Gravitational Lensing by Elliptical Galaxies and the Schwarz Function

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Basic Theory of Gravitational Lensing

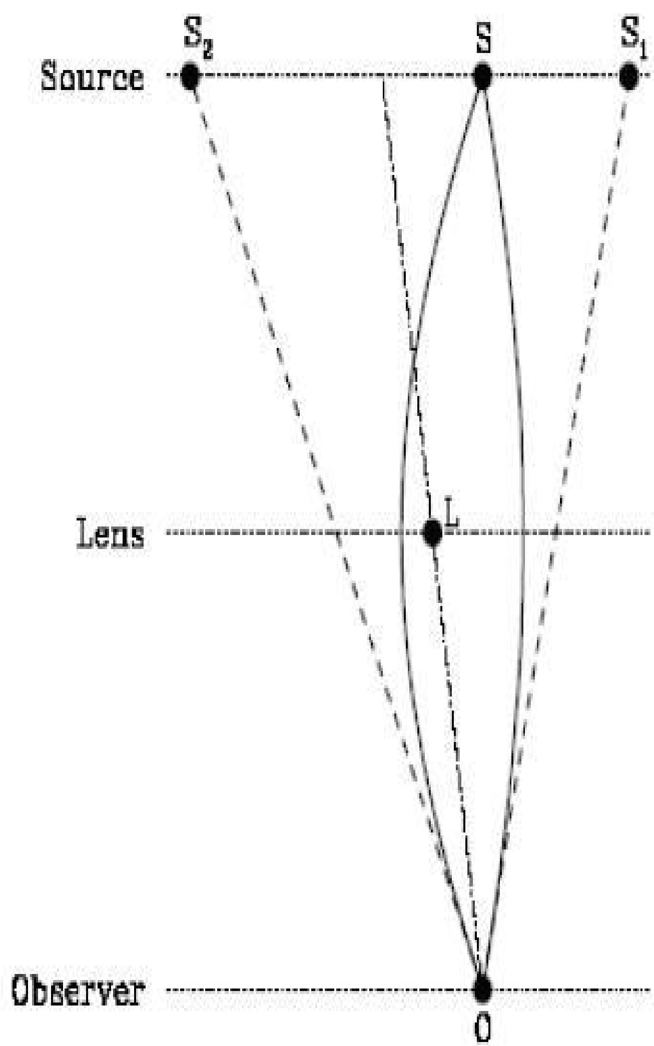


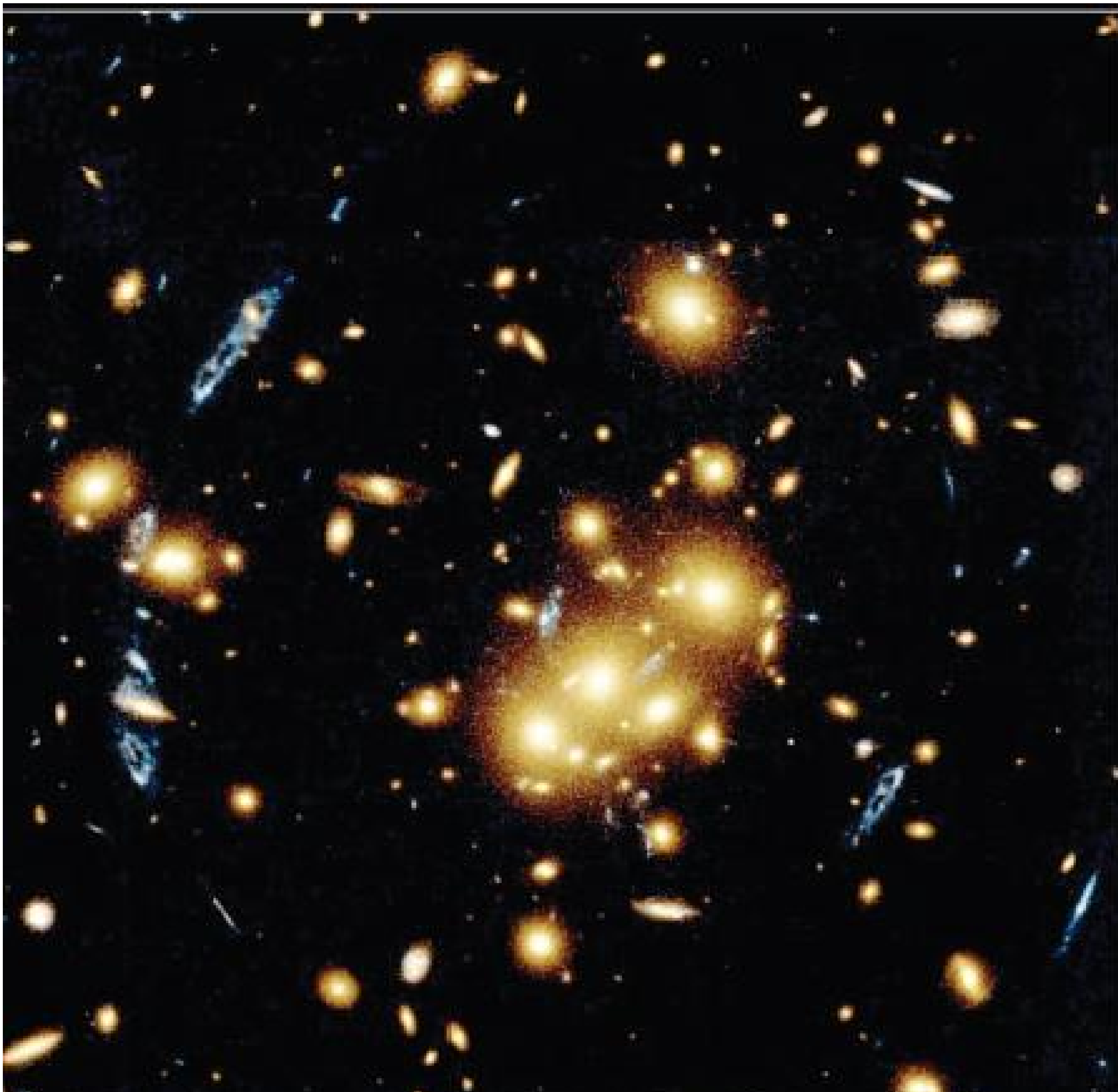
Figure 2: *Setup of a gravitational lens situation: The lens L located between source S and observer O produces two images S_1 and S_2 of the background source.*

Gravitational Microlensing

- n co-planar point-masses (e.g. condensed galaxies, black holes, etc.) in *lens plane* or *deflector plane*.

- Consider a light source in the plane parallel to the lens plane (*source plane*) and perpendicular to the line of sight from the observer.

- Due to deflection of light by masses multiple images of the source are formed. This phenomenon is known as *gravitational microlensing*.



Gravitational Lens
Galaxy Cluster 0024+1654

HST · WFPC2

PRC96-10 · ST ScI OPO · April 24, 1996

W.N. Colley (Princeton University), E. Turner (Princeton University), J.A. Tyson (AT&T Bell Labs) and NASA

Lens Equation for Co-Planar Point Masses

- Light source is located in the position w in the *source plane*.

- The lensed image is located at the position z in the *lens plane*.

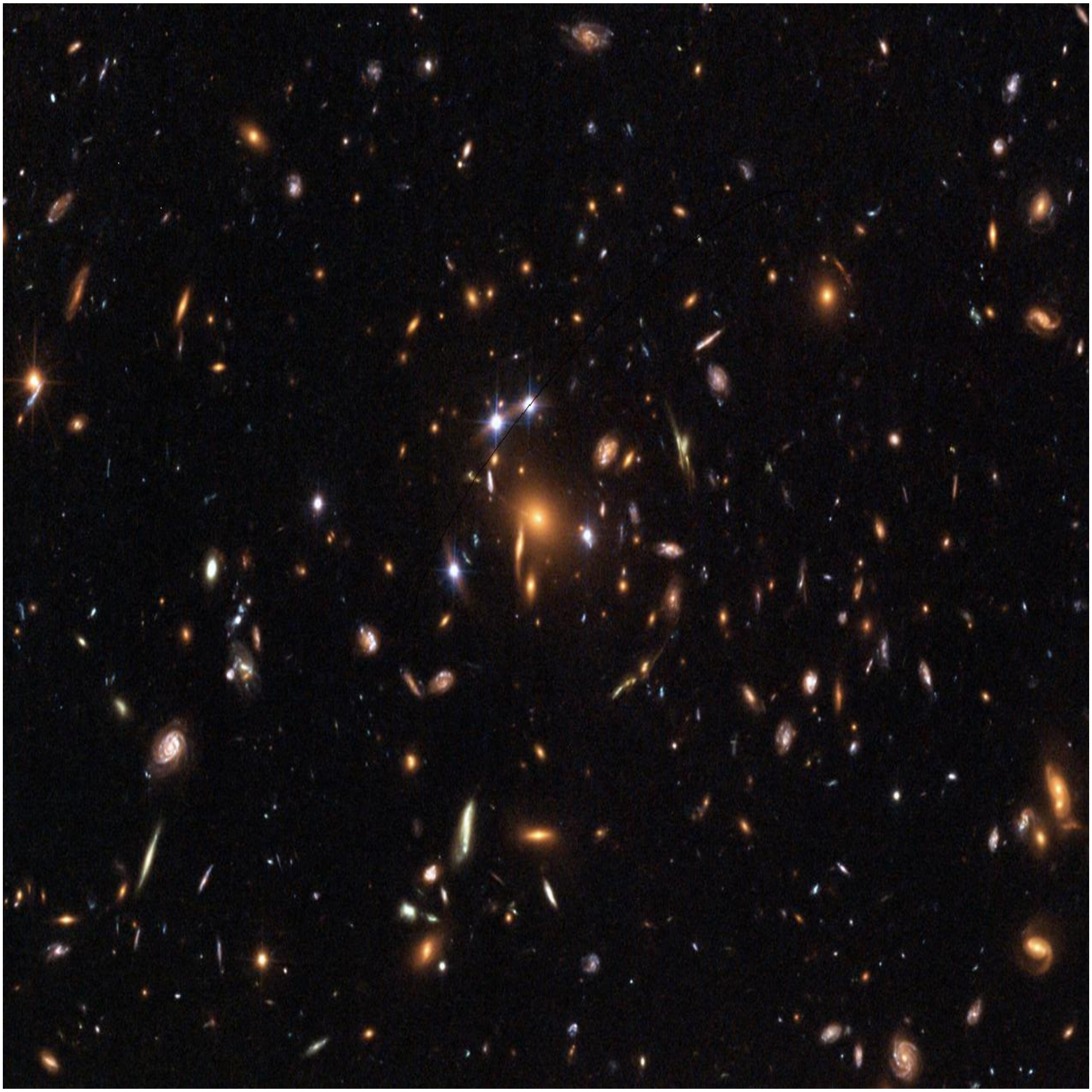
- The masses are located at the positions z_j in the *lens plane*.

$$w = z - \sum_1^n \sigma_j / (\overline{z} - \overline{z_j}),$$

where $\sigma_j \neq 0$ are real constants. Letting $r(z) = \sum_1^n \sigma_j/(z - z_j) + \overline{w}$, the lens equation becomes

$$z - \overline{r(z)} = 0, \, deg\, r = n.$$

The number of solutions = the number of "lensed" images.



History

• First calculations of the deflection angle based on Newton's corpuscular theory of light and the gravitational law (H. Cavendish and Reverend J. Michell circa 1784, P. Laplace - 1796, J. Soldner, 1804 - the first published calculation).

• $n = 1$ (one mass) A. Einstein (circa '33), either two images or the whole circle ("Einstein ring").

• H. Witt ('90) For $n > 1$ the maximum number of observed images is $\leq n^2 + 1$.

• S.H. Rhie ('01) conjectured the upper bound for the number of lensed images for an n-lens is $5n - 5$.

Solution

- (Mao - Petters - Witt, '97) The maximum is $\geq 3n + 1$

- $n = 2, 3$ ('97-'03)(Mao, Petters, Witt, Rhie) - the maximum is $5, 10$ respectively.

- $n = 4,$ is the maximum 15 or 17?

Theorem 1. *(G. Neumann-DK, '06). The number of lensed images by an n-mass lens cannot exceed $5n - 5$ and this bound is sharp (Rhie, '03). Moreover, it follows from the proof that the number of images is even when n is odd and vice versa.*

Quadratic vs. Linear Numbers of Images

A model problem: Let $p(z) := a_n z^n + ... + a_0$, $a_n \neq 0$ be a polynomial of degree $n > 1$.

Question. Estimate $\sharp\{z : z - \overline{p(z)} = 0\}$, or more generally,

$$\sharp\{(x, y) : A(x, y) + iB(x, y) = 0\},$$

where A, B are real polynomials of degree $\leq n$.

Bezout's theorem implies

$$\sharp\{(x, y) : A = B = 0\} \leq n^2.$$

Conjecture 1. *(T. Sheil-Small - A. Wilmshurst, '92)*

$$\sharp\{z : p(z) - \bar{z} = 0, n > 1\} \leq 3n - 2.$$

Results

- In the 1990s D. Sarason and B. Crofoot and, independently, D. Bshouty, A. Lyzzaik and W. Hengartner verified it for $n = 2, 3$.

- In 2001, G. Swiatek and DK proved Conjecture 1 for all $n > 1$.

- In 2003-2005 L. Geyer showed that $3n - 2$ bound is sharp for all n.

Examples

- One-point mass lens with source at $w = 0$.

$$z - \frac{c}{\overline{z} - \overline{a}} = 0.$$

Two images for $a \neq 0$, a circle ("Einstein ring") for $a = 0$, i.e., when the observer, the lens and the source coalesce.

- One-point lens with the tidal perturbation (a "shear") from a far away galaxy, a Chang-Refsdal lens.

$$z - \frac{c}{\overline{z}} - \gamma z = w.$$

The equation reduces to a quadratic and Bezout's theorem yields a bound of at most 4 images. Curves cannot occur!

Continuous Mass Distribution

For a continuous real-valued mass-distribution μ in a region Ω in the plane the lens equation with shear takes form

$$z - \int_{\Omega} \frac{d\mu(\zeta)}{\bar{z} - \bar{\zeta}} - \gamma \bar{z} = w.$$

- $\mu = n > 1$ non-overlapping radially symmetric masses. The number of images "outside" the masses $\leq 5n-5$ if $\gamma = 0$ and $\leq 5n$ if $\gamma \neq 0$ (DK-G. Neumann '06, refinements by J. H. An and N. W. Evans, '06).

- $\mu =$ uniform-mass distribution inside a quadrature domain Ω of order n, i.e. $\Omega = \phi(\mathbf{D}), \phi$ is a rational function with n poles univalent in $\mathbf{D} := \{|z| < 1\}$. The number of images outside Ω is $\leq 5n - 5$ (DK-GN, '06).

Smooth Mass Distributions

(W. L. Burke's Theorem, '81). *The number of images is always odd.* ($\gamma = 0$).

Take $w = 0$. Let n_+ be the number of sense preserving images and n_- - the number of sense reversing images. Argument principle yields

$1 = n_+ - n_-$, so the total number of images

$$N = n_+ + n_- = 2n_- + 1.$$

Einstein Rings are Ellipses

Theorem 2. *(CF-CK-DK,'07) For any lens μ, if the lensing produces an image "curve" surrounding the lens, it is either a circle when the shear $\gamma = 0$, or an ellipse.*

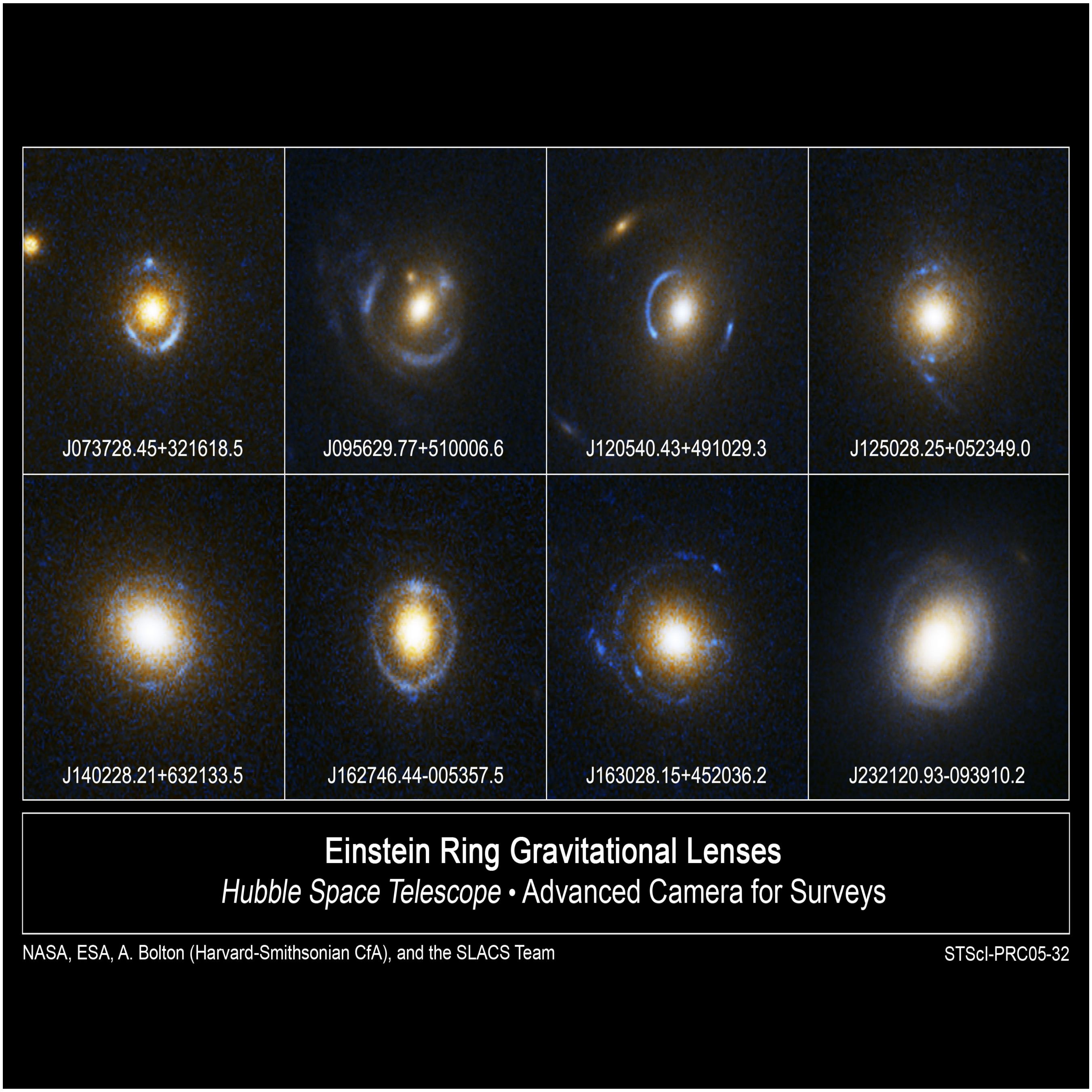
For an illustration assume the shear = 0. If the lens produces an image which is a curve Γ, then

$$\bar{z} = \int_{\Omega} \frac{d\mu(\zeta)}{z - \zeta} \text{ on } \Gamma$$

.

The integral is an analytic function in $\mathbf{C} \setminus \Omega$ vanishing at ∞. Hence $|z|^2$ matches on Γ a bounded analytic function in $\mathbf{C} \setminus \Omega$ and must be a constant.

Remark 1. *Using P. Divé's converse to the Newton's "no gravity in the ellipsoidal cavity" theorem, we can extend the above result to higher dimensions.*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J162746.44-005357.5
J163028.15+452036.2
J232120.93-093910.2
Einstein Ring Gravitational Lenses
Hubble Space Telescope • Advanced Camera for Surveys
NASA, ESA, A. Bolton (Harvard-Smithsonian CfA), and the SLACS Team
STScI-PRC05-32

Ellipsoidal Lens

Lens $\Omega = \{x^2/a^2 + y^2/b^2 \leq 1, a > b > 0\}$, with constant density. $c^2 = a^2 - b^2$.

$$\bar{z} - \frac{1}{\pi}\int_{\Omega} \frac{dA(\zeta)}{z - \zeta} - \gamma z = \bar{w},$$

or, using complex Green's formula,

$$\bar{z} - \frac{1}{2\pi i}\int_{\partial\Omega} \frac{\bar{\zeta} d\zeta}{z - \zeta} - \gamma z = \bar{w}.$$

The Schwarz Function of the Ellipse

The *Schwarz function* $S(\zeta) = \bar{\zeta}$ of $\partial\Omega$.

$$
\begin{aligned}
S(\zeta) &= \frac{a^2+b^2}{c^2}\zeta - \frac{2ab}{c^2}(\zeta - \sqrt{\zeta^2 - c^2}) \\
&= \frac{a^2+b^2-2ab}{c^2}\zeta + \frac{2ab}{c^2}(\zeta - \sqrt{\zeta^2 - c^2}) \\
&= S_1(\zeta) + S_2(\zeta),
\end{aligned}
$$

where S_1 analytic inside Ω, S_2 - outside Ω, $S_2(\infty) = 0$. This is the Plemelj-Sokhotsky decomposition of the Schwarz function of $\partial\Omega$.

- For z outside Ω the lens equation then reduces to

  $$\bar{z} + \frac{2ab}{c^2}(z - \sqrt{z^2 - c^2}) - \gamma z = \bar{w},$$

  that may have at most 4 solutions by Bezout's theorem.

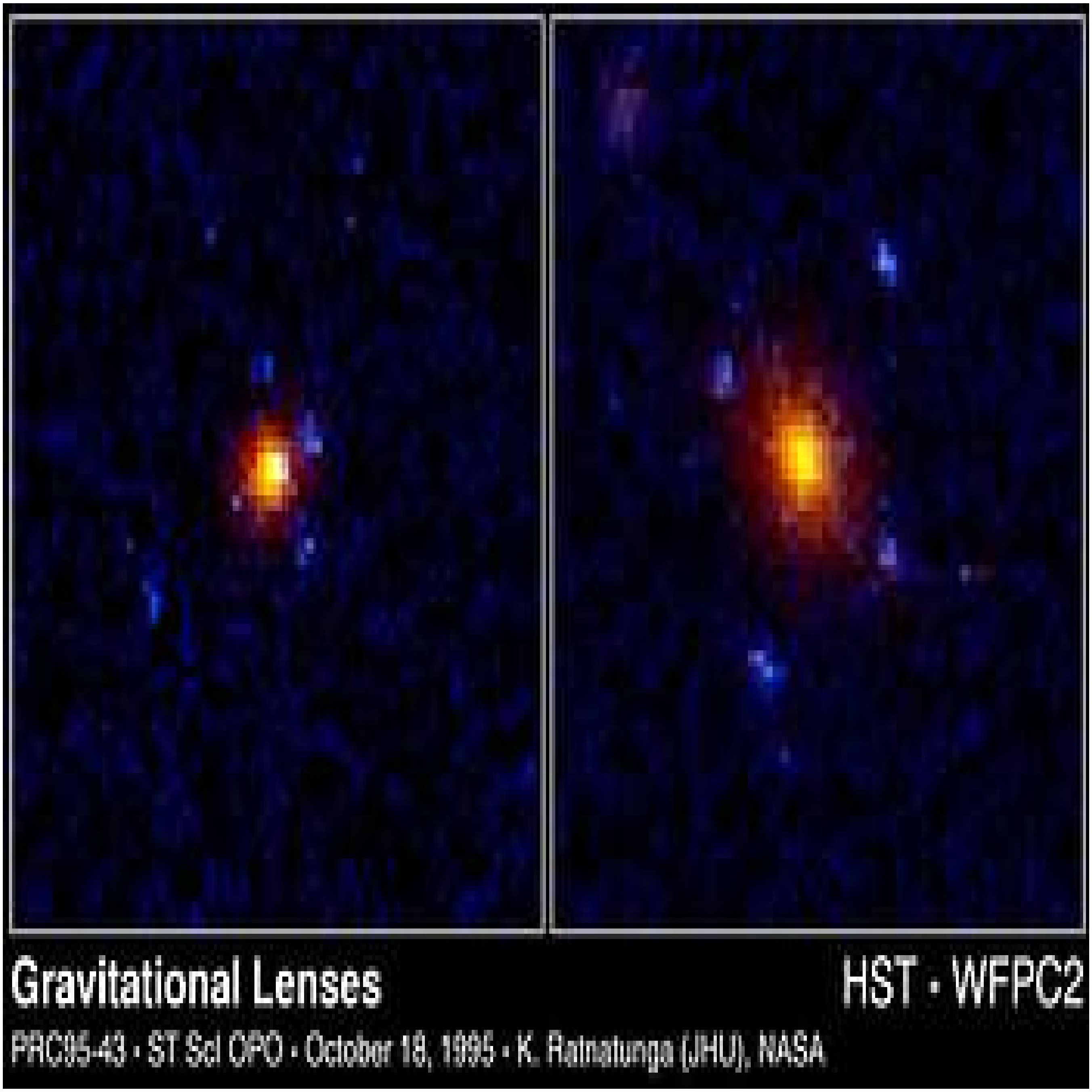
- For z inside Ω the lens equation reduces to a linear equation giving at most one solution.

  Theorem 3. *(CF-CK-DK) An elliptic galaxy Ω with a uniform mass density may produce at most* 4 *"bright" lensing images of a point light source outside Ω, and at most one "dim" image inside Ω, i.e., at most* 5 *lensing images altogether.*

Confocal Ellipses

MacLaurin's mean value theorem concerning potentials of confocal ellipsoids readily yields

Corollary 1. *An elliptic galaxy Ω with mass density that is constant on ellipses confocal with Ω, may produce at most* 4 *"bright" lensing images of a point light source outside Ω, and at most one "dim" image inside Ω, i.e., at most* 5 *lensing images altogether.*


Gravitational Lenses
HST • WFPC2
PRC95-43 • ST ScI OPO • October 18, 1995 • K. Ratnatunga (JHU), NASA

“Isothermal” Elliptical Lenses

- Density, inversely proportional to the distance from the origin, is constant on ellipses $\Gamma_t := \{x^2/a^2 + y^2/b^2 = t\}$ homothetic with $\partial\Omega$.

- Lens equation becomes transcendental:
$$z - const \int_0^1 \frac{dt}{\sqrt{\bar{z}^2 - c^2 t^2}} - \gamma \bar{z} = w.$$

- There are no more than 5 images ($4+1$) observed as of today.

- In 2000 Ch. Keeton, S. Mao and H. J. Witt constructed models with a strong tidal perturbation (shear) having 9, (8 bright+1 dim), images.

Remarks

- An isothermal sphere with a shear is covered by '06 K-N theorem (cf. also '06 paper by An - Evans on Chang-Refsdal lens)and may produce at most 4 images (observed).

- A rigorous proof that an isothermal elliptical lens may only produce finitely many images is still missing.

Critical Curves and Caustics

- Jacobian of the lens map $L(z) = z - \overline{p(z)} = w$ with potential $p(z)$

  $J(z) = 1 - |p'(z)|^2$.

- *Critical Curve* $C := \{z : J = 0\}$.

- *Caustic* $C' = L(C)$.

- $J(z)$ is the area distortion factor. Its reciprocal expresses the ratio of the apparent solid angle covering the lensed images z to that of the original source w, called *magnification*.

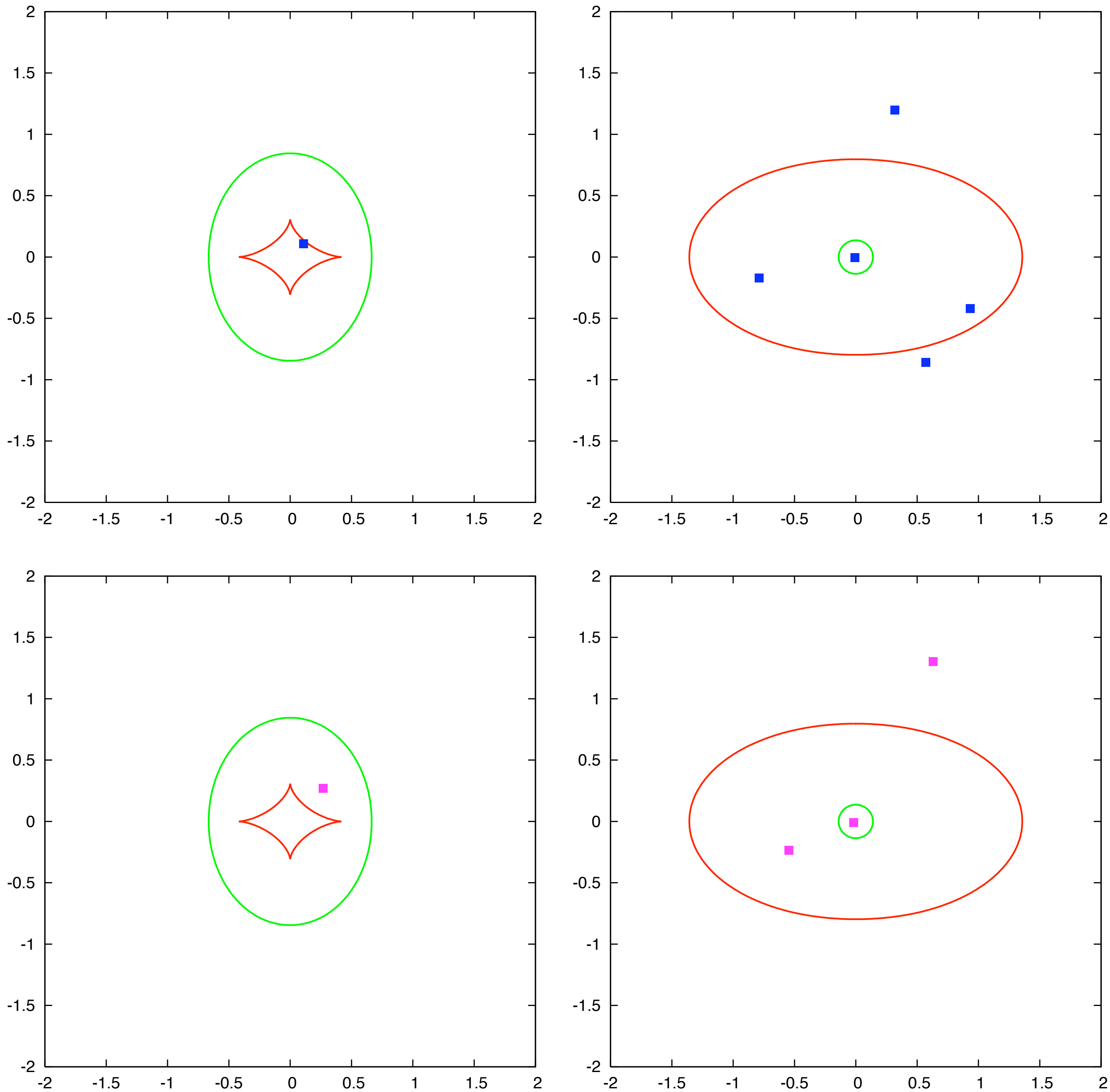
- Caustics indicate positions for the source where magnification tends to infinity.

Remarks

- Critical curves are *lemniscates*, caustics and their pre-images, "pre-caustics" , $L^{-}1(C')$ are much more complicated.

- Geometry of critical curves and caustics especially for $2, 3$ and 4 point lenses was modeled and studied by astrophysicists An, Evans, Keeton, Mao, Petters, Rhie, Witt to name just a few and, independently, by Bshouty, Hengartner, Lyzzaik, Neumann, Ortel, Suez, Suffridge, Wilmshurst. G. Neumann's thesis '03 has a variety of deep, novel geometric results.

- (K-N '06, conjectured by Rhie '01). The total number of "positive" $(J \geq 0)$ images produced by an $n >$ 1-point mass lens in absence of a tidal perturbation is $\leq 2n - 2$. Further refinements can be found in '06 work of An and Evans.



Isothermal Ellipsoid

QUESTIONS

Lensing by a uniform mass in a Q. D..

- Geometric interplay between critical curve(s) vs. the boundary of the q.d.

- Estimate the number of "dim" images inside the q.d. = in-depth study of the algebraic part of the Schwarz function.

- Valence of algebraic vs.transcendental harmonic mappings (cf. G. Neumann's papers '05, '07).

  Model Problem: Sharp estimate for $\sharp\{z : \bar{z}^m - p(z) = 0\}$, $n := deg\, p >> m$. Wilmshurst ('94) conjectured the upper bound $m(m-1) + 3n - 2$.

- Estimate the number of bright images for a polynomial mass density.

Elliptic Lenses

- Maximal number of images for the isothermal elliptical lens.

- Elliptical lens with a polynomial (rational) mass density
  (i) Maximal number of images
  (ii) Critical curves and pre-caustics
  (iii)Anomalies related to arbitrary continuous mass-densities

- Lensing by several elliptical masses (observed so far 2 galaxies lens giving 5 images and 3 galaxies lens with 6).

Three-Dimensional Lensing

The 3-dimensional lens equation with mass-distribution $dm(y)$ with source at $\vec{w}$ becomes

$$\vec{x} - \nabla_x \left(\int \frac{dm(y)}{|x-y|} \right) = \vec{w}.$$

- $dm = \sum_1^n c_j \delta_{y_j}$. There are rough estimates for the maximal number of images (Petters, '90s) based on Morse theory.

- A difficult Maxwell's problem concerns a number of stationary points of the Newtonian potential of n point-masses (conjectured $\leq (n-1)^2$). Most recent progress due to Eremenko, Gabrielov, D. Novikov, B. Shapiro.

- Ellipsoidal mass densities.

- Critical surfaces, caustics and pre-caustics of the lens map.

  (CF-CK-DK, '07: "Einstein" surfaces can only be either spherical in absence of a shear, or ellipsoidal.)

- Other mass-densities???

THANK YOU!