as well.

In the following section, we expand on this idea to answer the original question in the affirmative in R^n for all $n \geq 2$ and for slightly more general domains.

In Section 3 we gain what is hopefully the "higher ground" for the question at hand. Namely, we consider analytic continuation of the function u throughout the larger complex space $\mathbb{C}^n$, with $\mathbb{R}^n \subset \mathbb{C}^n$, $\mathbb{R}^n = \{z \in \mathbb{C}^n, z = (z_1, \ldots, z_n), z_j \in \mathbb{R}\}$. This idea rests on the notion of a *cell of harmonicity* (cf. [1, 2, 6, 7]), also known as *Vekua hulls* for $n = 2$. This allows us to extend our results to domains that are not extensively symmetric, e.g., certain shells between two heterogeneous ellipsoids. Moreover, this approach automatically yields the same results for solutions u of rather general analytic elliptic equations

$$Lu = f, \quad L = \Delta^m + \sum_{|\alpha| \leq 2m-1} a_\alpha(x)\, \partial^\alpha, \tag{1.2}$$

where $a_\alpha(x)$, f are, say, entire real-analytic functions (e.g., polynomials), $\Delta = \sum_1^n \partial^2/\partial x_j^2$. We use the standard multi-index notation $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_j \in \mathbb{N} \bigcup \{0\}$ and

$$\partial^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$$

(cf., e.g., [4, 5] or [6]).

We conclude with some remarks, examples and possible questions for further study.

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2. Exploiting Symmetry

The argument in Section 1 extends word-for-word to $\Omega \subset \mathbb{R}^n$, $n \geq 2$, if the line $y = 0$ in $\mathbb{R}^2$ is replaced by a hyperplane $H \subset \mathbb{R}^n$ and Ω is assumed to be symmetric about H.

Nadirashvili's question is more delicate, since we replace H by a much thinner set, a line, which prevents us from applying the reflection principle in all dimensions ≥ 3.

The following theorem addresses Nadirashvili's question in a slightly more general form.

Theorem 2. *Let Ω be a domain in $\mathbb{R}^n$ symmetric about the x_1-axis, i.e., Ω is symmetric about the x_1-axis when $n = 2$ and axially symmetric about the x_1 axis for $n \geq 3$. Then, if a function u, harmonic in Ω, vanishes on some interval I of the x_1-axis in Ω, it must vanish at all points of the x_1-axis which lie in Ω.*

Proof. The proof is based on some well known facts from the theory of *axially symmetric potentials* — cf., e.g., [9]. For the convenience of the reader we have incorporated most of them into our reasoning. Let $\rho = (x_2^2 + \cdots + x_n^2)^{1/2}$ be the distance to the x_1-axis. Let $x = (x_1, \ldots, x_n)$ be a point in Ω and define $v(x)$ to be the mean value of u over the $(n-2)$-dimensional sphere S_ρ^{n-2} centered at $(x_1, 0, \ldots, 0)$ of radius $\rho : \rho^2 = x_2^2 + \cdots + x_n^2$. Then, v is still harmonic in Ω and is invariant with respect to all rotations of $\mathbb{R}^n$ about the x_1-axis. Indeed, let G be the subgroup of the orthogonal group in $\mathbb{R}^n$ that leaves points on the x_1-axis fixed, i.e., G is the group of rotations around the x_1-axis. Let $m(g)$ be the Haar measure on G normalized so that $m(G) = 1$. Then,

$$v(x) = \int_G u(gx)\, dm(g).$$

In view of Weyl's lemma, to check the harmonicity of v it suffices to show that

$$\int_\Omega v(x)\, \Delta f(x)\, dx = 0$$

for every C^∞-function f in Ω with a compact support in Ω. The latter integral can be written as follows:

$$\begin{aligned}\int_\Omega \left\{ \int_G u(gx)\, dm(g) \right\} \Delta_x f(x)\, dx &= \int_G \left\{ \int_\Omega u(gx)\, \Delta_x f(x)\, dx \right\} dm(g) \\ &= \int_G \left\{ \int_\Omega u(y)\, \Delta_y f\left(g^{-1}y\right) dy \right\} dm(g).\end{aligned}$$

(We used the change of variables $gx = y$ and the fact that the Laplacian commutes with all orthogonal transformations.) Since $f(g^{-1}y) \in C_0^\infty(\Omega)$; the inner integral vanishes in view of the harmonicity of u and the harmonicity of v follows.

So, v is harmonic in Ω and is invariant with respect to all rotations of $\mathbb{R}^n$ about the x_1-axis. Thus, $v = V(x_1, \rho)$. Clearly, although V is initially defined on a one sided neighborhood of I, it extends to a full neighborhood of I in $\mathbb{R}^2$ by setting $V(x_1, -\rho) = V(x_1, \rho)$ and is real analytic there (cf. [9]). It is well known (cf., e.g., [9]), in view of the harmonicity of v, that, in the meridian (x_1, ρ)-plane, V satisfies the equation of axially symmetric potentials

$$\frac{\partial^2 V}{\partial x_1^2} + \frac{\partial^2 V}{\partial \rho^2} + \frac{n-2}{\rho}\frac{dV}{d\rho} = 0. \tag{2.1}$$

Since $V(x_1, 0) = u(x_1, 0, \ldots, 0)$ it follows from our hypothesis that V vanishes on the interval I of the x_1-axis and then, equation (2.1) yields that $\frac{dV}{d\rho}$ vanishes everywhere on I as well. Since $V(x_1, p)$ is real analytic and vanishes on I there is, unless V vanishes on a neighborhood of I in $\mathbb{R}^2$, a greatest positive integer k such that $V(x_1, \rho) = \rho^k W(x_1, \rho)$ with W real analytic and non-vanishing on I. But, substituting this into (2.1) and simplifying shows at once that W vanishes on I as well, which contradicts the maximality of k. This contradiction shows that V vanishes on a neighborhood of I. Since V is real analytic in Ω, $V \equiv 0$. But $u = V$ at all points on the x_1-axis inside Ω and the proof is now complete. ■

Remark 1.

(1) Instead of using (2.1) to see that V vanishes identically near I, one could alternatively expand $v(x_1, \rho)$ in some ball centered at a point of the x_1 axis in the series of zonal harmonics with the coefficients (Gegenbauer coefficients) that are expressible as integrals over the interval I on the x_1-axis, which all vanish since $v|_I = 0$ (cf. [9] for details).

(2) Of course, Theorem 2 remains true if we merely assume that Ω contains a domain Ω_1 that is axially symmetric about the x_1-axis and such that every interval of x_1-axis inside Ω intersects Ω_1 as well.

Returning to the two-dimensional situation we can generalize the above theorem slightly.

We say (cf. [7]) that a domain $\Omega \subset \mathbb{R}^2$ is symmetric with respect to a real-analytic curve γ given by $\bar{z} = S(z)$, where $S(z)$ is the Schwarz function of γ (see [3,8]) analytic near γ, if S is analytic and single-valued in Ω and the mapping $R(z) := \overline{S(z)}$, $R|_\gamma = id$, is a bijection of Ω onto itself.

Proposition 3. *Let Ω be symmetric with respect to γ. If u is harmonic in Ω and vanishes on some portion of γ, it vanishes at all points of the curve γ that lie in Ω.*

Proof. Following (1.1), define

$$v(z) := u(z) + u(R(z)). \tag{2.2}$$

Following the argument in Section 1 word-for-word and appealing to the Schwarz Reflection Principle for arbitrary analytic curves (cf. [3, 6, 8]), we conclude that $v \equiv 0$ in Ω. Hence $u|_{\Omega \bigcap \gamma} = v = 0$. ■

Remark 2. In fact, it suffices in the above proposition to assume that the open set $\Omega \cap R(\Omega)$ is merely connected in Ω, not necessarily coinciding with Ω.

3. A View from $\mathbb{C}^n$

Recall the notion of the harmonicity hull of a domain $\Omega \subset \mathbb{R}^n$ (the Vekua hull, for $n = 2$).

Let $z^0 \in \mathbb{C}^n$, $\Gamma_{z^0} := \left\{ z \in \mathbb{C}^n : \sum_1^n (z_j - z_i^0)^2 = 0 \right\}$ be the *isotropic cone* with the vertex at z^0.

The harmonicity hull $\hat{\Omega}$ of a domain $\Omega \subset \mathbb{R}^n$ is defined as

$$\mathbb{C}^n \setminus \bigcup_{x \in \mathbb{R}^n \setminus \Omega} \Gamma_x. \tag{3.1}$$

For examples, basic properties and extensive accounts of this concept, we refer the reader to [1] and [2], and also to [6, 7]. Note that in two dimensions, where

$$\Gamma_{z^0} = \left\{ z \in \mathbb{C}^2 : z_1 + iz_2 = z_1^0 + iz_2^0 \right\}$$
$$\cup \left\{ z \in \mathbb{C}^2 : z_1 - iz_2 = z_1^0 - iz_2^0 \right\}$$

is simply the union of two complex lines, the notion of harmonicity (Vekua) hull is especially geometrically transparent:

$$\hat{\Omega} = \left\{ z \in \mathbb{C}^2 : z_1 + iz_2 \in \Omega, \bar{z}_1 - i\bar{z}_2 \in \Omega \right\}.$$

Now if u satisfies the differential equation $Lu = f$ in Ω, with the differential operator L given by (1.2):

$$L := \Delta^m + \sum_{|\alpha| \leq 2m-1} a_\alpha (x) \, \partial^\alpha \tag{3.2}$$

with the coefficients a_α, f holomorphic in (a larger) domain $\hat{\Omega} \subset \mathbb{C}^n$, then u admits analytic continuation into $\hat{\Omega}$ (cf. [1, 2, 6, 7]). If $\hat{\Omega}$ is simply connected (as, for example, for convex Ω), u extends as a single-valued holomorphic function, otherwise it may have a nontrivial monodromy in $\hat{\Omega}$.

A proof of this result for polyharmonic functions can be seen more or less at once if one notices that these functions can be represented by integrals over $\partial\Omega$ with analytic kernels whose only singularities are restricted to isotropic cones Γ_x, $x \in \partial\Omega$ (cf. [1, 2]).

It is worth noticing that this approach extends *mutatis mutandis* to all Riesz potentials u in Ω:

$$u(x) = \int_{\mathbb{R}^n \setminus \bar{\Omega}} \frac{d\mu(y)}{|x-y|^\alpha}, \quad \alpha \in \mathbb{R} \tag{3.3}$$

where $d\mu$ is a compactly supported measure in $\mathbb{R}^n \setminus \bar{\Omega}$, and $|x-y|$ is the Euclidean distance. For solutions u of $Lu = f$, the strategy of proving analytic extendibility is roughly as follows. Fill up Ω as a union of convex domains, say balls B, and using the Bony-Schapira theorem (cf. [4, 6]), extend u to each $\hat{B}$. Then their union fills up $\hat{\Omega}$.

The following observation is now obvious.

Theorem 4. *Let $\Omega \subset \mathbb{R}^n$ be a domain and let $\hat{\Omega} \subset \mathbb{C}^n$ be its harmonicity hull (so $\Omega \subset \hat{\Omega} \subset \mathbb{C}^n$). Let L be as in (3.2) with all the coefficients holomorphic in $\hat{\Omega}$. Assume that the intersection of the complex z_1-line $\{z_2 = \cdots = z_n = 0\}$ with $\hat{\Omega}$ is path-connected. Then any solution u of $Lu = f$ in Ω that vanishes on some portion I of the x_1-axis in Ω, vanishes at all points of the x_1-axis that lie in Ω.*

Proof. Indeed, by the above remarks u extends as a multi-valued function to $\hat{\Omega}$. This extension has a single-valued branch in Ω that we still denote by u. Hence, this single-valued branch u is analytically continuable in the one-dimensional complex domain $D := \hat{\Omega} \cap \{z_2 = \cdots = z_n = 0\}$. Since it vanishes on an open segment of a real line in D, u must vanish identically in D, and the statement follows since D is assumed to be connected. ■

Corollary 5. *Let $\Omega := \{x \in \mathbb{R}^n : r < |x| < R\}$ be a spherical shell. L, f are as before. If a solution u of $Lu = f$ vanishes on some portion of the x_1-axis in Ω, it vanishes at all points of the x_1-axis that lie in Ω.*

Proof. For the proof we only need to show that the intersection of the complex line $M := \{z_2 = \cdots = z_n = 0\}$ with $\hat{\Omega}$ is path connected. Take any $c : r < c < R$. Let γ be the circle centered at 0 with radius c in M. Take any $x \in \mathbb{R}^n \setminus \Omega$ and consider the isotropic cone Γ with the vertex x, $\Gamma_x = \left\{ z \in \mathbb{C}^n : \sum\limits_1^n (z_i - x_i)^2 = 0 \right\}$. We only have to show that no point $\{(ce^{it}, 0, \ldots, 0), t \in \mathbb{R}\}$ on γ belongs to Γ_x. Indeed, if this were the case, we would have

$$\left(ce^{it} - x_1\right)^2 = -\rho^2, \quad \rho^2 = x_2^2 + \cdots + x_n^2,$$

or

$$ce^{it} = x_1 \pm i\rho,$$

i.e., $|c|^2 = x_1^2 + \rho^2 = |x|^2$. But this is impossible since $r < c < R$ while $x \in \mathbb{R}^n \setminus \Omega$ and hence $|x|$ is either $\geq R$, or $\leq r$. ■

Remark 3. The above corollary, of course, extends word-for-word if u is a Riesz potential (3.3).

Corollary 6. *The above corollary extends to shells $\Omega := \Omega_2 \setminus \Omega_1$, where Ω_j, $j = 1, 2$, are arbitrary solids homeomorphic to a ball, $\Omega_1 \ni \{0\}$, provided that $r := \max\{|x| : x \in \partial\Omega_1\} < \min\{|x| : x \in \partial\Omega_2\} =: R$.*

For the proof one needs only to notice that the shell Ω contains a spherical shell $\Omega' : \{x : r + \epsilon < |x| < R - \epsilon\}$ for sufficiently small $\epsilon > 0$ (cf. Remark (2) following the proof of Theorem 2).

4. Final Remarks

In view of Corollary 6, Corollary 5 extends to spherical shells Ω with the x_1-axis replaced by a parallel line sufficiently close to the center, i.e., by the line $m := (x_1, t_2, \ldots, t_n)$, where the t_j are fixed and $\left(\sum\limits_2^n t_j^2\right)^{1/2} = \rho < \frac{R-r}{2}$, R, r are as in Section 3. In the two-dimensional case, this can also be proved directly using the Schwarz reflection argument mentioned in the Introduction. The key point is that we need $\Omega \bigcap \Omega^*$ to be connected, where Ω^* denotes the reflection of Ω in the line in question, and this only holds for lines sufficiently close to the x_1-axis.

Obviously, if $\rho > r$, the intersection of m and Ω is connected, and Corollary 5 holds then as well.

For "thick shells", i.e., where $R/r > 3$, it is already true that $\frac{R-r}{2} > r$, hence unique continuation property holds for ALL lines. Yet, observe that when $R/r < 3$ and $r > \rho > \frac{R-r}{2}$, the intersection of the harmonicity shell $\hat{\Omega}$ with the complexification M of the line m becomes disconnected, indicating that the unique continuation property may fail, and there are say harmonic functions u in Ω vanishing on parts of m in Ω but not on the whole intersection of Ω and m. We have not been able to furnish an example of such u ourselves. The following simple and elegant example is due to Professor Th. Ransford.

Example 7. Let Ω be an annulus in $\mathbf{C} \setminus \{0, i, -i\}$ which separates $\{0, i\}$ from $\{-i\}$. Define $u(z) = Re\sqrt{z(z-i)(z+i)}$. Note that u is well-defined and harmonic in Ω (it does not matter which branch is chosen). A quick calculation shows that $u(x) = 0$ if $x < 0$ and $u(x) = \sqrt{x(x^2+1)} \neq 0$ if $x > 0$. It remains now to translate the whole picture, so that Ω is centered at the origin. Note that the above annulus is "thin", i.e., the ratio of the radii $R/r < 3$ and can be easily made arbitrary close to 3.

It is tempting to try and extend Proposition 3 to higher dimensions when a domain in $\mathbb{R}^n$ is intersected by, say, an algebraic variety of high-codimension, e.g., a curve, or a hypersurface. At this point we do not even have a reasonable conjecture to present here (Example 1 still remains somewhat of a mystery), yet we think that it is a worthy topic for future investigations.

References

[1] N. Aronszajn, T. M. Creese, and L. J. Lipkin. *Polyharmonic functions.* Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 1983.

[2] V. Avanissian. *Cellule d'harmonicité et prolongement analytique complexe.* Hermann, Paris, 1985.

[3] P. J. Davis. *The Schwarz function and its applications.* The Mathematical Association of America, Buffalo, N. Y., 1974. The Carus Mathematical Monographs, No. 17.

[4] L. Hörmander. *The analysis of linear partial differential operators. I.* Springer-Verlag, Berlin, 1983.

[5] F. John. *Partial differential equations.* Springer-Verlag, New York, second edition, 1975.

[6] D. Khavinson. *Holomorphic partial differential equations and classical potential theory.* Universidad de La Laguna, Departamento de Análisis Matemático, La Laguna, ISBN 84-600-9323-9, 1996.

[7] D. Khavinson and H. S. Shapiro. The Vekua hull of a plane domain. *Complex Variables Theory Appl.*, 14(1-4):117–128, 1990.

[8] H. S. Shapiro. *The Schwarz function and its generalization to higher dimensions*. University of Arkansas Lecture Notes in the Mathematical Sciences, 9. John Wiley & Sons Inc., New York, 1992.

[9] A. Weinstein. Generalized axially symmetric potential theory. *Bull. Amer. Math. Soc.*, 59:20–38, 1953.

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